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AN

INTRODUCTION

TO

THE THEORY AND PRACTICE

OF

PLANE AND SPHERICAL

TRIGONOMETRY,

AND

THE STEREOGRAPHIC PROJECTION OF THE SPHERE;

INCLUDING

THE THEORY OF NAVIGATION:

COMPREHENDING

A Variety of Rules, Formulæ, &c. with their Practical Applications to the Mensuration of Heights and Distances; to determining the Latitude by two Altitudes of the Sun, the Longitude by the Lunar Observations, and to other important Problems on the Sphere, and on Nautical Astronomy.

BY THOMAS KEITH.

THE SEVENTH EDITION, CORRECTED AND IMPROVED,

BY SAMUEL MAYNARD,

Author of the Key to Keith's Mensuration, and Editor of Keith's Arithmetic and Key, Geometry, Mensuration, &c. &c.

La Trigonométrie est sans contredit une des plus utiles applications de la Géométrie élémentaire: elle est la base de la Geodésie, de la Géographie, de l'Astronomie et de la Navigation. — Discours sur les Progrès des Sciences, rendu par l'Institut de France, 1809, page 6.

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1839.

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EDITOR'S PREFACE.

THE present edition of KEITH'S TRIGONOMETRY has undergone several important changes, and the work is now given in a form more in accordance with the modern state of the science. Various investigations and demonstrations have been pruned and remodelled, some important errors corrected, and the notation and enunciations throughout improved. The astronomical problems have been made to depend for their data on the Nautical Almanac for 1840. Some alteration has also been made in the Tables at the end of the volume, the logarithms, &c. being given in full, instead of having their leading figures suppressed, as in former editions. This arrangement gives to the computer a greater facility and confidence in the use of tables, and has in consequence been adopted. The examples and numerical operations have all been carefully examined, so that few errors of consequence are likely to have escaped.

SAMUEL MAYNARD.

8. *Earl's Court, Cranbourne Street,*
Leicester Square, February 20. 1839.

PREFACE.

TRIGONOMETRY is an important branch of the mathematical sciences: the speculative parts, like the Elements of Euclid, habituate the mind to close and demonstrative reasoning; and the practical parts are of extensive use in the common concerns of life.

By Trigonometry we determine the magnitudes of the earth and planets, and the positions of the fixed stars with respect to each other, by which we are enabled to depict the appearance of the heavens in a small compass.

The distances of the planets from the sun, their motions, eclipses, and other phenomena, are calculated by Trigonometry; as are likewise the distances and positions of places on the earth, with their latitudes and longitudes: it may, therefore, justly be considered as the basis of Astronomy and Geography.

Navigation, with all its modern improvements, depends entirely on Trigonometry, which is likewise the foundation of maritime surveying, and of almost every branch of practical mathematics; accordingly, we find this subject has been studied in the earliest ages of mathematical learning. Among the ancients were *Hipparchus*, *Theodosius*, *Menelaus*, *Ptolemy*, &c., who contributed to the advancement of this science.

The improvements in Astronomy, Navigation, and Trigonometry nearly kept pace with each other. The invention of Logarithms by Baron *Napier* was an invaluable acquisition to these sciences; and the improvements made by this illustrious person, in Spherical Trigonometry, will be a lasting monument of his penetration and judgment. From this period the history of Logarithms and that of Trigonometry are closely connected, and there is scarcely a writer on one of these subjects, who has not likewise introduced the other.

The first authors of note, after the invention of Logarithms, were *Briggs* and *Gellibrand.* — Dr. *Charles Hutton*, in the Introduction to his Logarithms, has given a very complete account of the different writers on that subject, which likewise includes the principal authors on Trigonometry; to this valuable work the Reader is, therefore, referred for further information in the history of the science.

The authors on Trigonometry may be divided into two classes, theoretical and practical; for none of them have combined the theory with the practice, in such a manner as to render the subject plain and intelligible to a learner: the most valuable and scientifical are, in general, too abstruse, and the practical scarcely furnish the student with the rationale of a single rule or operation.

The object of the ensuing treatise is to simplify the theory, yet to retain a methodical and accurate mode of investigation, and to exemplify this theory by a variety of important and useful examples.

The demonstrations are frequently founded on principles strictly Geometrical, especially where those principles can be rendered very plain and perspicuous by the assistance of simple diagrams; and sometimes the process of demonstration is conducted by algebraical signs, particularly where the Geometrical method would require a complicated figure, or a long and tedious process.

In the construction of various formulæ, the algebraical mode of deduction tends greatly to simplify the subject; yet the definitions and the elementary parts of the science must be acquired from Geometrical principles illustrated by diagrams; otherwise a student will never obtain a clear and satisfactory knowledge of the subject. Should any person attempt to teach the elementary principles of the science by the assistance of algebraical characters, and algebraical formulæ alone, without the aid of Geometry, he would most assuredly deceive both himself and his pupils.

It appears, then, if the above observations be correct, that Trigonometry cannot be learnt expeditiously and completely, either from principles purely geometrical, or purely algebraical, without sacrificing utility to the uniformity of system.

By way of illustration, let us compare some parts of the Trigonometry at the end of Dr. *Simson*'s Euclid, with similar parts of the Trigonometry at the end of *Legendre*'s Geometry.

FIRST, *Where the three sides of a spherical triangle are given, to find an angle.*

In Prop. XXVIII. of Dr. *Simson*'s work, the process is conducted from principles purely Geometrical, and the demonstration is long and tedious; whereas in *Legendre*'s Geometry, article LXXVIII, page 389, 6th edition (where the demonstration is similar to that given at page 185. of the ensuing treatise), it is short and perspicuous.

SECONDLY. In article LXXVII, page 387. of *Legendre*'s Geometry, where he has shown that *the sines of the sides of a spherical triangle are directly proportional to the sines of their opposite angles;* the demonstration, which is purely algebraical, is complicated, unnatural, and almost incomprehensible to a young student; whilst that given by Dr. *Simson* in Prop. XXIII, from Geometrical principles, is simple and obvious.* See likewise page 176. of the ensuing work, where the rule is deduced from Baron *Napier*.

These remarks are not introduced with a design to criticise the works of either of these eminent authors, but to show the insufficiency of the Geometrical and of the Algebraical mode of demonstration independent of each other.

The following work may, perhaps, at first view, appear to be too much extended for an introductory treatise; but a little attention to the subject will soon correct this supposition. The student is here supplied with THREE DISTINCT TREATISES, which are all essential towards his future progress in the science. *Firstly*, A treatise on Logarithms, and the necessary tables. *Secondly*, A treatise on Practical Trigonometry; and, *thirdly*, A theoretical treatise on the subject. A work which is deficient in any one of these three dependent parts, will oblige the student to have recourse to different authors; whereby the want of uniformity of reference, and of regularity of composition, will considerably retard his progress at the commencement of his studies.

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* It may be proper to remark, that *Legendre* has given a Geometrical demonstration, at page 385. of his treatise, which is equally as plain as that given by Dr. *Simson*.

"An Introduction" (says Lord *Bacon*) "ought to have two properties; the one, that of a perspicuous and clear method; the other, that of an universal latitude and comprehension; where the student may have a little pre-notion of every thing, like a model towards a great building."

Convinced of the propriety of these remarks, the author has employed much time and attention in the arrangement and execution of the subject; bearing constantly in mind the purpose for which he was writing, viz. *the Instruction of young Students.*

The Work is divided into FOUR BOOKS.

BOOK I. contains the nature and use of Logarithms, the explanation of the Tables, the construction and use of *Gunter's* Scale, and some preparatory Geometrical Problems.

BOOK II. comprises the principles and practice of Plane Trigonometry, illustrated by a variety of useful examples in the mensuration of distances, heights, &c.

The substance of this book is the same as that of Book I. of the first edition, with the exception of some additions, and an alteration in the arrangement of the several chapters. The easy parts are here placed together, and the demonstrations of the rules are separated from their practical applications; so that the student who wishes to acquire the practical parts only, may omit the demonstrations, and proceed regularly with those parts which are the sole object of his pursuit; whilst the theoretical reader may, if he thinks proper, omit the practical parts, and continue the theory without interruption.

The greater part of the *fourth Chapter* was added to the second edition, and will be found to contain information which will be useful on several occasions.

The *fifth Chapter* is nearly the same as *Chap. I.* Book I. of the first edition, except that the formulæ have been somewhat extended. The substance of the two general Scholia from pages 6th and 11th of the second edition of Mr. *Emerson's* Trigonometry (which were noticed in the 11th page of the first edition of this work,) has been introduced and commodiously arranged according to the notation of *Mauduit*, *Legendre*, &c. This chapter likewise contains a variety of formulæ, exclusive of those already mentioned, together with series for finding the sines, cosines, &c. of arcs, and the construction of a table of sines by the continual bisection of an arc. This method of construction was made use of on account of its simple and obvious principles, and is similar to that given in the Trigonometry of *Nikitin* and *Souvoroff*, inspectors of the Imperial Academy at Cronstadt, and to the method used by the late Dr. *Horsley*, Bishop of St. Asaph, at page 137. of his treatise.

BOOK III. comprehends the Theory and Practice of Spherical Trigonometry, and the Stereographic Projection of the Sphere.

The *first Chapter* contains Spherical Geometry, including various properties of Spherical Angles, Arcs, and Triangles, demonstrated in a plain and easy manner.

The *second Chapter* is wholly occupied with the Stereographical Projection of the Sphere, which was placed at the end of Spherical Trigonometry in the first edition: it has been brought forward, in this edition, in order that the constructions of the different practical cases of spherical triangles may appear with the calculations. The only alteration here is in the arrangement.

The *third Chapter* contains the investigation of general rules for calculating the sides and angles of right-angled Spherical Triangles. The XXIIId proposition, page 167, which is partly derived from the general figure page 151, is very extensive in its application. The original construction of this figure is ascribed, by Dr. *Horsley*, to *Corpernicus*, the celebrated astronomer. The demonstration of Baron *Napier*'s rule is derived from the XXIIId proposition. This rule is the most simple and comprehensive that ever was invented, for solving the different cases of right-angled spherical triangles, but has generally been explained in a confused and unintelligible manner. Valuable as this rule is, it has been censured as "too artificial and restricted to be generally employed in the present advanced state of the science." It is, notwithstanding this aspersion, susceptible of general application, and the species of the different cases may be as accurately ascertained from its equations, as from any rules, or formulæ, hitherto invented; in consequence of which, it will, undoubtedly, continue to be used, by those who know how to appreciate its merit, when the works of the author who has presumed to censure it shall be forgotten.

The *fourth Chapter* shows the manner of applying Baron *Napier*'s rule to the solutions of the different cases of oblique-angled spherical triangles, from which several useful rules are derived.

The *fifth Chapter* is, in substance, the same as the IVth Chapter of Book II. in the first edition, but the mode of demonstration has been varied. In this edition the method used by *Lagrange* and *Legendre* has been adopted, as being more simple than that used in the first edition. This chapter contains the whole of the formulæ given by *Legendre*, from page 386. to 399. of his Geometry, besides several others. These formulæ, however, are not copied from *Legendre*, each succeeding step being regularly deduced from the preceding, or from different propositions, to which references are given. These particulars, so essential to a learner, are entirely disregarded by *Legendre*: for illustration, compare article LXXXVIII. page 398. of *Legendre*'s Geometry, with U. 192. and W. 192. of the succeeding work, &c. At page 193, &c. the various formulæ relating to Spherical Trigonometry are concentrated, and classed under their appropriate heads. Some of these formulæ are the same with those given by *Delambre* in his preface to *Borda*'s trigonometrical tables.

The formulæ, where auxiliary angles are introduced (Q. 190, &c.) are, in effect, analogous to those which may be deduced from Baron *Napier*'s rule, by drawing a perpendicular from the vertical angle of a spherical triangle upon the base, as in proposition xxv, page 173, &c.

The *sixth*, *seventh*, *eighth*, and *ninth Chapters* contain the logarithmical solutions of the various cases of Spherical Trigonometry.

The *tenth*, *eleventh*, and *twelfth Chapters* include the substance of Book III. of the first edition, though the matter has undergone a considerable change: the original examples were all adapted to the year 1796, but in this edition those which depend on the Nautical Almanac are carried forward to the year 1822. The Nautical Almanac for that year can easily be procured. The necessary alteration has been attended with considerable labour, and, it is presumed, will be of much advantage to the young student. The method of finding the correct distance between the sun and the moon, or between the moon and a star, by having the apparent distance and the apparent altitudes of these objects given, is shorter than any that has hitherto appeared, where the common tables only are used, besides having the advantage of being perfectly correct; for it is not liable to those inaccuracies which frequently arise from taking the logarithmical, or natural, sines and cosines out of the tables; as the observed distance is never very small, nor ever near 180 degrees.

The Nautical Almanac was first published in the year 1767; from which time, no treatise on Trigonometry had appeared, containing astronomical examples adapted to that work, until the first edition of the following treatise; wherein the author, with great labour and attention, suited all his examples to the year 1796, and corrected the situations of all the stars to the same year: the *whole* of these examples, with the rules and observations upon them, were *copied** and republished, without a single acknowledgment, in less than five years after their first appearance.

The *thirteenth Chapter* contains the fluxional analogies of spherical triangles, with a few examples showing their use. This subject was first considered by Mr. *Cotes*, of Trinity College, Cambridge, in a treatise entitled De Æstimatione Errorum in Mixtâ Mathesi, which was published by Dr. *Robert Smith*, in the year 1722, six years after the author's decease.

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* In a *second edition* of the work into which these subjects were copied, it is stated that "nearly all the practical questions before given, *many of which had* "*passed, in the same state, from one work to another, for more than a century past,* "*are here reproposed with new data,*" &c. This insinuation will not have the effect intended, with those who are capable of examining the matter. However, in consequence thereof, I think it proper to state, that several of the rules, and likewise examples, stand in *that* second edition without alteration; they are, of course, precisely the same as those in the *first* edition of *my* work, and some of them are *incorrect*. The data will show how long they have been in use.

This branch of the science has hitherto received very little improvement, either from the English or foreign mathematicians, although a century has nearly elapsed since the first publication of Mr. *Cotes's* Treatise.

The substance of all the *differential equations* given by *Cagnoli*, *La Lande*, *Delambre*, &c. is included in the XIIIth Chapter; these differential equations may readily be deduced from *Simpson's Fluxions*.

The *fourteenth Chapter* consists of Miscellaneous Propositions, which could not easily be classed under different heads.

The first comprehends the French division of the circle. The sexagesimal division of the circle, which is ascribed to the Egyptians, has often been objected to by mathematicians, as by *Oughtred*, *Wallis*, &c. who recommend a decimal or centesimal division. Several tables have been constructed centesimally, particularly one by Dr. *John Newton*. Dr. *Charles Hutton*, in a memoir published in the Philosophical Transactions for 1783, proposed to divide the quadrant into equal decimal parts of the radius; by which means the degrees, or the divisions of the arc, would be the real lengths of the arcs, in terms of the radius.

The French have lately adopted the centesimal division of the circle in their calculations; and tables of greater extent than those by Dr. *Newton* have been published, as the Tables of *Hobert* and *Ideler*, printed at Berlin, in the year 1799; the tables of *Borda*, printed at Paris, an IX, with a preface by *Delambre*, &c.

The advantages of this new division of the circle, should it be generally adopted in practical calculations, are few and trifling, when compared with the confusion and perplexity it would occasion. It is true that degrees, &c. would be more readily turned into minutes or seconds, *et vice versâ*, and some other advantages of minor importance would be obtained, were the new division to be *universally* adopted: at the same time, all our valuable tables would be rendered useless; the many well-established trigonometrical and astronomical works, which from time to time have been published, would be little better than waste paper; the most valuable mathematical instruments, which have been constructed by celebrated artists, must be considered as lumber in the different observatories of Europe; the latitudes and longitudes of places must be changed, which change would render all the different works on Geography useless; or otherwise the Astronomers, and those in the habit of making trigonometrical calculations, must be perpetually turning the old division of the circle into the new, or the new into the old.

The best instruments in the observatories of Paris, Palermo, and of the different capitals of Europe, have been constructed by British artists, viz. by *Ramsden*, *Troughton*, &c. and are graduated according to the sexagesimal division of the circle: it is plain, therefore, that all observations made with these instruments must be reduced to the centesimal division of the circle before they can be used in calculation. Again, The logarithmical tables of sines, tangents, &c. which were originally constructed by the British mathematicians, have passed through so many hands, and have been so often examined, that they may be depended upon as correct; whilst the new tables would require great caution in using them.

In devising short methods of constructing the new tables, foreign mathematicians have, however, discovered several formulæ, which may be useful in examining or extending the old tables, and in some astronomical calculations. These formulæ comprehend the most beneficial parts of their labours, as their new division of the circle will doubtless, ere long, share the same fate as their new calendar; unless the works, which may hereafter be framed according to that model, should possess extraordinary and continued merit.

From an attentive investigation of this subject, it will appear, that, if any merit can be attached to the new division of the circle, the invention cannot, with any degree of justice, be ascribed to the French mathematicians, and that their anxious desire for its universal adoption is the natural result of the French revolution, the general tendency of which has invariably been to depreciate the merit of every subject which has not the glory of the French nation or the praise of Frenchmen for its basis.

The improvements made by the French in the various branches of mathematics, though highly extolled, will be found, in many cases, to be more specious than real; the modern analysis so *universally* adopted (and which in some instances is certainly illogical, if not unscientifical) has contributed greatly to vitiate the taste and explode that solid and accurate method of reasoning which is so conspicuous in the writings of the ancients. A flimsy mode of demonstration, grounded on a dexterous management of algebraical characters, has frequently been substituted for perspicuity and logical exactness. By making these remarks, the author does not mean to insinuate that the French have made no improvements in the mathematical sciences; he is convinced that they have made several, and that in a variety of instances their method of demonstration is neat and perspicuous; but his opinion is, that the value of those improvements has been greatly over-rated.

After considering the French division of the circle, the remainder of the *fourteenth Chapter* is occupied with problems which have been frequently applied in extensive trigonometrical surveys.

BOOK IV. treats of the Theory of Navigation; a subject intimately connected with Trigonometry, and immediately derived from the sphere.

The work CONCLUDES with a table of the logarithms of all numbers from an unit to ten thousand; a table of natural sines; a table of logarithmical sines and tangents to every degree and minute of the quadrant; with some other tables useful in the solutions of the different astronomical problems. These tables occupy only fifty pages, and will be found a valuable acquisition both to the teacher and the learner, as they will save the expense of a large set of tables, and answer every purpose of instruction with equal advantage.

ADVERTISEMENT

TO

THE FOURTH EDITION.

THIS edition of the ensuing work has been carefully revised. All the astronomical examples which depend on the Nautical Almanac are adapted to the year 1822, and they may be solved either with or without the Almanac; because the several articles necessary to be taken from that work are given at the end of the different examples. By these means the students will readily learn the use of the Nautical Almanac, and in a school where several are studying the same subject, their progress will not be retarded by waiting for the Almanac. The examples will be found sufficiently numerous and appropriate for the purpose of instruction. They might easily have been extended, *with different dates*, by a selection from various authors; but such examples are perfectly useless to a student, who has not in his possession a Nautical Almanac corresponding with each date.

No. 1. *York-buildings, New-road,*
St. Mary-le-bone, London, October, 1820.

MEMOIR

OF

THOMAS KEITH, ESQ.

THOMAS KEITH, Esquire, the subject of the present brief Memoir, professor of Mathematics, and Author of many distinguished works, was born in 1759, at Brandsburton, near Beverley, in the county of York.

His parents bestowed on him a respectable education; but by the death of his father he was thrown on the world with but slender pecuniary means, and left to make his way in it by his talents and industry.

He first engaged himself in a family, as private tutor; and after spending a few years in this employ, he was induced, from the precarious and scanty subsistence which was to be obtained in the *country*, as well as from the favourable opinion which his friends entertained of his abilities and acquirements, to seek his fortune in London.

He arrived in the metropolis in the year 1781, where he soon became known, and his merits as a mathematician duly appreciated, from the many works which his indefatigable industry produced.

In 1789 he published "The Complete Practical Arithmetician," to which "A Key" was added in 1790, for the use of students. In 1791, an Abridgement of "The Complete Practical Arithmetician *," for the use of young students, appeared; but after passing through one edition, it was suppressed. Shortly afterwards he published an "Introduction to the Science of Geography." Besides these works, Mr. Keith published, in 1801, "An Introduction to the Theory and Practice of Plane and Spherical Trigonometry;" "A Treatise on the Use of the Globes," in 1805; and "The Elements of Geometry," in 1814. He likewise sent many contributions to the various mathematical periodicals which were published towards the end of the latter, and at the commencement of the present, century.

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* In an advertisement at the end of this work Mr. Keith announced for publication, "The New Universal Spelling Book, or Young Gentlemen and Ladies' Preceptor." But whether it appeared in print or not we have not been able to ascertain.

Mr. Keith superintended several editions of "Hawney's Complete Measurer *," "Paterson's Roads," "Geography and History, by a Lady, for the Use of her Pupils," &c.

The mathematical works of Mr. Keith have been long known and esteemed in this country. His great experience as a mathematical teacher, and his persevering habits of mind, well qualified him for the task of supplying our schools with a good practical course of mathematical instruction. All his works bear incontestable marks of having been produced with much care and considerable labour. Not, indeed, that he developed fresh discoveries, or even laid out new views with old materials, for his productions have little or no pretensions to this kind of originality. But the labour displayed in his writings consists in the fulness and completeness with which he has discussed every subject to which he directed his mind, never evading any topic legitimately connected with the matter of inquiry merely on the score of its difficulty. His work on the Globes, his

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* The first edition of Mr. William Hawney's "Complete Measurer" was published in London in the year 1717; and was recommended to the public as a very useful work by Dr. John Harris, secretary and vice-president of the Royal Society, an eminent mathematician, and author of several mathematical works. (The "*Lexicon Technicum*, or a Universal Dictionary of Arts and Sciences, 2 vols. folio, 1708," afterwards augmented to 3 vols. folio, was Dr. Harris's principal literary labour: this work was the foundation of all our Cyclopædias.) It appears, from an advertisement which is printed on the fly-leaf at the commencement of the first edition of the "Complete Measurer," that Mr. W. Hawney kept a school, and was private teacher of the mathematics at Lydd, or Lidd, in the county of Kent. The last edition of this work, which Mr. Keith superintended, was published in January, 1824; and since the editor's decease, two editions of it, and also two editions of his own "Arithmetic" and "Key" have appeared, considerably enlarged, by *Samuel Maynard.*

Arithmetic, his Trigonometry, &c., are all examples of this comprehensiveness of plan, and together form a body of information which cannot be too strongly recommended to the study of those who value science more for its practical utility than for its merely speculative beauties. Although we cannot claim for Mr. Keith the merit of originality, yet he has praise of another kind, that of usefulness, which those who tread the same path of science will not fail to accord to him. And this is a distinction which, if not the most eminent, is certainly among the most honourable; for it will be readily acknowledged, that those who smooth the road to science or literature, and facilitate to others the attainment of knowledge, are often more permanently useful than such as are endowed with original genius. The greatest personal or mental acquirements die with their possessors; but they who labour that others may be wise benefit all posterity. The writers of good introductory treatises on the sciences are therefore entitled to no mean praise; and their fame deserves to live in the genius which they have excited, and in the facilities which they have afforded to the acquisition of knowledge.

Mr. Keith's reputation as an accountant led to his being appointed by George III., in 1804, to the situation of Secretary to the Master of his Majesty's Household.

As a teacher, too, Mr. Keith was very successful, being engaged in families of the first distinction, as well as in some of the most respectable ladies' establishments in the vicinity of the metropolis.

As a proof of his eminence in this department, and the estimation in which he was held, it is only necessary to mention his appointment, in 1810, to the Professorship of Geography and the Sciences to her Royal Highness the Princess Charlotte of Wales; from whom, as well as from her Royal Highness the Princess Sophia Matilda, (who, with many other distinguished personages, had the benefit of his instruction,) he received the most flattering marks of attention and respect.

In 1814 he was nominated by the Speaker of the House of Commons, the Right Honourable C. Abbot, afterwards

Lord Colchester, to the then vacant situation of Accountant to the British Museum, the duties of which he performed to the time of his death.

In the month of November, 1822, he was afflicted with an internal disorder, which ultimately caused his death, on the 29th of June, 1824, at No. 1. York Buildings, New Road, St. Mary-le-bone, London, in his 65th year.

He ended his life with perfect composure and resignation, and retained almost to his last hour the exercise of those strong mental faculties, and of those kind and gentle manners, which had so much endeared him to his family and friends. Mr. Keith left behind him, nearly completed, a new work on the science of Geography, intended for the use of schools, which was published in 1826.

Such is a short account of this useful and valuable man. Though the incidents of his life, like those of most other laborious students, afford very scanty materials for biography, being little more than a journal of studies and a catalogue of publications, curiosity may find ample compensation by surveying in his works the character of his mind. And to such of our readers as ardently desire to be useful in their day and generation, we would say, that though they may feel the impossibility of immortalising their names by great or splendid achievements, they may at least imitate the industry and perseverance of Mr. Keith, and emulate his usefulness.

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BOOK I.

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EXPLANATION OF THE CHARACTERS OR MARKS
USED IN THE FOLLOWING WORK.

$+$, *Plus* or *more*, the sign of addition; as $AD+DC$, signifies that the line AD is to be increased by the line DC; and $4+3$ signifies that the number 4 is to be increased by the number 3.

$-$, *Minus* or *less*, the sign of subtraction, and shows that the second quantity is to be taken from the first; as $CB-GB$ shows that the line CB is to be diminished by the line GB.

$\times$, *Into* or *by*, the sign of multiplication; as $ED\times DC$ signifies the rectangle formed by the lines ED and DC, and $a\times b$ expresses the product of the quantity a by the quantity b. Also $a.b$ or ab signifies the same thing.

$\div$, *Divide by*, as $PB\div CS$, or $\frac{PB}{CS}$ signifies that PB is to be divided by CS.

AB^2, AB^3, signify the square and the cube of AB; also $(14)^{\frac{3}{4}}$ signifies that 14 is to be involved to the third power, and then the fourth root is to be extracted.

$\sqrt{A}$ or $A^{\frac{1}{2}}$, $\sqrt[3]{A}$ or $A^{\frac{1}{3}}$, express the square and cube root of A.

$=$, *Equal to*, as $AB=CD$, shows that AB is equal to CD.

$\sim$, *Difference*, as $A\sim B$, shows that the difference between A and B is to be taken.

A vinculum, or *parenthesis*, serves to link two or more quantities together, as $\overline{A+B}\times m$, or $m(A+B)$, signifies that A and B are first to be added together, and then to be multiplied by the quantity *m*.

$\left\{\begin{array}{ccc} : & :: & : \\ \textit{is to} & \textit{as} & \textit{is to} \end{array}\right\}$ *Proportion*, A : B :: C : D signifies that A has to B the same ratio which C has to D, and is usually read A is to B as C is to D.

∴ *Therefore.*

∠ *Angle*, as ∠ A, signifies the angle A.

$>$ *Greater than*, as $A > B$, shows that A is greater than B.

$<$ *Less than*, as $A < B$, shows A to be less than B.

The other characters are explained among the definitions in the work.

AN

INTRODUCTION

TO

PLANE AND SPHERICAL TRIGONOMETRY.

BOOK I.

CHAPTER I.

THE NATURE AND PROPERTIES OF LOGARITHMS.

Article (1). LOGARITHMS are a series of numbers contrived to facilitate arithmetical calculations; the operation of multiplication being performed by addition, division by subtraction, involution by multiplication, and the extraction of roots by division.

Logarithms may therefore be considered as *indices* to a series of numbers in geometrical progression, where the first term is unity, or its symbolical equivalent r^0. Let r^0, r^1, r^2, r^3, r^4, r^5, r^6, &c. be such a series, increasing from $r^0=1$; or r^0, $\frac{1}{r}$, $\frac{1}{r^2}$, $\frac{1}{r^3}$, $\frac{1}{r^4}$, $\frac{1}{r^5}$, $\frac{1}{r^6}$, &c. decreasing from $r^0=1$; which last series, agreeably to the established notation in algebra, may be thus expressed, r^0, r^{-1}, r^{-2}, r^{-3}, r^{-4}, r^{-5}, r^{-6}, &c. Here the common ratio is r, and the indices 0, 1, 2, 3, &c. or 0, —1, —2, —3, &c. are logarithms. Hence it is obvious, that if a series of numbers be in geometrical progression, their logarithms will constitute a series in arithmetical progression. And, where the series is increasing, the terms of the geometrical progression are obtained by multiplication, and those of the arithmetical progression, or logarithms, by addition; on the contrary, if the series be decreasing, the terms of the geometrical progression are obtained by division, and those of the arithmetical progression by subtraction.

The same observations apply to logarithms when they are fractions; thus if $r^{\frac{1}{n}}$ denote any number, then will r^0, $r^{\frac{1}{n}}$, $r^{\frac{2}{n}}$, $r^{\frac{3}{n}}$, $r^{\frac{4}{n}}$, $r^{\frac{5}{n}}$, $r^{\frac{6}{n}}$, &c. constitute an increasing series of numbers in geometrical progression, of which the indices $0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \frac{4}{n}, \frac{5}{n}, \frac{6}{n}$, &c. are the logarithms; or $r^0, \frac{1}{r^{\frac{1}{n}}}, \frac{1}{r^{\frac{2}{n}}}, \frac{1}{r^{\frac{3}{n}}}, \frac{1}{r^{\frac{4}{n}}}, \frac{1}{r^{\frac{5}{n}}}, \frac{1}{r^{\frac{6}{n}}}$, &c. will constitute a decreasing series which may be thus expressed, r^0, $r^{-\frac{1}{n}}$, $r^{-\frac{2}{n}}$, $r^{-\frac{3}{n}}$, $r^{-\frac{4}{n}}$, $r^{-\frac{5}{n}}$, $r^{-\frac{6}{n}}$, &c. and of which $0, -\frac{1}{n}, -\frac{2}{n}, -\frac{3}{n}, -\frac{4}{n}, -\frac{5}{n}, -\frac{6}{n}$, &c. are the logarithms.

(2) Considering logarithms as *indices* to a series of numbers in geometrical progression, where r and n may represent any numbers whatever, it follows that there may be as many different kinds of logarithms as there can be taken different sorts of geometrical series: numbers in very different progressions may likewise have the *same* logarithms, and on the contrary, the same geometrical series may have different series of logarithms corresponding to them, but in every case the logarithm of 1 is 0.

(3) The tables of logarithms in common use are constructed upon a supposition that $r = 10$; hence it appears that the logarithm of any number whatever is the index of some power of 10.

Thus, the logarithm of 10 is 1, being the index of 10^1; the logarithm of 100 is 2, being the index of 10^2; the logarithm of 1000 is 3, being the index of 10^3; the logarithm of 10000 is 4, being the index of 10^4, &c. Hence the logarithms of all numbers between 1 and 10 will be greater than 0 and less than 1, that is, they will be decimals; between 10 and 100 they will be greater than 1 and less than 2, that is, they will be expressed by 1 with decimals annexed; between 100 and

1000 the logarithms are expressed by 2 with decimals annexed; between 1000 and 10000 they are expressed by 3 with decimals annexed.

Again, the logarithm of $\frac{1}{10}=\frac{1}{10^1}$ is -1, being the index of 10^{-1}; the logarithm of $\frac{1}{100}=\frac{1}{10^2}$ is -2, being the index of 10^{-2}; the logarithm of $\frac{1}{1000}=\frac{1}{10^3}$ is -3, being the index of 10^{-3}, &c. Hence the logarithms of all numbers decreasing from 1 to ·1 will be expressed by -1, with decimals annexed; decreasing from ·1 to ·01, they will be expressed by -2, with decimals annexed; decreasing from ·01 to ·001, by -3, with decimals annexed, &c.

And, *universally, if* $10^x = a$, *then* x *is the logarithm of* a; where a may be any number from 1 to 101000, the extent of our best modern tables.

If 10^x* $=2$, *then* $x=0{\cdot}30103$ *the log. of* 2.
If $10^x=3$, *then* $x=0{\cdot}47712$ *the log. of* 3.
If $10^x=4$, *then* $x=0{\cdot}60206$ *the log. of* 4.
If $10^x=50$, *then* $x=1{\cdot}69897$ *the log. of* 50.
If $10^x=94261$, *then* $x=4{\cdot}97433$ *the log. of* 94261.

The logarithm of $9{\cdot}4261 = {\cdot}97433 = x$, *viz.* $10^x = 10^{\cdot 97433}=9{\cdot}4261$.

Multiply by 10,

then $10^{x+1}=10^{1\cdot 97433}=94{\cdot}261$.

Multiply again by 10,

and $10^{x+2}=10^{2\cdot 97433}=942{\cdot}61$.

Multiply a third time by 10,

Then $10^{x+3}=10^{3\cdot 97433}=9426{\cdot}1$.

Multiply once more by 10,

Then $10^{x+4}=10^{4\cdot 97433}=94261$, and so on. If instead of multiplying by 10, we divide by 10 in the equation

$10^x=9{\cdot}4261$, then will

$$10^{x-1}=10^{\cdot 97433-1}=10^{-1\cdot 97433}={\cdot}94261,$$

---

* The design of this chapter is to show the nature and properties of logarithms, and not the different methods of constructing them.

Divide again by 10,

then $10^{x-2}=10^{-1{\cdot}97433-1}=10^{-2{\cdot}97433}={\cdot}094261$.

Divide a third time by 10,

then $10^{x-3}=10^{-2{\cdot}97433-1}=10^{-3{\cdot}97433}={\cdot}0094261$, &c. as far as you please.

(4) Hence it appears that all numbers whatever, whether they be whole numbers, mixed numbers, or decimals, have the decimal part of their logarithms the same, the only difference being in the index or whole number; and that the index to the logarithm of a whole number is *affirmative*, and the index to the logarithm of a pure decimal is *negative;* but, in every case, the *decimal* part of the logarithm is affirmative.

CHAP. II.

THE USE OF THE TABLE OF LOGARITHMS.

PROPOSITION I.

(5) *To find the logarithm of any whole number, or mixed decimal, consisting of one, two, three, or four figures.*

This proposition will appear plain from the following examples, observing that the decimal parts only of the logarithms are put down in the tables; because the index, or whole number, of the logarithm is always a unit less than the number of figures contained in the natural whole number to which it belongs. (3)

Required the logarithm of 5.

Look in the table in the column marked *No.* for 5, and against it in the colum *log.* stands ·69897. The natural number containing but one figure, the index is 0.

Required the logarithm of 74.

Look for 74 in the column marked *No.* and against it stands ·86923. The natural number containing two figures, the index is 1, therefore the logarithm of 74 is 1·86923.

Required the logarithm of 743.

Look for 743 in the column marked *No.* and against it stands ·87099. The natural number containing three figures, the index is 2, therefore the logarithm of 743 is 2·87099.

Required the logarithm of 7438.

Look for 743 in the column marked *No.* as before, and directing your eye to the numbers in the top column of the table, find the remaining figure 8, guide your hand down that column till you come in the same line with 743, against which you will find ·87146. The natural number containing four figures, the index is 3. Hence the logarithm of 7438 is 3·87146.

	No.	Log.	No.	Log.	No.	Log.
Examp.	37	=1·56820	3754	=3·57449	4980	=3·69723
	4	=0·60206	375·4	=2·57449	7986	=3·90233
	91·6	=1·96190	37·54	=1·57449	3700	=3·56820
	9·16	=0·96190	3·754	=0·57449	4000	=3·60206

PROPOSITION II.

(6) *To find the logarithm of any whole number, or mixed decimal, to five or six places of figures.*

RULE. Find the logarithm to the first four figures as above, then take the difference between this logarithm and the next greater in the table. Multiply this difference by the figures you have above four, and cut off so many figures from the right hand of the product as you multiply by, the rest must be added to the logarithm first taken out of the table.

Required the logarithm of 59684.

Logarithm of 5968 is 77583	7 diff.
The next greater is 77590	4 remaining figure
Diff. 7	2·8 product=3 nearly.

Then 77583+3=77586. The natural number consisting of five figures, the index is 4, therefore the logarithm of 59684 is 4·77586.

Examples. No.	Log.	No.	Log.
59·684	= 1·77586	65·439	= 1·81583
94268	= 4·97436	27863	= 4·44503
785920	= 5·89538	131755	= 5·11977

PROPOSITION III.

(7) *To find the logarithm of a pure decimal.*

RULE. Find the logarithm corresponding to the significant figures, as if they were whole numbers. Then, if the first significant figure be in the place of *tenths*, the index will be—1; if in the place of hundredths it will be—2; if in the place of thousandths it will be—3, and so on.

Thus, the logarithm of ·3754 is — 1·57449
The logarithm of ·03754 = — 2·57449
The logarithm of ·003754 = — 3·57449
The logarithm of ·0003754 = — 4·57449, &c.

PROPOSITION IV.

(8) *To find the logarithm of a vulgar fraction.*

Rule. Reduce compound fractions, mixed fractions, &c. to simple fractions. Then subtract the logarithm of the denominator from the logarithm of the numerator; if you carry 1, add it to the index of the logarithm of the denominator; then take the difference of the indices for the index of the logarithm of the fraction. For a proper fraction this index will be −, for an improper fraction it will be +.

Required the logarithm of $\frac{5}{18}$.

The logarithm of 5 = 0·69897
The logarithm of 18 = 1·25527

The logarithm of $\frac{5}{18}$ = − 1·44370

In the same manner the logarithm of $\frac{93}{94}$ = − 1·99535; the log. of $\frac{55}{72}$ = − 1·88303; the log. of $\frac{3}{5}$ = − 1·77815; the log. of $\frac{5349}{999}$ = + 0·72870, &c.

PROPOSITION V.

(9) *To find the number answering to any logarithm to four places of figures.*

Rule. Look in the column of logarithms marked 0 at the top of the table; if you do not find your logarithm there, take the next less to it which you can find in that column; the first three figures in the natural number will stand in the column marked *No.*, and looking in the same line from the left hand towards the right, you will either find your logarithm or the nearest to it in the table; the fourth figure of the natural number will stand at the top of the table.

Required the natural number answering to the logarithm 3·90233.

Rejecting the index 3, I look in the column 0, where the first three figures of the logarithm are 902, and against 90200 I find the natural number 798, and looking along that line, under 6, I find 90233 the given logarithm; therefore the natural number is 7986, and the index being three, the natural number is a whole number.

Examples.	Log.	No.	Log.	No.
	1·76343 =	58	2·85776 =	720·7
	2·77056 =	589·6	3·86681 =	7359

PROPOSITION VI.

(10) *To find the number answering to any logarithm, to five or six places of figures.*

RULE. Find the next less logarithm in the tables to the given one, and take out the four figures answering to it as before. Take the difference between this logarithm and the next greater in the tables, and also between this logarithm and the given one; divide the latter difference, with ciphers annexed to it, by the former, and place the quotient to the right hand of the natural number already found.

Required the natural number answering to the log. 4·59859.

Given log. 59859 | next less log. 59857
Next less 59857 nat. no. 3968 | next greater 59868
diff. 2 annex ciphers. | diff. 11.

then 200, &c. ÷ 11 = 18, &c. the natural number is therefore 39681·8, the index being 4, the integral part must have five figures.

Examples.

Log.	No.	Log.	No.
4.97436	= 94267	2·50181	= 317·55
2·97436	= 942·67	4·81584	= 65440
4·89538	= 78592	5·37081	= 234861

PROPOSITION VII.

(11) *To multiply numbers together.*[*]

RULE. Add the logarithms of the numbers together, the natural number answering to the sum will be the product required.

When some of the numbers are less than a unit, add the logarithms of the numbers together; when you come to the indices, add the affirmative indices and what you carry into one sum, and the negative indices into another. Take the difference between these sums for the index of the product, prefixing the sign of the greater sum.

Required the product of 84 *and* 56.

Logarithm of 84 = 1·92428
Logarithm of 56 = 1·74819
Product 4704 Log. = 3·67247

Required the product of 76·5 *and* 5·5.

Logarithm of 76·5 = 1·88366
Logarithm of 5·5 = 0·74036
Product 420·75 Log. = 2·62402

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[*] Let $10v = a$; $10^{-x} = b$; $10^{-y} = c$; $10^{z} = d$. Then $10^{v-x-y+z} = abcd$. Where v is the logarithm of a; $-x$ the logarithm of b; $-y$ the logarithm of c; z the logarithm of d. And $v-x-y+z$ is the logarithm of $abcd$. (1) *and* (3).

Required the product of ·84, ·056, *and* 37.

The logarithm of ·84 = —1·92428

of ·056 = —2·74819

of 37 = + 1·56820

Sum of the decimals and affirmative index + 3·24067

Sum of the negative indices — 3.

Logarithm of the product = 0·24067

The number answering to which is 1·7405, the product required.

Required the product of ·37, 426, ·5, ·004, ·275, *and* 336.

Answer 29·128.

PROPOSITION VIII.

(12) *To divide one number by another.**

Subtract the logarithm of the divisor from the logarithm of the dividend, and the remainder will be the logarithm of the quotient.

If any of the indices be negative, or if the divisor be greater than the dividend, change the index of the divisor: then if the indices have unlike signs take their difference, and prefix the sign of the greater; if they have like signs take their sum, and prefix the common sign. When there is a unit to carry from the decimal part of the logarithm of the divisor, subtract it from the index of that logarithm if it be negative, otherwise add it, before the signs are changed.

Divide 3450 *by* 23.

Logarithm of 3450 = 3·53782

Logarithm of 23 = 1·36173

Quotient 150 Log. = 2·17609

Divide 420·75 *by* 5·5.

Logarithm of 420·75 = 2·62402

Logarithm of 5·5 = 0·74036

Quotient 76·5 Log. = 1·88366

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* Let $10^x = a$, and $10^y = c$, then x is the logarithm of a, and y is the logarithm of c. Now $\frac{10^x}{10^y} = 10^{x-y} = \frac{a}{c}$, where $x-y$ is the logarithm of $\frac{a}{c}$.

Divide ·00375 *by* ·0678
Logarithm of ·00375 = − 3·57403
Logarithm of ·0678 = − 2·83123

Quotient ·05531 log. = − 2·74280

Divide $\frac{5}{9}$ *by* 937$\frac{1}{2}$
Logarithm of $\frac{5}{9}$ = − 1·74473
Logarithm of 937·5 = + 2·97197

Quotient ·0005926 log. = − 4·77276

PROPOSITION IX.

(13) *To involve a number to any power; that is to square or cube any number, &c.**

RULE. Multiply the logarithm of the number by the number expressing the power, viz. by 2 for the square, 3 for the cube, 4 for the biquadrate, &c.; and the product will be the logarithm of the power required.

If the index to the given logarithm be negative; multiply this index and the decimal part of the logarithm separately by the index of the given power, and subtract the former product from the latter; when you come to the indices, or whole numbers of these products, their difference must be taken, but if there be a unit to carry from the decimal part of the lower product, it must be added to the index of that product before you take the difference. The result will in every case be negative.

Required the cube of 1·2.
Logarithm of 1·2 = 0·07918
3

The cube is 1·728 log. = 0·23754

Required the square of 25.
Logarithm of 25 = 1·39794
2

The square is 625 log. = 2·79588

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* Let $10^x = a$, and n = the index of the power, then $10^{nx} = a^n$, where x is the logarithm of a, and nx the logarithm of a^n.

Required the third power of ·0725.

The logarithm of ·0725 is —2·86034
Then ·86034 × 3 = 2·58102 product of the decimal.
and 2 × 3 = 6· product of the indices.

Power ·000381 log.= —4·58102

Required the 6·25 *power of* ·0032.

The logarithm of ·0032 is — 3·50515.

Then ·50515 × 6.25 = 3.15719 {product of the dec.
and 3 × 6·25 = 18·75 {product of the ind.

Power·000000000000000025538log.= —16·40719

PROPOSITION X.

(14) *To extract the square or cube root, &c. of any number.**

Divide the logarithm of the number by 2 for the square root, 3 for the cube root, &c. and the quotient will be the logarithm of the root.

If the index to the logarithm be negative, and does not exactly contain the divisor, increase it by such a number as will make it exactly divisible, and increase the logarithm also by the same number before you begin to divide.

What is the square root of 3·24?

Logarithm of 3·24 is 0·51055, which divided by 2 gives 0·25527, the number answering to which is 1·8.

What is the cube root of 10648?

Logarithm of 10648 is 4·02727, which divided by 3 gives 1·34242, the number answering to which is 22.

What is the cube root of ·0003811?

The logarithm of ·0003811 is—4·58104=—4+0·58104; and by adding 2 to each part, it is=—6+2·58104; divide by 3, then —2·86035 is the logarithm of the root, the number answering to which is ·0725, the root sought.

What is the ·72 *root of* ·096?

The logarithm of ·096 is —2·98227 =—2+0·98227, and by adding ·16 to each part (in order that the negative index may be divisible by ·72) it becomes —2·16+1·14227; divide by ·72, then $-3\cdot + 1\cdot58649 = -2\cdot58649$ is the logarithm of the root; hence the root sought is ·03859.

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* Let $10^x = a$, and n = the index of the root. Then $10^{\frac{x}{n}} = a^{\frac{1}{n}}$, where x is the logarithm of a, and $\frac{x}{n}$ is the logarithm of $a^{\frac{1}{n}}$.

PROPOSITION XI.

(15) *To find the value of a quantity having a vulgar fraction for its exponent.*

RULE. Multiply the logarithm of the given number by the numerator of the exponent, and divide the product by the denominator; the quotient will be the logarithm of the quantity required.

The multiplication must be performed as directed in the 9th Proposition, and the division according to the directions given in the 10th Proposition: for, the numerator denotes the power to which the given number is to be raised, and the denominator shows what root of that power is to be extracted.

What is the value of $\overline{\cdot096}|^{\frac{25}{18}}$?

The logarithm of $\cdot096 = -2\cdot98227$, which multiplied by 25 (Prop. IX.) produces $-26\cdot55675$; this divided by 18 (Prop. X.) gives $-2\cdot58649$, the number answering to which is ·03859. *Answer.*

PROPOSITION XII.

(16) *To find a fourth proportional to three given numbers; or to work a question in the rule of three by logarithms.*

RULE. Add the logarithms of the second and third terms together, and from the sum subtract the logarithm of the first term, the remainder will be the logarithm of the fourth term.

What is the fourth proportional to ·75, 36, *and* ·008?

Logarithm of 36 = 1·55630

Logarithm of ·008 = − 3·90309

Sum = − 1·45939

Logarithm of ·75 = − 1·87506 subtract.

− 1·58433 the number

answering to which is ·384 the fourth proportional required. For, ·75 : 36 :: ·008 : ·384.

PROMISCUOUS EXAMPLES, EXERCISING ALL THE PROPOSITIONS.

1. Find the logarithm of $\frac{36\frac{2}{3}}{48}$. *Ans.* − 1·88303.
2. Required the logarithm of ·563′ or ·56⅓. *Ans.* − 1·75077.
3. Required the logarithm of ·0628803. *Ans.* − 2·79851.
4. Find the logarithm of ·008497133. *Ans.* − 3·92928.

5. Required the product of $56\frac{1}{3} \times \cdot 7 \times \frac{19}{20} \times \frac{44\frac{2}{3}}{47} \times \frac{43}{68\frac{1}{3}}$ *Ans.* Logarithm of the product is 1·34735, and the product 22·251.

6. Required the product of ·0594405 × ·583′ × ·0322916 × ·428571 × ·1′8′. *Ans.* Log. of the product is $-5{\cdot}94074$, and the product = ·000087244.

7. Divide ·0565 by ·25. *Quotient* = ·226.

8. Divide ·00375 by ·0678. *Quotient* = ·05531.

9. Divide 54498 by ·093. *Quotient* = 586000.

10. Divide $\frac{182}{369}$ by $\frac{1528}{13}$. *Quotient* = ·0041962.

11. Involve 1·05 to the 40th power. *Ans.* 7·0405.

12. Required the 3·75 power of 14·79; or find the value of $\overline{14{\cdot}79}|^{\frac{15}{4}}$. *Ans.* 24400.

13. Required the ·34′54′ power of 94·75′; or find the value of $\overline{94{\cdot}7\frac{5}{9}}|^{\frac{3451}{9990}}$. *Ans.* 4·8173.

14. Involve ·09475′ to the ·34′54′ power. *Ans.* ·44307.

15. Find the cube root of ·000381078. *Ans.* ·0725.

16. What is the ·625 root of ·027588? *Ans.* ·0035606.

17. Find a fourth proportional to 58^{m} 13^{s}; $11^{s}{\cdot}75$; and 24 hours. *Ans.* 4^{m} $50^{s}{\cdot}6$.

18. Find a fourth proportional to 23^{h} 12^{m} 37^{s}; 24 hours; and 7^{h} 59^{m} 34^{s}. *Ans.* 8^{h} 15^{m} 53^{s}.

CHAP. III.

THE USE OF THE TABLES OF SINES AND TANGENTS.*

PROPOSITION I.

(17) *To find the natural sine or cosine of an arc, also the lorgarithmic sine, tangent, secant, &c.*

Rule. If the degrees in the arc be less than 45, look for them at the top of the table, and for the minutes (if any) in the left hand column marked (′), against which, in the column signed at the top of the table with the proposed name, viz. *sine*, *cosine*, &c. stands the sine, cosine, &c. required. If the degrees are more than 45, they must be found at the bottom of the table, and the minutes (if any) must be found in the right hand column. The name in this case, viz. sine, tangent, &c. must be taken at the bottom of the table. To find the secants, see the first page of Table III.

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* The construction of these tables will be found at the end of Book II. Chap. V. Before the student reads this and the following chapter, it will be proper for him to read the definitions, &c. in Book II. Chap. I.

The natural sines must be looked for in the table entitled *natural sines;* and the logarithmic sines in the table entitled *logarithmic sines and tangents.*

Required the natural and logarithm. sine and cosine of 39° 42′.
Natural sine of 39° 42′ = ·63877, cosine = ·76940.
Logarithmic sine of 39° 42′ = 9·80534, cosine = 9·88615.

Required the natural and logarithm. sine and cosine of 73° 27′.
Natural sine of 73° 27′ = ·95857, cosine = ·28485.
Logarithmic sine of 73° 27′ = 9·98162, cosine = 9·45462.

If the sine, tangent, &c. be wanted to any number of degrees above 90, subtract those degrees from 180° and find the sine, tangent, &c. of the remainder; or subtract 90° from the given number of degrees, and find the *cosine, cotangent,* &c. of the remainder, which is the same thing.

Required the logarithmic sine, tangent, secant, cosine, cotangent, and cosecant of 137° 29′.

180°
137° 29′

rem. 42° 31′, sine = 9·82982, cosine = 9·86752, tangent = 9·96231, cotangent = 10·03769, secant = 10·13248, cosecant = 10·17018, and these are respectively equal to the cosine, sine, cotangent, tangent, cosecant, and secant of 47° 29′ = 137° 29′ − 90°.

PROPOSITION II.

(18) *To find the logarith. sine, cosine, &c. of an arc to seconds.*

Find the logarithm to the degrees and minutes as in Proposition I.; take the difference between this logarithm and the next greater or less in the same column, according as you want a sine or cosine, tangent or cotangent, &c.; multiply this difference by the number of seconds given, and divide the product by 60; add the quotient to the given logarithm if it be a sine, tangent, or secant, but subtract the quotient from the given logarithm if it be a cosine, cotangent, or cosecant, and the sum, or remainder, will be the logarithm required.

Required the logarith. sine, tangent, and secant of 35° 44′ 24″.

Log. sine 35° 44′	= 9·76642	tan = 9·85700	sec = 10·09058
next *greater* sine	= 9·76660	tan = 9·85727	sec = 10·09067
	diff. 18	diff. 27	diff. 9
	24	24	24
	60 | 432	60 | 648	60 | 216
Add the propor. part	7	11	4

Then, sine 35° 44′ 24″ = 9·76649, tan = 9·85711, sec = 10·09062.

In the same manner the *natural* sine is found, being ·58411.

Required the logarithmic cosine, cotangent, and cosecant of 35° 44′ 24″.

Log cos 35° 44′=9·90942 cotan=10·14300 cosec=10·23358
next *less* cosine=2·90933 cotan=10·14273 cosec=10·23340

	diff.	9	diff.	27	diff.	18
		24		24		24
	60	216	60	648	60	432
Subtract prop. part		4		11		7

Then cosine 35° 44′ 24″ = 9·90938, cotan = 10·14289, cosecant = 10·23351.

In a similar manner the *natural* cosine is found, being ·81167.

PROPOSITION III.

(19) *To find the degrees, minutes, or degrees, minutes, and seconds, corresponding to any given logarithmic sine, tangent, &c.*

Rule. Find the nearest logarithm to the given one in the table, and the degrees answering to it will be found at the top of the column if the name be there, and the minutes on the left hand; but if the name be at the bottom of the table, the degrees must be found at the bottom of the table, and the minutes on the right hand. To find the arc to seconds, take the difference between the two logarithms which include the given one, also the difference between the given logarithm and the next less. Multiply the latter difference by 60, and divide the product by the former difference, the quotient will give the number of seconds, which must be added to the degrees and minutes corresponding to the next less number in the tables, if your given logarithm be a sine, tangent, or secant; but if your given logarithm be a cosine, cotangent, or cosecant, the number of seconds must be subtracted from the degrees and minutes corresponding to the next less number in the tables.

Find the degrees, minutes, and seconds, corresponding to the logarithmic sine 9·43299.

Sine less than the given one	9·43278		21
Sine greater than the given one	9·43323		60
Former difference	45	45	1260
Given sine	9·43299		
Next less	9·43278 answering to 15° 43′		28 quot.
Latter diff.	21		

Therefore the required arc is 15° 43′ 28″.

The same manner of proceeding must be observed in finding a tangent, secant, or natural sine.

Find the degrees, minutes, and seconds, corresponding to the logarithmic cosine 9·43297.

Cosine less than the given one 9·43278
Cosine greater than the given one 9·43323
——— 19
Former difference 45 60
——— ———
Given cosine 9·43297 45 | 1140
Next less 9·43278 answering to 74° 17′. ———
——— 25 quot.
Latter difference 19 ———
———

Therefore the required arc is 74° 16′ 35′.

PROPOSITION IV.

(20) *To find the natural or logarithmic versed sine of an arc, by the help of a table of natural or logarithmic sines.*

To find the natural versed sine, subtract the natural cosine from unity if the arc be less than 90°, but if greater than 90 add it to unity.

To find the logarithmic versed sine, find the logarithmic sine of half an arc, double it, and subtract 9·69897 from the product.

Required the natural versed sine of 65° 45′.

Radius = 1·
Nat cosine 65° 45′ = ·41072
———
Nat ver sine 65° 45′ = ·58928
———

Required the natural versed sine of 115° 35′.

Natural cosine 115° 35′ or
cosine 64° 25′ = ·43182
To which add 1·
———
vers sine of 115° 35′ = 1·43182
———

Required the logarithm. versed sine of 72° 14′.

Half the arc is 36° 7′
Log sine 36° 7′ = 9·77043
2
———
19·54086
9·69897
———
Log vers sine 72° 14′ = 9·84189
———

Required the log. versed sine of 37° 53′.

Here half the arc is 18° 56½′
Log sine 18° 56′ = 9·51117
Log sine 18° 57′ = 9·51154
———
double sine 18° 56′½ = 19·02271
9·69897
———
Log vers sine 37° 53′ = 9·32374
———

CHAP. IV.

THE CONSTRUCTION AND USE OF THE PLANE SCALE.

(21) *The Plane Scale* is a mathematical instrument of extensive use. The scale generally used at sea is two feet in length, having drawn upon it equal parts, chords, sines, tangents, secants, &c. These are contained on one side of the scale, and the other side contains their logarithms.

(22) Describe a semicircle with any convenient radius CB (Fig. I. Plate II.); from the centre C draw CD perpendicular to AB, and produce it to F, &c.; draw BE parallel to CF, and join AD and BD.

(23) *Rhumbs.* Divide the quadrantal arc AD into eight equal parts, with one foot of the compasses in A, transfer the distances A1, A2, A3, &c. to the straight line AD, and it will be a line of rhumbs containing eight points of the compass, or one fourth of the whole circumference of the compass. By subdividing each of the divisions A1; 1–2; &c. into four equal parts, and transferring them in the same manner to the line AD, it will contain the points, and half and quarter points.

(24) *Chords.* Divide the arc BD into nine equal parts, with one foot of the compasses in B and the distances B–10, B–20, B–30, &c., transfer them to the straight line BD, which will be a line of chords constructed to every ten degrees. The single degrees are constructed by subdividing the arcs B–10, 10–20, &c. into ten equal parts, and transferring the divisions in the same manner to the line BD.

(25) *Sines.* Through each of the divisions of the arc BD draw lines parallel to CD, such as 80–10, 70–20, &c., and the line CB will be divided into a line of sines reckoning from C to B (for CG is the cosine of the arc B–80, or the sine of the arc D–80, which is ten degrees); if this line be numbered from B towards C, it will become a line of *versed sines.*

(26) *Tangent.* From the centre C draw straight lines through the several divisions of the quadrantal arc BD, to touch the straight line BE, which will become a line of tangents.

(27) *Secants.* Transfer the distances between the centre C and the divisions of the line of tangents, to the line DF, and it will become a line of secants which must be numbered from D towards F, as in the figure.

(28) *Semi-tangents.* From A draw lines through the several divisions of the arc BD, and they will divide the line CD into

semi-tangents, which must be marked with the corresponding figures of the arc 10, 20, &c.

The preceding are some of the principal lines on the plane scale, and to render them convenient for practice, they are transferred from a general figure, such as Fig. I., to a plane scale or ruler, such as Fig. II., and arranged in the same order.

Besides the lines already mentioned, there are, on several scales, other lines derived from the circle, such as

(29) *Miles of Longitude.* Divide the line AC into 60 equal parts; through each of these draw lines *a* 10, *b* 20, &c., parallel to DC, and every tenth of them will cut the arc AD in the points *a*, *b*, *c*, *d*, &c.; from A as a centre, the several divisions of the arc AD may be transferred to the line AD, which will give the divisions of the line of longitude. If this line be laid upon the scale, close to the line of chords (as in Fig. III.) both inverted, so that 60 on the scale of longitude be against 0 on the chords, &c., and any degree of latitude be counted on the chords, there will stand opposite to it, on the line of longitude, the miles contained in one degree of longitude in that latitude, a degree at the equator being 60 miles.

These lines are generally drawn to a larger radius than the lines above described, and consequently are applied to a larger scale of chords, as appears by Fig. III.

(30) *Equal parts.* The divisions of the line AC to form the miles of longitude may be considered as a scale of equal parts, and on some plane scales are laid down as such, marked *Leag.*, viz. leagues or equal parts.

Some scales are divided into equal parts, as in Fig. IV.

The outer one is generally a scale of inches. The others are divided similar to those marked A, B, C, &c. The scale A contains 40 equal parts to an inch, the scale B 35, C 30, &c. But the most useful scale of equal parts is

(31) *The Diagonal Scale.* Draw eleven lines parallel to and equidistant from, each other, as in Fig. V.; cut them at right angles by the lines BC – EF. 1 – 9, 2 – 7, &c., then will HG, &c. be divided into ten equal parts; divide the line OP into ten equal parts, also the line *m n*. From the points of division on the line *m n*, draw diagonals to the points of division on the line OP, viz. join *m* and the first division on OP, the first division on *m n*, and the second on OP, &c.

The chief use of such a scale as this, is to lay down any line from a given measure, or to measure any line. In doing which, the units are counted from *m* towards 0, the tens from *m* towards *n*, and the hundreds from *m* towards H.

Thus, for example, to take off the number 242, extend the compasses from *m* to 2, towards H; with one leg fixed in the point 2, extend the other till it reaches 4 in the line *mn;* move one leg of the compasses along the line 2, 7, and the other along the line 4, till you come to the line marked 2 in BC, and you will have the number required.

OF THE LOGARITHMIC LINES, OR GUNTER'S SCALE.

Mr. Edmund Gunter was the first who applied the logarithms of numbers, and of sines and tangents, to straight lines drawn on a scale or rule; with which, proportions in common numbers and trigonometry may be solved by the application of a pair of compasses only. The method is founded on this property, *That the logarithms of the terms of equal ratios are equi different.* This was called Gunter's proportion, and Gunter's line; hence the scale is generally called the Gunter.

The logarithmic lines, on *Gunter's* scale, are the eight following:

(32) *S. Rhumb*, or sine rhumbs, is a line containing the logarithms of the natural sines of every point and quarter point of the compass, numbered from a brass pin on the right hand towards the left, with 8, 7, 6, 5, 4, 3, 2, 1.

(33) *T. Rhumb*, or tangent rhumbs, also corresponds to the logarithms of the tangents of every point and quarter point of the compass. This line is numbered from near the middle of the scale with 1, 2, 3, 4, towards the right hand, and back again with the numbers 5, 6, 7, from the right hand towards the left. To take off any number of points below 4, we must begin at 1 and count towards the right hand; but to take off any number of points above 4, we must begin at 4 and count towards the left hand.

(34) * *Numbers*, or the line of numbers, is numbered from the left hand of the scale towards the right with 1, 2, 3, 4, 5, 6, 7, 8, 9, 1, which stands exactly in the middle of the scale; the

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* This line being the principal one on the scale, has been contrived various ways. Gunter first applied it to the two feet rulers, or rather to the cross-staff. Wingate drew the logarithms on two separate rules sliding against each other, to save the use of compasses. Oughtred applied the logarithms to concentric circles. Mr. William Nicholson proposed another disposition of them on concentric circles (vide Philosophical Transactions 1787, page 251); his instrument consists of three concentric circles engraved and graduated on a plate of brass; from the centre two legs proceed as radii, having straight edges; these

numbers then go on 2, 3, 4, 5, 6, 7, 8, 9, 10, which stands at the right-hand end of the scale. These two equal parts of the scale are divided equally, the distance between the first or left hand 1, and the first 2, 3, 4, &c. is exactly equal to the distance between the middle 1 and the numbers 2, 3, 4, &c. which follow it. The subdivisions of these scales are likewise similar, viz. they are each one-tenth of the primary divisions, and are distinguished by lines of about half the length of the primary divisions.

These subdivisions are again divided into ten parts, where room will permit; and where that is not the case, the units must be estimated, or guessed at, by the eye, which is easily done by a little practice.

The primary divisions on the second part of the scale, are estimated according to the value set upon the unit on the left hand of the scale: If you call it one, then the first 1, 2, 3, &c. stand for 1, 2, 3, &c., the middle 1 is 10, and the 2, 3, 4, &c. following stand for 20, 30, 40, &c., and the ten at the right hand is 100: If the first 1 stand for 10, the first 2, 3, 4, &c. must be counted 20, 30, 40, &c., the middle one will be 100, the second 2, 3, 4, 5, &c. will stand for 200, 300, 400, 500, &c., and the ten at the right hand for 1000.

If you consider the first 1 as $\frac{1}{10}$ of an unit, the 2, 3, 4, &c. following will be $\frac{2}{10}$, $\frac{3}{10}$, $\frac{4}{10}$, &c., the middle 1 will stand for a unit, and the 2, 3, 4, &c. following will stand for 2, 3, 4, &c., also the division at the right-hand end of the scale will stand for 10. The intermediate small divisions must be estimated according to the value set upon the primary ones.

(35) *Sine.* The line of sines is numbered from the left hand of the scale towards the right, 1, 2, 3, 4, 5, &c. to ten; then 20, 30, 40, &c. to 90, where it terminates just opposite 10 on the line of numbers.

(36) *Versed Sine.* This line is placed immediately under the line of sines, and numbered in a contrary direction, viz. from the right hand towards the left, 10, 20, 30, 40, 50, to about 169; the small divisions are here to be estimated according to the number of them to a degree.

(37) *Tangent.* The line of tangents begins at the left hand, and is numbered 1, 2, 3, &c. to 10, then 20, 30, 45, where

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legs or rulers are movable about the centre either singly or together, so that if one be placed at the first term of any proportion, and the other at the second, and they be then fixed at this angle and moved together till the leg which was fixed at the first term coincides with the third, the other leg will point out the fourth.

there is a little brass pin just under 90 in the line of sines; because the sine of 90° is equal to the tangent of 45°. It is numbered from 45° towards the left hand 50, 60, 70, 80, &c. The tangents of arcs above 45° are therefore counted backward on the line, and are found at the same points of the line as the tangents of their complements.

Thus, the division at 40 represents both 40 and 50, the division at 30 serves for 30 and 60, &c.

(38) *Meridional Parts.* This line stands immediately above a line of equal parts, marked *Equal Pt.*, with which it must always be compared when used. The line of equal parts is marked from the right hand to the left with 0, 10, 20, 30, &c.; each of these large divisions represents 10 degrees of the equator, or 600 miles. The first of these divisions is sometimes divided into 40 equal parts, each representing 15 miles.

The extent from the brass pin on the scale of meridional parts to any *division* on that scale, applied to the line of equal parts, will give (in degrees) the meridional parts answering to the latitude of that *division.* Or the extent from any *division* to another on the line of meridional parts, applied to the line of equal parts, will give the meridional difference of latitude between the two places denoted by the *divisions.* These degrees are reduced to leagues by multiplying by 20, or to miles by multiplying by 60.

THE CONSTRUCTION OF THE LOGARITHMIC LINES ON GUNTER'S SCALE.

(39) *The line of numbers*, on which most of the others depend, sometimes called the *Line of Lines*, is constructed thus: Let a line equal in length to half the line of numbers be divided into 1000 equal parts; then since the logarithm of 1 is 0, the distance of 1 from the beginning of the line is 0, viz. 1 stands at the beginning of the line. And because the logarithm of 2 is ·301, when the logarithm of 10 is 1, or, which is the same thing, the logarithm of 2 is 301, when the logarithm of 10 is 1000; therefore the distance between 1 and 2 is 301 equal parts taken from a plane scale of equal parts; the distance 477, the logarithm of 3, must be set off from 1 to 3; and 602 equal parts, the logarithm of 4, must be set off from 1 to 4, &c. Thus are all the primary divisions on the line formed; and the intermediate divisions are formed in a similar manner, by taking the logarithms of the intermediate numbers, as for example, the logarithm of $1\frac{1}{10}$, is 41, the logarithm of $1\frac{2}{10}$, is 79, of $1\frac{3}{10}$ is 114, &c. These numbers, set off in a successive order from 1, will divide the primary divisions into ten parts, &c.

(40) *The line of sines* is constructed by taking the* arithmetical complements of the logarithmic sines from the same scale of equal parts from which the line of numbers was constructed, and setting them off from 90 backwards, or towards the left hand; thus the arithmetical complement of the logarithmic sines of 80°, 70°, 60°, 50°, 40°, 30°, 20°, 10°, (or, which is the same thing, their cosecants rejecting the indices,) are 7, 27, 62, 116, 192, 301, 466, 760, and these taken from the same scale of equal parts with which the line of numbers was constructed, and set off from 90 towards the left hand, give the sines of the above-mentioned degrees respectively. The reason of setting off these numbers thus is obvious, for the sine of 90° is equal to the radius; and the several arithmetical complements are what the sines of the arcs want of radius, viz. their distances from radius or 90°.

In the same manner the intermediate degrees, and divisions of the degrees are found.

(41) *The line of sine rhumbs* is constructed in a similar manner to the line of sines, by taking the arithmetical complements of the logarithmic sines of the degrees and minutes which are contained in the several points, and quarter-points of the compass; and setting them off from the right hand end of the scale towards the left.

(42) *The line of tangents* is constructed in the same manner as that of the sines, by setting off the arithmetical complements of the tangents under 45°, backward from 45° towards the left hand. For the tangent of 45° is equal to the radius, and the arithmetical complement of any logarithmic tangent under 45°, is what that tangent wants of radius. It has been observed, that the division at 40 serves both for 40 and 50, that at 30 for 30 and 60, &c. The reason of this will appear if we consider that the tangent of any arc is to the radius, as the radius to the cotangent; or, which is the same thing, $\frac{\text{tangent}}{\text{radius}} = \frac{\text{radius}}{\text{cotangent}}$, or in logarithms, the difference between the tangent and radius is equal to the difference between the radius and cotangent. But the tangent of 45° is equal to the radius, therefore the difference between the tangent of an arc (below 45°) and 45°, is equal to the difference between the cotangent of that arc and 45°; viz. they are both equidistant from 45°.

(43) *The line of tangent rhumbs* is constructed in a similar

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* The arithmetical complement of any logarithm is what that logarithm wants of 10; or what a sine, &c. wants of the logarithmic radius. Thus the arithmetical complement of 2·56820, the logarithm of 370, is 7·43180; the arithmetical complement of 9·53405, the logarithmic sine of 20, is 0·46595.

manner to the line of tangents, by taking the arithmetical complements of the logarithmic tangents of the degrees and minutes contained in the first four points of the compass, and setting them from the end of the line towards the left hand.

(44) *The line of versed sines* is constructed by the help of a table of logarithmic versed sines extending to 180°. Take the logarithmic versed sines of the supplements of the arcs, and subtract the logarithm of 2 from them, the arithmetical complements of the remainders, taken from the same scale of equal parts as the other lines were constructed from, and applied from the right hand towards the left, will give the divisions of the line of versed sines.

(45) The nature and use of this line are, I believe, very imperfectly understood; and in order to explain them clearly, we must have recourse to the inventor, viz. *Gunter;* he says, p. 231. Leybourn's edition, 1673, that he contrived this line "for the more easy finding of an angle having three sides, or "a side having three angles of a spherical triangle given." He then gives the following proportions:

As radius
Is to sine of one of the sides containing any angle, so is the sine of the other containing side, to a fourth sine.

As this fourth sine
Is to sine of half the sum of the three sides, so is the sine of this half sum diminished by the side opposite the given angle, to a seventh sine.

The mean proportional between this seventh sine and the radius, gives the sine of the complement of half the angle required.—By the scale,

(46) Extend the compasses from the sine of 90 to the sine of one of the sides containing any angle; that extent applied the same way will reach from the sine of the other side to a *fourth sine.* From this fourth sine extend the compasses to the sine of half the sum of the three sides, and this extent applied the same way will reach from the sine of the difference between the half sum of the three sides and the side opposite the angle taken, to a *seventh sine;* immediately under which, stands the angle required in the line of versed sines.

It is for this reason that the line of sines and versed sines are placed so close together. This is* *Gunter*'s Rule, and it is

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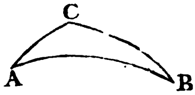
* As *Gunter*'s works are merely practical, perhaps an investigation of these proportions will be acceptable to some readers: —

Let ABC be any spherical triangle, denote the sides opposite the angles A, B, C, by a, b, c, and put $s = \frac{1}{2}(a+b+c)$.

Then Gunter's proportions are,

$$\text{rad} : \sin a :: \sin b :: 4th\ sine = \frac{\sin a.\ \sin b,}{\text{rad}}$$



more simple in its application than that which *Robertson* has given in his Navigation, without demonstration (Art. 29. Book IX.), for working an Azimuth.

(47) *The line of meridional parts* is constructed by the help of a table of meridional parts. Take the meridional parts corresponding to the several degrees of latitude from the table, and divide them by 60; take these quotient from the scale of equal parts, already described under the article meridional parts, and set them off on the line of meridional parts from the right hand towards the left.

THE USE OF THE LOGARITHMIC LINES ON GUNTER'S SCALE.

By these lines and a pair of compasses, all the problems in Trigonometry, Navigation, &c. may be solved.

(48) These problems are all solved by proportion: Now, in natural numbers, the quotient of the first term by the second is equal to the quotient of the third by the fourth; therefore, in logarithms, the difference between the first and second term is equal to the difference between the third and fourth; consequently on the lines on the scale, the distance between the first and second term will be equal to the distance between the third and fourth. And for a similar reason, because four proportional quantities are alternately proportional, the distance between the first and third terms, will be equal to the distance between the second and fourth. Hence the following: —

$$\begin{aligned}
&\text{and } \frac{\sin a.\ \sin b}{\text{rad}} : \sin s :: \sin (s-c) : 7th\ sine\,; \\
&\therefore\ 7th\ sine = \text{rad.}\ \frac{\sin s.\ \sin (s-c)}{\sin a.\ \sin b} \\
&\text{and } 7th\ sine \times \text{rad} = \text{rad.}^2\ \frac{\sin s.\ \sin (s-c)}{\sin a.\ \sin b} \\
&\qquad = \cos^2 \tfrac{1}{2}\, C \text{ (by art. 426)} \\
&\qquad = \tfrac{1}{2} \text{ suvers C. rad (by art. 228)} \\
&\therefore\ 7th\ sine = \tfrac{1}{2} \text{ suvers C.}
\end{aligned}$$

Hence the mean proportional between the 7*th sine* and *radius* gives the *cosine* of the required angle; and this 7*th sine* is also equal to half the *suversed sine* of the required angle.

"But because the finding the mean proportional between the radius or sine of 90° and *the* 7*th sine*, is somewhat troublesome," says Gunter, "I have added this "line of versed sines, that having found *the* 7*th sine*, you might look over against "it and there find the angle." *Gunter* has neither shown the construction of the line of versed sines, nor given the investigation of the above proportions; but he has illustrated them by a practical example, wherein the three sides of a spherical triangle are given to find an angle.

(49) General Rule. The extent of the compasses from the first term to the second will reach, *in the same direction*, from the third to the fourth term. Or, the extent of the compasses from the first term to the third, will reach *in the same direction*, from the second to the fourth.

By the same direction is meant, that if the second term lie on the right hand of the first, the fourth will lie on the right hand of the third, and the contrary. This is true, except the first two or last two terms of the proportion are on the line of tangents, and neither of them under 45°; in this case the extent on the tangents is to be made in a contrary direction: For had the tangents above 45° been laid down in their proper direction, they would have extended beyond the length of the scale towards the right hand; they are therefore as it were folded back upon the tangents below 45°, and consequently lie in a direction contrary to their proper and natural order.

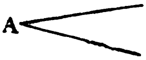
(50) If the last two terms of a proportion be on the line of tangents, and one of them greater and the other less than 45°, the extent from the first term to the second, will reach from the third beyond the scale. To remedy this inconvenience, apply the extent between the first two terms from 45° backward upon the line of tangents, and keep the left hand point of the compasses where it falls; bring the right hand point from 45° to the third term of the proportion; this extent now in the compasses applied from 45° backward will reach to the fourth term, or the tangent required. For, had the line of tangents been continued forward beyond 45°, the divisions would have fallen above 45° forward; in the same manner as they fall under 45° backward.

CHAP. V.

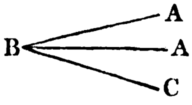
GEOMETRICAL DEFINITIONS AND INTRODUCTORY PROBLEMS,

DEFINITIONS, ETC. OF ANGLES.

(51) An angle is the inclination or opening of two straight lines meeting in a point as A.

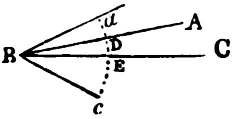


(52) One angle is said to be less than another, when the lines which form it are nearer to each other. Take two lines AB and BC meeting each other in the point B; conceive these two lines to open like the legs of a pair of com-



passes, so as always to remain fixed to each other in B. While the extremity A moves from the extremity C, the greater is the opening or angle ABC; and, on the contrary, the nearer you bring them together, the less the opening or angle will be.

(53) The magnitude of an angle does not consist in the length of the lines which form it, but in the extent of their opening or inclination to each other. Thus the angle ABC is less than the angle *a*B*c*, though the lines AB and CB which form the former angle, are longer than the lines *a*B and *c*B which form the latter.

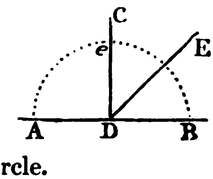


(54) When an angle is expressed by three letters, as ABC, the middle letter always stands at the angular point, and the other two letters at the extremities of the lines which form the angle; thus the angle ABC is formed by the lines AB and CB, and that of *a*B*c* by the lines *a*B and *c*B, &c.

(55) Every angle of a triangle is measured by an arc of a circle described about the angular point as a centre; thus the arc *ac* is the measure of the angle *a*B*c*, and the arc DE is the measure of the angle ABC.

(56) The circumference of every circle is supposed to be divided into 360 equal parts called degrees, each degree into 60 equal parts called minutes, each minute into 60 equal parts called seconds. The angles are measured by the number of degrees cut from the circle by the lines which form the angles; thus, if the arc DE contain 20 degrees, or the 18th part of the circumference of the circle, the measure of the angle ABC is 20 degrees. Degrees, &c. are thus marked, 44° 32′ 21″ 14‴, &c. and read 44 degrees, 32 minutes, 21 seconds, 14 thirds, &c.

(57) When a straight line CD standing upon a straight line AB, makes the angles CDB and CDA on each side equal to one another, each of these equal angles is said to be a right angle, and the line CD is perpendicular to AB. The measure of a right angle is therefore 90°, or a quarter of a circle.

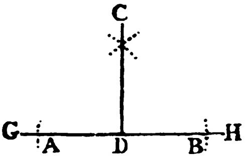


(58) An *acute* angle is less than a right angle, or 90°, as EDB.

(59) An *obtuse* angle is greater than a right angle, or 90°, as ADE.

(60) If ever so many angles are formed at the point D, on the same side of the line AB, they are altogether equal in measure to two right angles, or 180°.

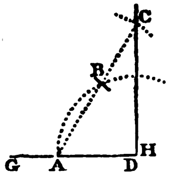
PROBLEM I.



(61) *To erect a perpendicular from a given point* D *in a given line* GH, *or to make a right angle.*

On each side of D take the equal distances AD and BD.

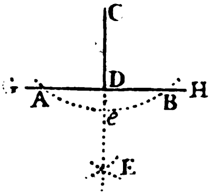
With any extent of the compasses greater than AD, and centres A and B, describe arcs crossing each other in C; a line CD, drawn through C and D, will be the perpendicular required.



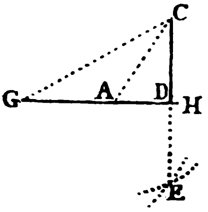
Otherwise. When the point D is at the end of the line GH; with the centre D and any opening of the compasses describe an arc; set off the distance AD from A to B; with B as a centre, and the distance AB in your compasses, describe another arc; through A and B draw the line ABC, cutting the second arc in C; lastly, through C and D draw the line CD, and it will be the perpendicular required.

PROBLEM II.

(62) *From a given point* C, *not in the straight line* GH, *to draw a straight line* CD *perpendicular to* GH.



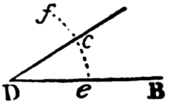
Take any point *e* on the contrary side of GH to which the point C is, and with the distance C*e* and centre C describe an arc cutting GH in A and B; with A and B as centres, describe arcs crossing each other in E, a line CDE drawn through C and E will be the perpendicular required.



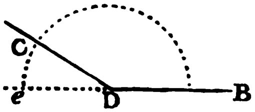
Otherwise. When the point C is near the end of the line GH. Take any points A and G in the line GH, with the centres A and G, and the distances AC and GC, describe arcs crossing each other in E, the line CDE drawn through C and E, will be the perpendicular required.

PROBLEM III.

(63) *To make an angle of any proposed number of degrees upon a given straight line, by the scale of chords.*



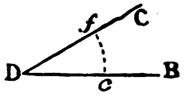
Upon the line DB to make an angle of 30°. Take the extent of 60° from the line of chords, with which and the centre D, describe the arc *e f*. Take 30° from the same scale of chords and set them off from *e* to C; through C draw the line DC, then CDB is the angle required.

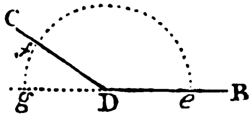


To make an angle of 150°. Produce the line BD to *e*, with the centre D and the chord of 60° describe a semicircle, take the given obtuse angle from 180° and set off the remainder, viz. 30° from *e* to C, through C draw CD, then CDB is the angle required.

PROBLEM IV

(64) *An angle being given, to find how many degrees it contains, by a scale of chords.*





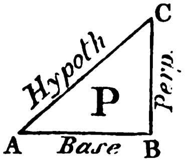
With the chord of 60° in your compasses and centre D describe an arc *e f*, cutting DB and DC in *e* and *f*. Then take the distance *e f* in your compasses, and setting one foot on the brass pin at the beginning of the chords on your scale, observe how many degrees the other foot reaches to, and that will be the number of degrees contained in the arc *e f*, or angle CDB.

If the extent *e f* reach beyond the scale, which will always be the case when the angle is obtuse, extend the line BD from D towards *g*, and measure the arc *g f* in the same manner; the degrees it contains deducted from 180°, will give the measure of the angle CDB.

DEFINITIONS, ETC. OF TRIANGLES.

(65) *A plane triangle* is a space included by three straight lines, and contains three angles.

(66) Triangles are of various kinds, but in trigonometry only two kinds are considered, viz. right-angled triangles, and oblique-angled triangles; which indeed include the whole.

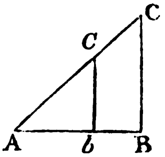


(67) *A right-angled triangle** is that which has one right angle in it, as P. The longest side AC, or that opposite to the right angle B, is called the hypothenuse; the other two, AB and BC, are called the legs, or sides, or the base and perpendicular.

(68) The sum of the three angles of every plane triangle is equal to 180°; hence in a right-angled plane triangle, if either acute angle be taken from 90°, the remainder will be the other acute angle.

(69) *The complement* of an arc, or angle less than 90°, is what that angle wants of a quadrant, or 90°.

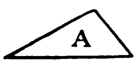
(70) If one acute angle of a right-angled triangle be 45°, or half a right angle, the other acute angle will also be 45°, or half a right angle; and the base and perpendicular will be equal to each other.

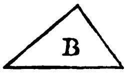


(71) If two right-angled triangles ABC and A*bc*, have the angle A common, they are equiangular and similar; that is, the sides about the equal angles are proportional, viz.

AB : BC :: A*b* : *bc*, and AB : BC :: AC : AC, &c.

(72) If a straight line be drawn parallel to one of the sides of a plane triangle, it will cut the other two sides proportionally.





(73) *An oblique-angled triangle* is that which has not a right angle in it; hence two of its angles must necessarily be acute, or each less than a right angle, but the remaining angle may be either greater or less than a right angle; as A and B.

(74) Any one angle of an oblique plane triangle subtracted from 180°, leaves the sum of the other two angles. Or the sum of any two angles subtracted from 180°, leaves the third angle.

(75) *The supplement* of any angle is what that angle wants of 180°. Hence the supplement of any one angle is always equal to the sum of the other two.

(76) Any two sides of a triangle added together are greater than the third side. The greatest side of any triangle is opposite to the greatest angle; and the contrary, the greatest angle is opposite to the greatest side.

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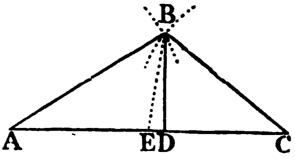
* An *equilateral* triangle has three equal sides; an *isosceles* triangle has two of its sides equal; an *acute-angled* triangle has three acute angles; and an *obtuse-angled* triangle has two acute angles, and one obtuse.

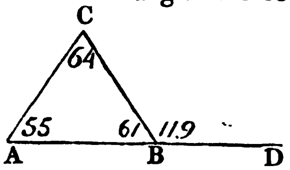
(77) Every triangle has two of its angles acute, and if the third angle be either a right angle, or an obtuse angle, it is opposite to the greatest side.

(78) If a perpendicular BD be drawn upon the longest side of any triangle, from the opposite angle, it will fall within the triangle; and the greater segment AD, will meet the greater (AB) of the other two sides and the less segment DC, will meet the less of these sides (BC).



(79) In an equilateral or isosceles triangle, a perpendicular BD drawn from the vertical angle, will bisect both the base and the vertical angle.

(80) If any one side of a plane triangle be produced or extended beyond the angular point, the outward angle will be equal to the two inward angles, opposite to the angular point where the side is extended. Thus in the triangle ABC, if the measure of the angles be as expressed in the triangle, and the side AB be produced to D; then will the angle CBD be equal to the angles BAC and BCA together.



(81) In a right-angled plane triangle, the hypothenuse is equal to the square-root of the sum of the squares of the base and perpendicular: and the square-root of the difference of the squares of the hypothenuse and either of the other sides, is equal to the remaining side.

(82) Each of the angles of an equilateral triangle is 60 degrees.

(83) Each of the angles at the base of an isosceles triangle is equal to half the supplement of the vertical angle, or to the complement of half the vertical angle.

(84) If two straight lines intersect each other, the vertical or opposite angles will be equal.

(85) If a straight line intersect two parallel straight lines, it makes the alternate angles equal to each other, and the exterior angle equal to the interior and opposite on the same side of the line.

(86) All angles in the same segment of a circle are equal to each other.

(87) An angle at the centre of a circle is double to an angle at the circumference when they both stand on the same arc.

(88) Every angle in a semicircle is a right angle.

BOOK II.

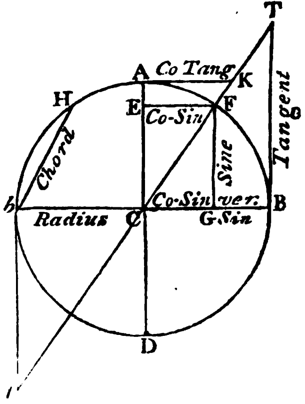
CHAPTER I.

DEFINITIONS OF PLANE TRIGONOMETRY, WITH THE INVESTIGATION OF GENERAL RULES FOR CALCULATING THE SIDES AND ANGLES OF PLANE TRIANGLES.

(89) PLANE TRIGONOMETRY is the art of measuring and calculating the sides and angles of triangles described on a plane surface, or of such triangles as are composed of straight lines. It likewise includes the relation between the radius of a circle and certain other straight lines described in and about a circle.

(9) The theory of triangles is the very foundation of all geometrical knowledge, for all straight-lined figures may be reduced to triangles. The angles of a triangle determine only its relative species, and are measured in *degrees*, *minutes*, and *seconds* (56); but the sides determine its absolute magnitude, and are expressed in *yards*, *feet*, *chains*, or any other lineal measure.

(91) *A circle* is a plain figure contained under one line, called the circumference, to which all lines drawn from the centre are equal. Thus ABDBHA is the circumference; C the centre, and CA, CD, C*b*, CB, CF, are all equal to each other.



(92) The distance from the centre of a circle to the circumference is called the *radius:* thus CA, CB, CD, &c. are radii.

(93) A straight line drawn through the centre of a circle to touch the circumference in two points, is called a diameter, and is always double the radius. Thus AD and B*b* are diameters, and are each of them double of AC or of *b*C.

(94) The exact ratio between

the radius and the circumference of a circle being unknown, mathematicians were at a loss to form a comparison between the sides and angles of a triangle, since they could not compare a straight line with any part of the circumference of a circle. They were therefore under the necessity of determining the relation between the radius of a circle, and certain other straight lines, described in and about a circle, called *chords*, *sines*, *tangents*, &c.

(95) *The chord* of an arc is a straight line drawn from one extremity of the arc to the other. Thus *b*H is the chord of the arc *b*H, or of the arc HAFBD*b*.

The chord of an arc of 60° *is equal to the radius of the circle.**

(96) *The complement* of any arc is the difference between that arc and a quadrant. Or it is the difference between any angle and 90°. Thus the arc AF is the complement of the arc BF, or the angle ACF is the complement of the angle FCB.

(97) *The supplement* of any arc is the difference between that arc and a semi-circle. Or it is the number of degrees which any angle wants of 180°. Thus the arc BF is the supplement of the arc FAH*b*, or the angle FCB is the supplement of the angle FC*b*.

(98) *The sine* of an arc is a straight line drawn from one end of that arc, perpendicular to a diameter passing through the other end of the same arc. Thus FG is the sine of the arc BF, or it is the sine of the supplemental arc FAH*b*.

The sine of an arc of 90° *is equal to the radius, for* AC *is the sine of the arc* BA. *The sine of an arc of* 30° *is equal to half the radius.*†

(99) *The tangent*‡ of an arc is a straight line drawn from one extremity of the arc, perpendicular to the diameter, and is terminated by a straight line drawn through the centre of the circle and the other extremity of the arc. Thus BT is the tangent of the arc BF, or of the angle BCF.

The tangent of any arc is equal to the tangent of the supplement

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* For (Euclid 15. of IV.) the side of a hexagon, which is the chord of 60°, is equal to the radius of the circumscribing circle.

† *The sine of any arc is equal to half the chord of double that arc;* thus let BF and BH (Plate I. Fig. 1.) be equal arcs, then FGH is the chord of the double arc FBH; and FGH is bisected in G (Euclid 3. of III.). Therefore if FB be an arc of 30°, FG its sine will be half the chord of 60°, and the chord of 60° has been shown to be equal to the radius, therefore the sine of 30° is equal to half the radius.

‡ *The semi-tangent* of an arc is the tangent of half that arc. Let BF be any arc (Plate I. fig. 1.) then *mc* is the semi-tangent of that arc. For the angle FCB is double the angle F*b*B (Euclid 20. of III.), consequently the arc *oc*, of which *cm* is the tangent, is the half of the arc BF.

of that arc.[*] *The tangent of* 45° *is equal to the radius*[†], *and an arc of* 90° *has no tangent.*

(100) *The secant* of an arc is a straight line drawn from the centre of the circle, and produced till it meets the tangent. Thus CT is the secant of the arc BF, or of the angle BCF.

The secant of any arc is equal to the secant of the supplement of that arc.[‡] *The secant of* 45° *is equal to double the sine of* 45° [§], *and an arc of* 90° *has no secant.*

(101) *The versed sine* of an arc is that part of the diameter which is contained between the sine and the arc. Thus BG is the versed sine of the arc BF, and bG is the versed sine of the supplemental arc FAHb.

The versed sine of the supplement of an arc, is the difference between the versed sine of that arc and the diameter.

For $bG = bB - GB$; or, $GB = bB - bG$.

(102) *The cosine* of an arc is the sine of the complement of that arc; or it is that part of the diameter contained between the centre of the circle and the sine. Thus FE is the cosine of the arc BF, being the sine of AF, which is the complement of BF; or CG is the cosine of BF, because $CG = FE$.

The cosine of an arc is equal to the cosine of its supplement.

The cosine of an arc, less than a quadrant, is equal to the radius diminished by the versed sine. For $CG = CB - GB$.

(103) *The cotangent* of an arc is the tangent of the complement of that arc. Thus AK is the cotangent of the arc BF, being the tangent of the arc AF, which is the complement of BF.

The cotangent of an arc is equal to the cotangent of its supplement.

(104) *The cosecant* of an arc is the secant of the complement of that arc. Thus CK is the cosecant of the arc BF, being the secant of the arc AF.

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[*] The supplemental tangent will fall on the contrary side of the diameter. Let BF be any arc (Plate I. fig. 1.), its supplement is FAfb, bt is the tangent of the supplement, and it is equal to BT. For the right-angled triangles CBT and cbt are equi-angular, and have the side CB equal to cb.

[†] For in this case the angle BCT, measured by the arc BF (Plate I. fig. 1.), will be half a right angle; consequently the angle BTC will be half a right angle; the triangle is therefore isosceles, and $BC = BT$.

[‡] Though the secant of an arc, and the secant of its supplement, be the same in quantity, they have contrary directions, like the tangents. For bt (Plate I. fig. 1.) has been shown to be the tangent of the supplemental arc FAfb, and equal to TB; it is plain therefore that $CT = ct$.

[§] For if the arc $BF = 45°$, $BC = BT$; and (Euclid 47. of I.) $CB^2 + BT^2 = CT^2$ the square of the secant, $= CB^2 + AC^2 = AB^2$; therefore $AB = CT$, that is, the chord of 90° is equal to the secant of 45°. But the sine of 45° is equal to half the chord of 90°, art. (98), and Note, therefore the sine of 45° is equal to half its secant.

The cosecant of an arc, and the cosecant of its supplement, are of the same magnitude.

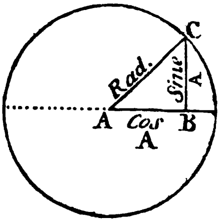
(105) *The coversed sine* of an arc is the versed sine of the complement of that arc. Thus, AE is the coversed sine of the arc BF, being the versed sine of the arc AF.

(106) *The cochord* of an arc is the chord of the complement of that arc.

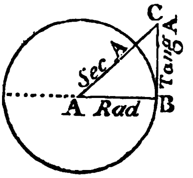
(107) Let A denote any arc or angle; the trigonometrical functions sine A, cosine A, tangent A, cotangent A, secant A, cosecant A, versed sine A, coversed sine A, are abbreviated into sin A, cos A, tan A, cot A, sec A, cosec A, vers A, covers A, respectively. The square or second power of sin A is written sin ^{2}A, the cube or third power sin ^{3}A, &c.

(108) From the preceding definitions it appears that the sine and cosine of any arc can never exceed the radius; the secant and cosecant always exceed the radius, and the tangent and cotangent admit of all possible degrees of magnitude. Also, that any side of a right-angled triangle may be considered as the radius of a circle, and that the other sides will necessarily become either sines, tangents, or secants.

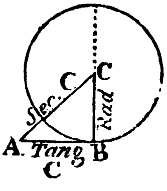
1st. *If the hypothenuse* AC be made the radius of a circle, it is evident that the perpendicular BC is the sine of the angle A, and that AB is the cosine thereof. But the sine of either of the acute angles of a right-angled triangle is the cosine of the other, and the contrary; therefore BC is the cosine of C, and AB is the sine thereof.



2dly. *If the base* AB be considered as the radius of a circle, BC is evidently the tangent of the angle A, and AC is the secant thereof. But the tangent of one angle is the cotangent of the other; also the secant of one angle is the cosecant of the other: therefore BC is the cotangent of C, and AC is the cosecant of C.



3dly. *If the perpendicular* BC be considered as the radius of a circle, it is evident that the base AB will be the tangent of the angle C, and the hypothenuse AC will be the secant thereof. But because the tangent, secant, &c. of one angle of a right-angled triangle is the cotangent, cosecant, &c. of the other; the base AB will be the cotangent of the angle A, and the hypothenuse AC will be its cosecant.



INVESTIGATION OF RULES FOR CALCULATING THE SIDES AND ANGLES OF PLANE TRIANGLES.

PROPOSITION I. (*Plate I. Fig.* 4.)

(109) In any right-angled plane triangle, *if any of the three sides be made the radius of a circle; the other sides will be either sines, tangents, or secants, of the respective angle, correspondent to that radius; and will be similar to the sines, tangents, or secants in the tables, when compared with the tabular radius.*

The sines, tangents, &c. of all angles whatever, are calculated to the logarithmic radius of *ten thousand millions*, and arranged in tables for the convenience of calculation. Now, as no triangle can be formed but another may be formed similar to it, and that the sides about the equal angles of similar triangles are proportional; if A*d*, C*e*, A*b*, and C*b'*, each represent the radius in the tables, then will AF and CG be the secants, *b*F and G*b'* the tangents, and D*d* and E*e* the sines in the tables. Then,

I. *If the hypothenuse be the radius of a circle.*

Because the triangles *e*EC and ABC are similar,

C*e* : AC : : E*e* : AB
viz. rad : *hypoth.* : : *sin* C = *cos* A : *base.*

Again, because the triangles AD*d* and ABC are similar,

A*d* : AC : : D*d* : BC
viz. rad : *hypoth.* : : *sin* A = *cos* C : *perp.*

(110) THEREFORE, the sine of any angle is to the side opposite to it, as the sine of any other angle is to its opposite side.

II. *If the base be the radius of a circle.*

Because the triangles A*b*F and ABC are similar,

AF : AC : : A*b* : AB
And AF : AC : : *b*F : BC
viz sec A = *cosec* C : *hypoth.* : : *rad* : *base,*
and, *sec* A = *cosec* C : *hypoth.* : : *tan* A = *cot* C : *perp.*

(111) HENCE. Radius : base : : tangent of the angle A, or cotangent of the angle C : perpendicular.

III. *If the perpendicular be the radius of a circle.*

Because the triangles G*b'*C and ABC are similar,

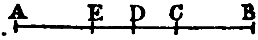
CG : AC : : *b'*G : AB
And, CG : AC : : C*b'* : BC
viz. sec C = *cosec* A : *hypoth.* : : *tan* C = *cot* A : *base.*
and, *sec* C = *cosec* A : *hypoth.* : : *rad* : *perpend.*

(112) HENCE. Tangent of angle C, or cotangent of A base : : radius : perpendicular.

Any of the foregoing analogies are proportional by inversion, &c.

PROPOSITION II.

(113) *The excess of the greater of two given magnitudes above half their sum, is equal to half the difference between those magnitudes: or half their sum increased by half their difference, gives the greater magnitude; and being diminished by half their difference, leaves the less magnitude.*



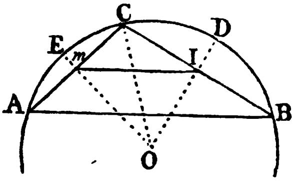
Let AC and CB be two unequal magnitudes, of which AC is the greater.

Bisect AB in D and make AE = CB, then will AD = half the sum of AC and CB, and DC = half the difference; for, since AB is bisected in D, and AE = CB, it follows that EC is bisected in D, and EC is the difference between AC and CB.

Hence AC − AD = DC; or AD + DC = AC, and AD − ED(= DC) = AE = BC.

PROPOSITION III.

(114) In any plane triangle, *the sine of any angle is to the side opposite to it, as the sine of any other angle is to the side opposite to it, and the contrary.*



Since any triangle whatever may be inscribed in a circle, let ABC be the given triangle and O the centre of the circle. Draw OD and OE perpendicular to the sides of the triangle, then (3 and 30 Euclid III.) the sides of the triangle will be bisected, and also the arcs which these sides subtend. Now an angle at the centre of a circle is double to an angle at the circumference when they stand on the same arc, (20 Euclid III.) therefore an angle at the centre is equal to an angle at the circumference standing on the double arc; or, in other words, *an angle at the circumference of a circle is measured by half the arc on which it stands, and an angle at the centre by the whole arc on which it stands:* hence the angle COD = the angle CAB, and COE = CBA, &c. Now CI is the sine of the angle COD, and C*m* is the sine of the angle COE. Draw *m*I, which will be parallel to AB, then per similar triangles CI : CB :: C*m* :

AC, viz. the sine of the angle at A, is to the side BC, as the sine of the angle at B, is to the side AC, and the contrary by inversion.

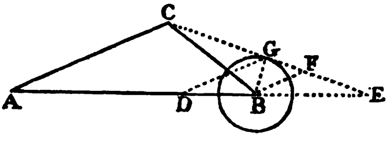
COROL. The sides of triangles are to each other as the chords of double their opposite angles.

PROPOSITION IV.

(115) In any plane triangle, *the sum of any two sides is to their difference, as the tangent of half the sum of their opposite angles is to the tangent of half their difference.*

Let ABC be any triangle; make $BE = BC$ and join CE, then the angle CBE, being the exterior angle of the triangle ABC, is equal to $CAB + BCA$. (32 Euclid I.)

Bisect AE in D and CE in G then join DG and it will be parallel to AC: draw BF parallel to DG, join BG, and with the centre B and radius BG describe a circle.



The angle $CBG =$ the angle EBG equal to half the sum of the angles CAB and BCA; for the triangles CBG and EBG have the two sides BC and CG, equal to the two sides BE and EG, and the side BG common to both, therefore they are equal in all respects. (8 Euclid I.)

BF being parallel to AC, the angle $FBE =$ the angle CAB, and the angle $CBF =$ the angle ACB. (29 Euclid I.)

The angle $GBF =$ half the difference between the angles CBF and FBE, for it is the excess of the greater above half their sum GBE (113); hence $GE =$ tangent of the half sum, and $GF =$ tangent of the half difference between the two angles CBF and FBE. For the same reason, DB being the excess of the greater of the two given lines AB and BE or BC, above half their sum AD, it is equal to half their difference.

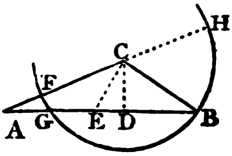
By similar triangles, $DE = \frac{1}{2}(AB + BC) : DB = \frac{1}{2}(AB - BC) :: GE = \text{tangent } \frac{1}{2}(ACB + CAB) : GF = \text{tangent } \frac{1}{2}(ACB - CAB)$.

Half the sum of the angles ACB and CAB added to half their difference gives the angle opposite to the greater side (113), for the greater *angle* is opposite to the greater side: and half the sum diminished by half the difference, gives the angle opposite to the less side; for the *less* angle is opposite to the less side. (18 Euclid I.)

NOTE. Instead of the tangent of half the sum of the opposite angles, the tangent of half the supplement of the contained angle may be used; or the cotangent of half the contained angle[*], which are all equal to each other.

PROPOSITION V.

(116) In any plane triangle, *double the base or longest side, is to the sum of the other two sides, as their difference is to the distance of a perpendicular from the middle of the base; this distance, added to half the base, gives the greater segment, and being subtracted leaves the less.*



Let ABC be the triangle, with C as a centre, and the radius CB = the least side, describe a circle, produce AC to H; then because CF = CB = CH; AH = AC + CB the sum of the sides, and AF = AC − BC the difference between the sides.

Bisect AB in E, and draw CD perpendicular to GB, then GD = BD (3 Euclid III.), therefore AG is the difference between the segments of the base; but ED = ½ AG (113), for it is the excess of the greater segment AD above half their sum AE.

Then (36 Euclid III. corol.) AH × AF = AB × AG, therefore AB : AH :: AF : AG; that is, AB : AC + CB :: AC − CB : 2ED, or, 2AB : AC + CB :: AC − CB : ED, the distance of the perpendicular CD from the middle E of the base; and that ED added to half the base AE gives AD the greater segment, &c. is obvious.

PROPOSITION VI.

(117) *From half the sum of the three sides of any triangle subtract the side opposite to the angle sought, and note the half sum and remainder. Then, the rectangle of the sides containing that angle, is to the square of the radius, as the rectangle of the half sum and remainder is to the square of the cosine of half the angle taken.*

Let ABC be the triangle; bisect the angle CAB by the indefinite line AH, and draw EC perpendicular to AD. Draw DG parallel and equal to CB, join GB and produce it to H.

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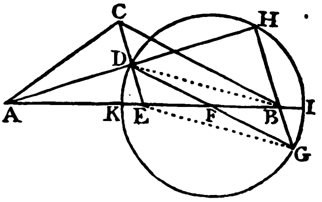
[*] Since the sum of the three angles of every triangle is equal to two right angles, it is plain that half the sum of any two angles is half the supplement of the third angle. And, that the tangent of half the supplement of any angle is equal to the cotangent of half that angle, may be thus shown:

Let HBF (Fig. I. Plate 1.) be any arc bisected in B, and FA*f*, its supplement, bisected in A; then AK, the cotangent of the half arc BF, is evidently the tangent of the half supplement AF.

Now the triangles ADE and ADC have the two sides DC and DE equal to each other (by construction), and the side AD common, also the angles about D are right angles; therefore $AC = AE$, (4 Euclid I.) and consequently EB is equal to the difference between the sides AC and AB.

Because the side EC of the triangle CEB is bisected in D, and DF is parallel to CB (by construction), $EF = FB$, therefore $FB =$ half the difference between the sides AC and AB, and $AF =$ half their sum (113).

By construction CB is equal and parallel to DG; therefore (33 Euclid I.) GB is equal and parallel to CD or DE, for the same reason DBGE is a parallelogram, and $FD = FG$, for the diagonals of a parallelogram bisect each other. And, since GB is parallel to ED, GH is also parallel to it, and the angle at D being a right angle, the angle at H is also a right angle. With F as a centre, and radius $FD = FG = \frac{1}{2}$ CB describe a circle, which will pass through the point H. (31 Euclid III.)



Since $FG = \frac{1}{2}$ CB, $FI = \frac{1}{2}$ CB, but $AF = \frac{1}{2}(AC + AB)$, to each of these equals add FI or the half of CB, then will $AI = \frac{1}{2}(AC + AB + BC) = s$, making $AC + AB + BC = 2s$; but $KI = DG = CB$; therefore $AK = AI - CB = s - BC$.

$$\text{Now (109) rad} : AE :: (\sin AED =) \cos \tfrac{1}{2} BAC : AD,$$
$$\text{and rad} : AB :: (\sin ABH =) \cos \tfrac{1}{2} BAC : AH;$$
$$\text{hence rad}^2 : AE \cdot AB :: \cos^2 \tfrac{1}{2} BAC : AD \cdot AH.$$
$$\text{But } AD \cdot AH = AK \cdot AI \text{ (36 Euclid III.), and } AE = AC;$$
$$\text{therefore rad}^2 : AC \cdot AB :: \cos^2 \tfrac{1}{2} BAC : AK \cdot AI$$
$$\text{or } AC \cdot AB : \text{rad}^2 :: AK \cdot AI : \cos^2 \tfrac{1}{2} BAC;$$
$$\text{hence } AC \cdot AB : \text{rad}^2 :: s(s - BC) : \cos^2 \tfrac{1}{2} BAC.$$

PROPOSITION VII.

(118) *From half the sum of the three sides of any plane triangle subtract the side opposite to the angle sought, and note the remainder: Also from the said half sum subtract each of the other sides of the triangle.* Then,

The rectangle contained by half the sum of the sides and the first remainder is to the square of the radius; as the rectangle contained by the other two remainders is to the square of the tangent of half the angle sought.

The same construction being made as in the preceding proposition, $AI = s$, and $AK = s - BC$.

In the right-angled triangles ADE and AHB

$$\text{rad} : AD :: (\tan DAE =)\ \tan \tfrac{1}{2} BAC : DE,$$
$$\text{rad} : AH :: (\tan DAE =)\ \tan \tfrac{1}{2} BAC : BH;$$

Hence $\text{rad}^2 : AD \,.\, AH :: \tan^2 \tfrac{1}{2} BAC : DE \,.\, BH$,

or $AD \,.\, AH : \text{rad}^2 :: DE \,.\, BH : \tan^2 \tfrac{1}{2} BAC$.

Again, $AD \,.\, AH = AI \,.\, AK$ (36 Euclid III.) and because $DE = GB$ being opposite sides of a parallelogram, we have $DE \times BH = GB \times BH = KB \times BI$. (35 Euclid III.) But because KI is bisected in F, and that FE and FB are equal, KE is equal to BI, and therefore KB is equal to EI; but $EI = AI - AE = s - AC$, consequently $KB = s - AC$, $BI = s - AB$, and $DE \,.\, BH = KB \,.\, BI = (s - AC) \,.\, (s - AB)$.

Hence $AI \,.\, AK : \text{rad}^2 :: (AI - AC) \times (AI - AB) : \tan^2 \tfrac{1}{2} BAC$,

or $s \,.\, s - BC : \text{rad}^2 :: (s - AC) \,.\, (s - AB) : \tan^2 \tfrac{1}{2} BAC$.

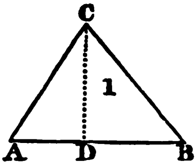
PROPOSITION VIII.

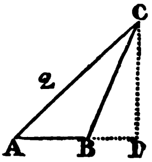
(119.) *Twice the rectangle of any two sides of a plane triangle is to the sum of the squares of the same two sides diminished by the square of the third side, as radius is to the cosine of the angle contained between those sides.*

Let ABC be any plane triangle and CD a perpendicular from the vertical angle upon the base or upon the base produced.

1. If the perpendicular falls within the triangle $AC^2 = AB^2 + BC^2 - 2AB \,.\, BD$. (13 EUCLID II.)

2. If the perpendicular falls without the triangle $AC^2 = AB^2 + BC^2 + 2AB \,.\, BD$. (13 EUCLID II.)





From the first $BD = \dfrac{AB^2 + BC^2 - AC^2}{2AB}$.

From the second $BD = \dfrac{AC^2 - AB^2 - BC^2}{2AB}$.

But $BC : \text{rad} :: BD : \cos B$ (109).

$\therefore BD = \dfrac{BC \,.\, \cos B}{\text{rad}}$, the $\angle B$ being acute in Fig. 1., and obtuse in Fig. 2. By substitution, we have

for Fig. 1. $\dfrac{BC \,.\, \cos B}{\text{rad}} = \dfrac{AB^2 + BC^2 - AC^2}{2AB}$,

and for Fig. 2. $\dfrac{-BC \,.\, \cos B}{\text{rad}} = \dfrac{AC^2 - AB^2 - BC^2}{2AB}$:

from either of these equations, by reduction,

$$\cos \angle B = \frac{\text{rad}.(AB^2 + BC^2 - AC^2)}{2AB \cdot BC},$$

$$\text{or, } 2AB \cdot BC : AB^2 + BC^2 - AC^2 :: \text{rad} : \cos B.$$

Note. If the sides of the triangle be large, this analogy is inconvenient for arithmetical calculation.

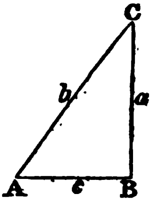
SCHOLIUM.

120. Various other propositions might be given, but the preceding are sufficient for solving every case of plane trigonometry. The analogies of plane trigonometry are only particular cases of spherical trigonometry, and may, in general, be deduced from them by simple inferences only.

Vide Book III. Chap. V. Prop. xxviii. and the *Scholium*.

SOLUTIONS OF THE DIFFERENT CASES OF RIGHT-ANGLED PLANE TRIANGLES.

Rules for solving the different cases both of right-angled plane triangles, and oblique-angled plane triangles, have already been investigated. The solutions here given are adapted to the table of natural sines[*]; and the notation is that used by Lagrange and Legendre, the three angles of the triangle being represented by A, B, C, and their opposite sides by a, b, c, as in the figure annexed; r = radius = sine of 90°.



Case I. Given the angles and the hypothenuse, to find the base and perpendicular.

Solution. $$a = \frac{b \sin A}{r} = \frac{b \cos C}{r} = \frac{b \tan A}{\sec A} = \frac{br}{\operatorname{cosec} A}.$$

$$c = \frac{b \cos A}{r} = \frac{b \sin C}{r} = \frac{br}{\sec A} = \frac{b \cot A}{\operatorname{cosec} A}.$$

Case II. Given the angles and the base, to find the hypothenuse and the perpendicular.

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[*] Logarithmic formulæ are easily supplied, thus in the first case log. sin A + log. $b - 10 =$ log. a. The introduction of such formulæ would only increase the size of the book, without any real advantage to the student.

SOLUTION. $b=\frac{cr}{\cos A}=\frac{cr}{\sin C}=\frac{c \sec A}{r}=\frac{c \operatorname{cosec} A}{\cot A}$

$$a=\frac{c \sin A}{\cos A}=\frac{c \cos C}{\sin C}=\frac{c \tan A}{r}=\frac{cr}{\cot A}$$

CASE III. Given the angles and the perpendicular, to find the base and the hypothenuse.

SOLUTION. $c=\frac{a \cos A}{\sin A}=\frac{a \sin C}{\cos A}=\frac{ar}{\tan A}=\frac{a \cot A}{r}$.

$$b=\frac{ar}{\sin A}=\frac{ar}{\cos C}=\frac{a \sec A}{\tan A}=\frac{a \operatorname{cosec} A}{r}$$

CASE IV. Given the hypothenuse and the base, to find the angles and the perpendicular.

SOLUTION. $\cos A=\sin C=\frac{cr}{b}$; $\sec A=\operatorname{cosec} C=\frac{br}{c}$

$\tan \frac{1}{2} A=r\sqrt{\frac{b-c}{b+c}}$.

Having found the angles, the perpendicular may be found by Case I. or by Case II. OR, $a=\sqrt{b^2-c^2}=\sqrt{(b+c)(b-c)}$.

CASE V. Given the hypothenuse and the perpendicular, to find the angles and the base.

SOLUTION. $\sin A=\cos C=\frac{ar}{b}$; $\sec C=\operatorname{cosec} A=\frac{br}{a}$;

$\tan\frac{1}{2} C=r\sqrt{\frac{b-a}{b+a}}$.

Having found the angles, the base may be found by Case I. Or, $c=\sqrt{b^2-a^2}=\sqrt{(b+a)\ (b-a)}$.

CASE VI. Given the base and the perpendicular, to find the angles and the hypothenuse.

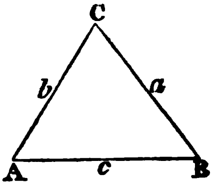
SOLUTION. $\tan A=\cot C=\frac{ar}{c}$; $\cot A=\tan C=\frac{cr}{a}$;

$$b=\sqrt{a^2+c^2}=a\sqrt{1+\frac{c^2}{a^2}}=c\sqrt{1+\frac{a^2}{c^2}}.$$

Or, having found the angles, the hypothenuse may be found by Case II. or by Case III.

SOLUTIONS OF THE DIFFERENT CASES OF OBLIQUE-ANGLED PLANE TRIANGLES.

Case I. Given the angles and one side, to find the other two sides.



$$a=\frac{c\sin A}{\sin C};\ b=\frac{\sin B}{\sin C}$$

$$b=\frac{a\sin B}{\sin A};\ c=\frac{a\sin C}{\sin A}\ \Big|\ a=\frac{b\sin A}{\sin B};\ c=\frac{b\sin C}{\sin B}.$$

Case II. Given two sides and an angle opposite to one of them, to find the other angles, and the third side.

$$\sin B=\frac{b\sin A}{a};\ \sin A=\frac{a\sin B}{b}\ \Big|\ \sin C=\frac{c\sin A}{a};$$

$$\sin A=\frac{a\sin C}{c}\ \Big|\ \sin C=\frac{c\sin B}{b};\ \sin B=\frac{b\sin C}{c}$$

Note. If the *given* ∠ be obtuse; or if the side opposite to the given ∠ be the *greater* of the two given sides, then the ∠ sought is always acute: in every other case it is ambiguous.

Two angles being now known, their sum deducted from 180°, leaves the third ∠; hence the remaining side may be found by Case I.

Case III. Given two sides and the angle contained between them, to find the other angles, and the third side.

$\tan\frac{1}{2}(A-B)=\frac{a-b}{a+b}\cdot\cot\frac{1}{2}C$ (115), in which we consider $a>b$, and consequently $A>B$.

Then $\frac{1}{2}(A-B)+(90°-\frac{1}{2}C)=\angle$ opposite to the greater side.

And $(90°-\frac{1}{2}C)-\frac{1}{2}(A-B)=\angle$ opposite to the less side.

OR THUS,

Suppose $a > b$, and let $b : a :: r : \tan \theta$,
then $r : \tan (\theta - 45^\circ) :: \cot \frac{1}{2} C : \tan \frac{1}{2} (A - B)$,
and proceed as before for the angles.

The remaining side may be found from the angles, by Case I;

$$\text{or from } c = \sqrt{a^2 + b^2 - \frac{2ab \cos C}{r}} \quad (119).$$

Case IV. Given the three sides, to find the angles.

Let $s = \frac{1}{2}(a + b + c)$, then

$$\cos \tfrac{1}{2} A = r \sqrt{\frac{s(s-a)}{bc}} \quad (117).$$

$$\tan \tfrac{1}{2} C = r \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \quad (118).$$

$$\cos A = \frac{r(b^2 + c^2 - a^2)}{2bc} \quad (119).$$

$$\text{vers } A = r - \cos A = \frac{r\{a^2 - (b-c)^2\}}{2bc}.$$

CHAP. II.

I. PRACTICAL RULES FOR THE SOLUTION OF ALL THE DIFFERENT CASES IN RIGHT-ANGLED PLANE TRIANGLES, WITH THEIR APPLICATION BY LOGARITHMS.

(121) In every right-angled plane triangle there must be two *given* quantities, one of which must be a side, (together with the right angle,) to find the rest.

TO FIND A SIDE.

Call any one of the sides of the triangle radius, and write upon it the word radius; observe whether the other sides become sines, tangents, or secants, and write these words on them accordingly. Call the word written upon each side the name of that side. Then say,

As the *name* of the given side is to the given side, so is the name of the required side to the required side.

TO FIND AN ANGLE.

Call either of the *given* sides of the triangle radius, and write upon it the word radius; observe whether the other sides become sines, tangents, or secants, and write these words on them accordingly. Call the word written upon each side the name of that side. Then say,

As the side made radius is to radius, so is the other *given* side to its name. The angles being found, the remaining side must be found by the first part of this rule.

OR; IN ANY RIGHT-ANGLED TRIANGLE.

1. The sine of any angle is to the side opposite to it, as the sine of any other angle is to its opposite side. *And,*

Any side is to the sine of its opposite angle, as any other side is to the sine of its opposite angle.

2. Radius is to the tangent of either of the acute angles, as the side adjacent to that angle is to the side opposite to it.

And, The side adjacent to either of the acute angles is to the side opposite to that angle, as radius is to the tangent of that angle.

3. Radius is to the cotangent of either of the acute angles, as the side opposite to that angle is to the side adjacent to it.

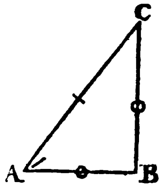
Also, The side opposite to either of the acute angles is to the side adjacent to that angle, as radius is to the cotangent of that angle.

(122) Case I. *Given the angles*[*] *and the hypothenuse, to find the base and perpendicular.*

Given { The hypothenuse AC = 480
The angle A = 53° 8′ } Required the base AB and perpendicular BC.

BY CONSTRUCTION.

Draw the line AB of any length, make the angle CAB = 53° 8′ (63) by a scale of chords, draw the hypothenuse AC = 480 from a scale of equal parts, from C let fall the perpendicular CB (62), then ABC is the triangle required. AB measured by the same scale of equal parts, which the hypothenuse was measured by, will be 288, and BC will be 384.



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[*] The *given* parts of a triangle are generally marked with a dash –, and the *required* parts with an O, as in the figures annexed to the constructions

BY CALCULATION.

I. *The hypothenuse radius*, BC will be the sine of the angle A, and AB the cosine.

To find BC.		To find AB.	
radius, sine of 90°,	10·00000	radius, sine of 90°	10·00000
: hypoth. AC=480	2·68124	: hypoth. AC=480	2·68124
: : sin A=53° 8′	9·90311	: : cos A 53° 8′	9·77812
: perpendicular BC=384	2·58435	: base AB=288	2·45936

II. *The base radius*, BC will be the tangent of the angle A, and AC will be the secant thereof.

To find BC.		To find AB.	
sec A=53° 8′	10·22188	sec A=53° 8′	10·22188
: hypoth. AC=480	2·68124	: hypoth. AC=480	2·68124
: : tan A=53° 8′	10·12499	: : radius, sine of 90°,	10·00000
: perpendicular BC=384	2·58435	: base AB=288	2·45936

III. *The perpendicular radius*, the base AB will be the tangent of the angle C, or cotangent of A, and the hypothenuse AC will be the secant of C, or the cosecant of A.

To find BC.		To find AB.	
cosec A=53° 8′	10·09689	cosec A=53° 8′	10·09689
: hypoth. AC=480	2·68124	: hypoth. AC=480	2·68124
: : radius, sine of 90°	10·00000	: : cot A=53° 8′	9·87501
: perpendicular BC=384	2·58435	: base AB=288	2·45936

BY GUNTER'S SCALE.

In working the several cases by Gunter's scale, we shall always suppose the hypothenuse radius, (where it can be done,) being the most simple of the three.

1. Extend the compasses from 90° to 53° 8′ on the line of sines, that extent will reach from 480 to 384 on the line of numbers.

2. Extend the compasses from 90° to 36° 52′ the complement of the angle A, that extent will reach from the hypothenuse AC=480, to the base AB=288, on the line of numbers.

PRACTICAL EXAMPLES.

1. In the right-angled plane triangle ABC,

$$\text{Given}\begin{cases}\text{Hypoth. AC}=645\\ \angle\text{A}=39^\circ\ 10'\end{cases}\qquad \textit{Ans.}\begin{cases}\text{AB}=500{\cdot}08\\ \text{BC}=407{\cdot}37\end{cases}$$

Required the base and perpendicular.

2. In the right-angled plane triangle ABC,

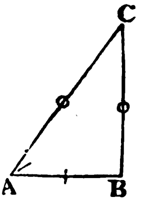
$$\text{Given}\begin{cases}\text{Hypoth. AC}=98\\ \angle\text{A}=56^\circ\ 48'\end{cases}\qquad \textit{Ans.}\begin{cases}\text{AB}=53{\cdot}66\\ \text{BC}=82{\cdot}00\end{cases}$$

Required the base and perpendicular.

(123) Case II. *Given the angles and the base, to find the hypothenuse and the perpendicular.*

Given { The angle A=53° 8′ } Required the hypoth. AC,
{ The base AB=288 } and the perpend. BC.

BY CONSTRUCTION.



Draw the base AB, which make=288, from a scale of equal parts, at B erect the perpendicular BC (61), make the angle A=53° 8′ (63), and draw the hypothenuse AC to cut the perpendicular BC in the point C. Then will AC measure 480, and BC 384.

BY CALCULATION.

I. *The hypothenuse radius*, BC will be the sine of the angle A, and AB the cosine.

To find AC.		To find BC.	
cos A=53° 8′	9·77812	cos A=53° 8′	9·77812
: base AB=288	2·45939	: base AB=288	2·45939
: : radius, sine of 90°	10·00000	: : sin A=53° 8	9·90311
: hypoth. AC=480	2·68127	: perpend. BC=384	2·58438

II. *The base* AB *radius*, BC will be the tangent of A, and AC the secant thereof.

To find AC.		To find BC.	
radius	10·00000	radius	10·00000
: base AB=288	2·45939	: base AB=288	2·45939
: : sec A =53° 8′	10·22188	: : tan A=53° 8′	10·12499
: hypoth. AC=480	2·68127	: perpend. BC=384	2·58438

III. *The perpendicular* BC *radius*, AB will be the tangent of C, or cotangent of A, and AC will be the secant of C, or cosecant of A.

To find AC.		To find BC.	
cot A=53° 8′	9·87501	cot A=53° 8′	9·87501
: base AB=288	2·45939	: base AB=288	2·45939
: : cosec A=53° 8′	10·09689	: : radius	10·00000
: hypoth. AC=480	2·68127	: perpend. BC=384	2·58438

BY GUNTER'S SCALE.

1. Extend the compasses, from 36° 52′ the complement of A to 90, on the line of sines; that extent will reach from 288 to 480=AC on the line of numbers.

2. Extend from 36° 52′ to 53° 8′ on the line of sines; that extent will reach from 288 to BC=384 on the line of numbers.

PRACTICAL EXAMPLES.

1. In the right-angled plane triangle ABC,

$$\text{Given}\begin{cases}\angle A=39^\circ\ 10'\\ \text{Base } AB=500\cdot08\end{cases}\qquad \textit{Ans.}\begin{cases}AC=645\cdot00\\ BC=407\cdot37\end{cases}$$

Required the hypothenuse and the perpendicular.

2. In the right-angled plane triangle ABC,

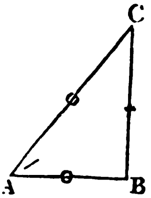
$$\text{Given}\begin{cases}\angle A=56^\circ\ 48'\\ \text{Base } AB=53\cdot66\end{cases}\qquad \textit{Ans.}\begin{cases}AC=98\\ BC=82\end{cases}$$

Required the hypothenuse and the perpendicular.

(124) CASE III. *Given the angles and perpendicular, to find the base and hypothenuse.*

$$\text{Given}\begin{cases}\text{The angle } A=53^\circ\ 8'\\ \text{The perpend. } BC=384\end{cases}\Big\}\ \text{Required the hypothenuse } AC\text{, and the base } AB.$$

BY CONSTRUCTION.



Draw the line AB of a sufficient length, at any point B erect the perpendicular BC (61), which make equal to 384 by a scale of equal parts, at C make the angle ACB=36° 52′ (63), the complement of A, from C draw the hypothenuse, and it will cut the base AB in the point A. Then will AC measure 480, and AB 288.

BY CALCULATION.

I. *The hypothenuse radius,* BC will be the sine of A, and AB the cosine thereof.

To find AC.		To find AB.	
sin A = 53° 8′	9·90311	sin A = 53° 8′	9·90311
: perpend. BC = 384	2·58433	: perpend. BC = 384	2·58433
: : radius	10·00000	: : cos A = 53° 8′	9·77812
: hypoth. AC = 480	2·68122	: base AB = 288	2·45934

II. *The base* AB *radius,* BC will be the tangent of A, and AC the secant thereof.

To find AC.		To find AB.	
tan A = 53° 8′	10·12499	tan A = 53° 8′	10·12499
: perpend. BC = 384	2·58433	: perpend. BC = 384	2·58433
: : sec A = 53° 8′	10·22188	: : radius	10·00000
: hypoth. AC = 480	2·68122	: base AB = 288	2·45934

III. *The perpendicular* BC *radius,* the base AB will be the tangent of C, or the cotangent of A; AC will be the secant of C, or cosecant of A.

To find AC.		To find AB.	
radius	10·00000	radius	10·00000
: perpend. BC = 384	2·58433	: perpend. BC = 384	2·58433
: : cosec A = 53° 8′	10·09689	: : cot A = 53° 8′	9·87501
: hypoth. AC = 480	2·68122	: base AB = 288	2·45934

BY GUNTER'S SCALE.

1. Extend he compasses from 53° 8′ to 90° on the line of sines, that extent will reach from 384 to 480, the hypothenuse on the line of numbers.

2. Extend from 53° 8′ to 36° 52′, the complement of the angle A, on the line of sines, that extent will reach from 384 to 288, on the line of numbers.

PRACTICAL EXAMPLES.

1. In the right-angled plane triangle ABC,

Given { ∠ A = 39° 10′ ; Perp. BC = 407·37 } *Ans.* { AC = 645·00 ; AB = 500·08 }

Required the hypothenuse and the base.

2. In the right-angled plane triangle ABC,

Given { ∠ A = 56° 48′ ; Perp. BC = 82 } *Ans.* { AC = 98·00 ; AB = 53·66 }

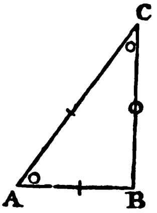
Required the hypothenuse and the base.

(125) Case IV. *Given the hypothenuse and base, to find the angles and the perpendicular.*

Given { The hypothenuse AC = 480 ; The base AB = 288 } { Required the angles A and C, and the perp. BC. }

BY CONSTRUCTION.

Draw the base AB equal to 288 from a scale of equal parts, at B erect the perpendicular BC (61); with the distance AC = 480, taken from a scale of equal parts, cross BC in the point C. Then BC measured by a scale of equal parts will be 384, and the angles A and C measured by a scale of chords (64) will be 53° 8′ and 36° 52′



BY CALCULATION.

I. *The hypothenuse radius*, BC will be the sine of the angle A, and AB the cosine thereof.

To find the angle A.		To find the perpend. BC.	
hypoth. AC = 480	2·68124	radius	10·00000
: radius	10·00000	: hypoth. AC = 480	2·68124
: : base AB = 288	2·45939	: : sin A = 53° 8′	9·90311
: cos A = 53° 8′	9·77815	: perpend. BC = 384	2·58435

II. *The base radius*, BC will be the tangent of A, and AC the secant thereof.

To find the angle A.		To find BC.	
base AB = 288	2·45939	radius	10·00000
: radius	10·00000	: base AB = 288	2·45939
: : hypoth. AC = 480	2·68124	: : tan A = 53° 8′	10·12499
: sec A = 53° 8′	10·22185	: perpend. BC = 384	2·58438

NOTE. The side BC may be found (47 by Euclid I.) thus, $\sqrt{AC^2 - AB^2} = BC = 384$.

(126) From the examples hitherto given, it appears that when we want to find a side, the proportions being with a *given* angle; and in the first three cases all the angles are given: therefore any side may be considered as the radius of a circle. But when we want to find an angle, the proportions begin with a *given* side; and as only two sides are given at once, it follows that these two *given* sides *only* can be considered as the radii of circles.

BY GUNTER'S SCALE.

1. Extend the compasses from the hypothenuse 480 to the base 288 on the line of numbers, that extent will reach from 90° to 36° 52′, on the line of sines, the complement of the angle A.

2. Extend the compasses from 90° to 53° 8′ on the line of sines, that extent will reach from the hypothenuse 480 to BC the perpendicular 384.

PRACTICAL EXAMPLES.

1. In the right-angled plane triangle ABC,

$$\text{Given}\begin{cases}\text{Hypoth. AC}=645\\ \text{Base AB}=500{\cdot}08\end{cases}\qquad \textit{Ans.}\begin{cases}\angle A=39^\circ\ 10'\\ \angle C=50^\circ\ 50'\\ BC=407{\cdot}37\end{cases}$$

Required the angles and the perpendicular.

2. In the right-angled plane triangle ABC,

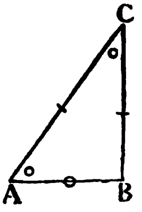
$$\text{Given}\begin{cases}\text{Hypoth. AC}=98\\ \text{Base AB}=53{\cdot}66\end{cases}\qquad \textit{Ans.}\begin{cases}\angle A=56^\circ\ 48'\\ \angle C=33^\circ\ 12'\\ BC=82\end{cases}$$

Required the angles and the perpendicular.

(127) CASE V. *Given the hypothenuse and perpendicular, to find the angles and the base.*

$$\text{Given}\begin{cases}\text{The hypothenuse AC}=480\\ \text{The perpendicular BC}=384\end{cases}\text{Required the angles A and C, and the base AB.}$$

BY CONSTRUCTION.



Draw the base AB of an indefinite length, at B erect the perpendicular BC (61) which make equal to 384 by a scale of equal parts; take AC =480 from the same scale, with this extent in your compasses and centre C cross the base AB in A. Then AB measured by a scale of equal parts, will be 288, and the angles A and C measured by a scale of chords (64) will be 53° 8′, and 36° 52′.

BY CALCULATION.

I. *The hypothenuse radius,* BC will be the sine of the angle A, and AB the cosine thereof.

To find the angle A.		To find the base AB.	
hypoth. AC = 480	2·68124	radius	10·00000
: radius	10·00000	: hypoth. AC = 480	2·68124
: : perpend. BC = 384	2·58433	: : cos A = 53° 8′	9·77812
: sin A = 53° 8′	9·90309	: base AB = 288	2·45936

II. *The perpendicular* BC *radius,* AB will be the tangent of C, or the cotangent of A, and AC will be the secant of C or cosecant of A.

To find the angle A.		To find the base AB.	
perpend. BC = 384	2·58433	radius	10·00000
: radius	10·00000	: perpend. BC = 384	2·58433
: : hypoth. AC = 480	2·68124	: : cot A = 53° 8′	9·87501
: cosec A = 53° 8′	10·09691	: base AB = 288	2·45934

Or AB may be found thus, $\sqrt{AC^2 - BC^2} = AB = 288$.

BY GUNTER'S SCALE.

1. Extend the compasses from the hypothenuse 480 to the perpendicular 384 on the line of numbers, that extent will reach from 90° to 53° 8′ on the line of sines.

2. Extend the compasses from 90° to 36° 52′, the complement of A, on the line of sines, that extent will reach from 480 to 288 on the line of numbers.

PRACTICAL EXAMPLES.

1. In the right-angled plane triangle ABC,

Given		*Ans.*
Hypoth. AC = 645		∠ A = 39° 10′
Perp. BC = 407·37		∠ C = 50° 50′
		AB = 500·08

Required the angles and the base.

2. In the right-angled plane triangle ABC,

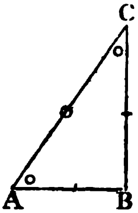
Given		*Ans.*
Hypoth. AC = 98		∠ A = 56° 48′
Perpen. BC = 82		∠ C = 33° 12′
		AB = 53·66.

Required the angles and the base.

(128) CASE VI. *Given the base and perpendicular, to find the angles and the hypothenuse.*

Given {The base AB = 288; The perpend. BC = 384} Required the angles A and C, and hypothenuse AC.

BY CONSTRUCTION.



Make the base AB equal to 288 by a scale of equal parts, at B erect the perpendicular BC, which make equal to 384 from the same scale; join AC. Then AC measured by a scale of equal parts will be 480, and the angles A and C measured by a scale of chords (64) will be 53° 8′ and 36° 52′.

BY CALCULATION.

I. *The base radius*, BC will be the tangent of the angle A, and AC will be the secant thereof.

To find the angle A.		To find the hypoth. AC.	
base AB = 288 -	2·45939	radius - -	10·00000
: radius - -	10·00000	: base AB = 288 -	2·45939
: : perpend. BC = 384 -	2·58433	: : sec A = 53° 8′ -	10·22188
: tan A = 53° 8′ -	10·12494	: hypoth. AC = 480 -	2·68127

II. *The perpendicular radius*, AB will be the tangent of C, or the cotangent of A; and AC will be the secant of C, or the cosecant of A.

To find the angle A.		To find the hypoth. AC.	
perp. BC = 384 - -	2·58433	radius - -	10·00000
: radius - -	10·00000	: perpend. BC = 384 -	2·58433
: : base AB = 288 -	2·45939	: : cosec A = 53° 8′ -	10·09689
: cot A = 53° 8′ -	9·87506	: hypoth. AC = 480 -	2·68122

Or AC may be found thus, $\sqrt{AB^2 + BC^2} = AC = 480$.

BY GUNTER'S SCALE.

1. Extend the compasses from 384 to 288 on the line of numbers, that extent will reach from 45° to 53° 8′ on the line of tangents.

2. Extend the compasses from 53° 8′ to 90° on the line of sines, that extent will reach from 384 to 480 on the line of numbers.

PRACTICAL EXAMPLES.

1. In the right-angled plane triangle ABC,

$$\text{Given}\begin{cases}\text{Base AB} = 500\cdot08\\ \text{Perp. BC} = 407\cdot37\end{cases}\qquad \textit{Ans.}\begin{cases}\angle A = 39^\circ\ 10'\\ \angle C = 50^\circ\ 50'\\ AC = 645\end{cases}$$

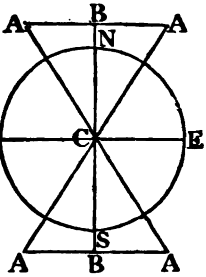
Required the angles and the hypothenuse.

2. In the right-angled plane triangle ABC,

$$\text{Given}\begin{cases}\text{Base AB} = 53\cdot66\\ \text{Perp. BC} = 82\end{cases}\qquad \textit{Ans.}\begin{cases}\angle A = 56^\circ\ 48'\\ \angle C = 33^\circ\ 12'\\ AC = 98\end{cases}$$

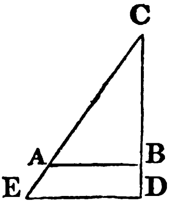
Required the angles and the hypothenuse.

SCHOLIUM.



(129) Plane sailing in navigation is nothing more than the practice of right-angled trigonometry; calling the hypothenuse the distance sailed, the perpendicular the difference of latitude, the base the departure, and the angle opposite to the base the course.

In the annexed figure, WESN represents the horizon of the place C, whence the ship sails; CA the point of the compass or rhumb she sails on, and A the place sailed to. Then WE represents the parallel of latitude sailed from, and AA the parallel of latitude arrived at. Hence BC becomes the difference of latitude, AB the departure, CA the distance sailed; the angle ACB the course, and the angle BAC the complement of the course. And this will always be the case whether the ship sails between the north and west, viz. between N and W; the north and east, viz. between N and E; the south and west, viz. between S and W; or south and east, viz. between S and E.



(130) The practice of Mercator's sailing is only the application of right-angled trigonometry, and similar triangles (109). What is called the difference of latitude in plane sailing, is here called the *proper* difference of latitude, to distinguish it from the line CD, which is called the meridional difference of latitude. The part BD, by which CB is enlarged, is found by help of a table of meridional parts; being the sum or difference of the meridional parts answering to the latitude sailed from and bound to, according as they are on different sides or on the same side of the equator. ED is the difference of longitude.

II. PRACTICAL RULES FOR SOLVING ALL THE CASES OF OBLIQUE TRIANGLES, WITH THEIR APPLICATION BY LOGARITHMS.

(131) I. *When two of the three* given parts are a side and its opposite angle.*

Any one side of a triangle
Is to the sine of its opposite angle,

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* In every plane triangle there must be three parts given, to find the rest; and of these three given parts, one, at least, must be a side, because the same angles are common to an infinite number of triangles.

As any other side
Is to the sine of its opposite angle.
And, The sine of any angle
Is to its opposite side,
As the sine of any other angle
Is to its opposite side. (114.)

(132) An *angle* found by this rule is *sometimes* ambiguous, for trigonometry gives us only the sine of an angle, and not the angle itself, and the sine of every angle is also the sine of its supplement. The table gives only the *acute* value of an angle; the *obtuse* value is the supplement thereof.

When the *given* side, opposite to the *given* angle, is greater than the other *given* side, then the angle opposite to that other given side is always acute. But when the *given* side opposite to the *given* angle, is less than the other given side, then the angle opposite that other given side may be either acute or obtuse, and consequently it is ambiguous.

(133) II. *When two sides and the angle contained between them are given.*[*]

The sum of the two given sides
Is to their difference,
As the cotangent of half the contained angle
Is to the tangent of half the difference between the other two angles.

This half difference added to the complement of half the contained angle, gives the angle opposite to the greater side; and being subtracted, leaves the angle opposite to the less side,

OR,

The sum of the two given sides
Is to their difference,
As the tangent of half the supplement of the contained angle
Is to the tangent of half the difference between the other two angles.

This half difference, added to the half supplement, gives the angle opposite to the greater side; and being subtracted, leaves the angle opposite to the less side. (115.)

The remaining side of the triangle may be found by Rule I.

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[*] If the two given sides be equal to each other, the triangle is isosceles, and each of the remaining angles will be equal to half the supplement of the given angle. If the given angle be 90°, the required parts may be found by Case VI. of right-angled triangles.

(134) III. *When the three sides of a triangle are given, to find the angles.*[*]

Double the base, or longest side,
Is to the sum of the other two sides,
As the difference between those two sides
Is to the distance of a perpendicular from the middle of the base.

This distance added to half the base gives the greater segment, and being subtracted therefrom, leaves the less segment. (113.)

The triangle being thus divided into two right-angled triangles, each of which contains two given sides, the remaining angles may be found by Rule I.

OR,

From half the sum of the three sides of any plane triangle, subtract the side opposite to the angle sought, and note the half sum and the remainder. Then, to the logarithm of the half sum, add the logarithm of the remainder, and increase the index of the sum by 20; from the sum thus increased, subtract the sum of the logarithms of the two sides containing the required angle, the remainder, divided by two, will give the log cosine of half the angle required. (117.)

The remaining angles may either of them be found by Rule I.

OR,

From half the sum of the three sides of any plane triangle subtract each of the sides containing the angle sought, and note the two remainders.

Then, add the logarithms of these two remainders together, and increase the index of their sum by 20; also, from half the sum of the three sides, subtract the side opposite to the angle required, and add the logarithm of the half sum to the logarithm of the remainder; the difference between these logarithmic sums divided by 2, will give the log tangent[†] of half the angle sought. (118.)

Either of the remaining angles may be found by Rule I.

---

[*] If the triangle be equilateral, each of its angles will be 60°; if it be isosceles, the perpendicular will bisect the base, and the angles may be found by Case IV. of right-angled triangles.

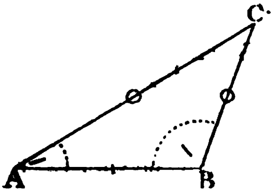
[†] If an angle be near 90°, or if it be a very small angle, it ought to be determined by a rule which produces a tangent or a cotangent. For the sines of arcs near 90° vary by almost imperceptible increments; and the cosines of very small arcs, or those near 180°, vary by decrements which are not easily assigned. To illustrate these observations by an example; suppose that in working any problem, by a table calculated to seconds, and carried to 7 places of decimals, the result of the operation should produce 9·9999998 a sine or cosine. By the tables this is

(135) CASE I. *Given the angles and one side of a triangle, to find the rest.*

Given { The angle A = 32° 15′
The angle B = 114° 24′
The side AB = 98 } Required the sides AC and BC.

BY CONSTRUCTION.

Make AB equal to 98 by a scale of equal parts, draw BC making the angle B equal to 114° 24′ (63); also make the angle A equal to 32° 15′, draw AC meeting BC in C. Then AC measured by the same scale of equal parts is 162, and BC 95.



BY CALCULATION. *See Rule I.*

To the angle A = 32° 15′ add the angle B = 114° 24′, the sum is 146° 39′, which subtract from 180° leaves the angle C = 33° 21′.

To find AC.		To find BC.	
sin C = 33° 21′ -	9·74017	sin C = 33° 21′ -	9·74017
: AB = 98 - -	1·99123	: AB = 98 - -	1·99123
· : sin B = 114° 24′ -	9·95937	: : sin A = 32° 15′ -	9·72723
side AC = 162·34 -	2·21043	: side BC = 95·12 -	1·97829

BY GUNTER'S SCALE.

1. Extend the compasses from 33° 21′ to 65° 36′ the supplement of B, on the sines, that extent will reach from 98 to 162 on the line of numbers.

2. Extend the compasses from 33° 21′ to 32° 15′ on the line of sines, that extent will reach from 98 to 95 on the line of numbers.

PRACTICAL EXAMPLES.

1. In the plane triangle ABC,

Given { ∠A = 27° 59′
∠B = 44° 12′
AC = 104 }   *Ans.* { AB = 142·02
BC = 69·99
∠C = 107° 49′ }

Required the other parts.

---

the sine of any arc between 89° 56′ 19″, and 89° 57′ 8″; or the cosine of any arc between 2′ 52″ and 3′ 41″; here is therefore a choice of 49″ or arcs, and no reason can be given why one should be preferred to another: but the logarithmic differences between the tangents and cotangents are never so small as to leave any doubt respecting the accuracy of the result, the smallest difference for 1″ being 42, at which time the arc is 45°.

2. In the plane triangle ABC,

$$\text{Given}\left\{\begin{array}{l}\angle A=79^\circ\ 23'\\ \angle B=54^\circ\ 22'\\ BC=125\end{array}\right. \qquad \textit{Ans.}\left\{\begin{array}{l}AC=103{\cdot}36\\ AB=\ 91{\cdot}87\\ \angle C=46^\circ\ 15'\end{array}\right.$$

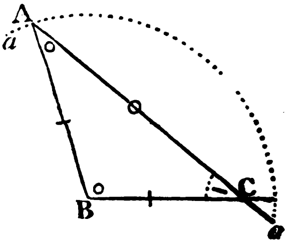
Required the other parts.

(136) CASE II. *Given two sides and an angle opposite to one of them, to find the rest.*

$$\text{Given}\left\{\begin{array}{l}\text{The angle } C=33^\circ\ 21'\\ \text{The side } AB=98\\ \text{The side } BC=95{\cdot}12\end{array}\right\}\ \text{Required the angles A and B and the side AC.}$$

BY CONSTRUCTION.

Make BC=95·12 by a scale of equal parts, draw CA making the angle C=33° 21′ (63). With the side AB in your compasses, taken from the same scale of equal parts, and B as a centre, describe the arc *aa* cutting AC in the point A. Then AC measured by the scale of equal parts will be 162, and the angles A and B measured by a scale of chords (64) will be 32° 15′ and 114° 24′.



Here agreeably to the observation (132) the angle A is acute and not ambiguous; but had AB been less than BC, the arc *aa* would evidently have cut AC in two points on the same side of BC.

BY CALCULATION. *See Rule I.*

AB=98	1·99123	sin C=33° 21′	9·74017
: sin C=33° 21′	9·74017	: AB=98	1·99123
: : BC=95·12	1·97827	: : sin B=114° 24′	9·95937
: sin A=32° 15′	9·72721	: side AC=162·34	2·21043
hence ∠B=114° 24′			

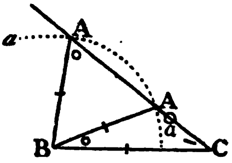
BY GUNTER'S SCALE.

1. Extend the compasses from 98 to 95 on the line of numbers, that extent will reach from 33° 21′ to 32° 15′ on the line of sines.

2. Add the angles A=32° 15′ and C=33° 21′ together, and the sum will be 65° 36′; then extend the compasses from 33° 21′ to 65° 36′ on the line of sines, that extent will reach from 98 to 162·3 on the line of numbers.

$$\text{Given}\left\{\begin{array}{l}\text{The angle } C=33^\circ\ 21'\\ \text{The side } BC=95{\cdot}12\\ \text{The side } AB=60\end{array}\right\}\ \text{Required the angles A and B and the side AC.}$$

BY CONSTRUCTION.



This construction is exactly the same as in the preceding example; only AB, being shorter than BC, cuts AC in two points on the same side of BC, hence the angle A may be either acute or obtuse.

BY CALCULATION. *See Rule I.*

AB=60 - - -	1·77815
: sin C=33° 21 - -	9·74017
: : BC=95·12 - -	1·97827
: sin A $\left\{\begin{array}{l} 60^\circ\ 38'\ \textit{acute} \\ 119^\circ\ 22'\ \textit{obtuse} \end{array}\right\}$ -	9·94029

The sum of the angles C and A subtracted from 180° leaves the angle B=86° 1′, if A be acute; or 27° 17′ if A be obtuse.

sin C=33° 21′ - -	9·74017
: AB=60 - -	1·77815
: : sin B=86° 1′ - -	9·99895
: AC=108·87 - - -	2·03693

OR,

sin C=33° 21′ - -	9·74017
: AB=60 - - -	1·77815
: : sin B=27° 17′ - -	9·66124
: AC=50·03 - - -	1·69922

PRACTICAL EXAMPLES.

1. In the plane triangle ABC,

$$\text{Given}\left\{\begin{array}{l} AC=104 \\ BC=70 \\ \angle B=44^\circ\ 12' \end{array}\right. \qquad \textit{Ans.}\left\{\begin{array}{l} \angle A=\ 27^\circ\ 59' \\ \angle C=107^\circ\ 49' \\ AB=142{\cdot}02 \end{array}\right.$$

Required the other parts.

2. In the plane triangle ABC,

$$\text{Given}\left\{\begin{array}{l} AC=104 \\ AB=142{\cdot}02 \\ \angle B=44^\circ 12' \end{array}\right. \qquad \textit{Ans.}\left\{\begin{array}{l} \angle A=63^\circ\ 37' \textit{ or } 27^\circ\ 59' \\ \angle C=72^\circ\ 11' \textit{ or } 107^\circ\ 49' \\ BC=133{\cdot}63 \textit{ or } 69{\cdot}99 \\ \quad \text{ambiguous.} \end{array}\right.$$

Required the other parts.

3. A man travels from A to B, 3·7 miles, then turning a little to the left hand goes from B to C, which is 4·7 miles: at C he observes that A and B make an angle of 29° 16′. What is the distance from A by the shortest cut?

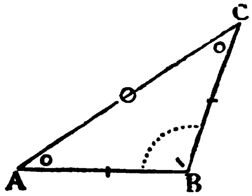
Answer. The side AB, opposite the given angle C, being greater than BC, the triangle belongs to the *ambiguous* case: and AC is either 7 or 2 miles.

(137) CASE III. *Given two sides and the angle contained between them, to find the rest.*

$$\text{Given}\left\{\begin{array}{l} \text{The side } AB=98 \\ \text{The side } BC=95{\cdot}12 \\ \text{The angle } B=114^\circ\ 24' \end{array}\right\} \text{Required the angles A and C, and the side AC.}$$

BY CONSTRUCTION.

Make AB=98 by a scale of equal parts, and the angle B=114° 24′ by a scale of chords (63); draw BC, which make equal to 95·12, and ABC is the triangle required. The angles A and C will measure 32° 15′ and 33° 21′, and AC will be 162.



BY CALCULATION. *See Rule II.*

To find the angles.

AB + BC = 193·12 - - 2·28583
: AB—BC = 2·88 - - 0·45939
: : cot ½ B = 57° 12′ - 9·80919
: tan ½ (C—A) = 0° 33′ - 7·98275
comp. ½ B 32° 48′

sum 33° 21′ = ∠C

diff. 32° 15′ = ∠A

Or thus,

AB + BC = 193·12 - - 2·28583
: AB—BC = 2·88 - - 0·45939
: : tan ½ supp. B = 32° 48′ - 9·80919
: tan ½ (C—A) = 0° 33′ - 7·98275

sum 33° 21′ = ∠C

diff. 32° 15′ = ∠A

To find the side AC.

sin C = 33° 21′ - - - 9·74017
: AB = 98 - - - - 1·99123
: : sin B = 114° 24′ - - - 9·95937
: AC = 162·34 - - - 2·21043

BY GUNTER'S SCALE.

1. Extend the compasses from 193·12 to 2·88 on the line ot numbers, that extent will reach from 32° 48′ to 0° 33′ on the line of tangents. This is the method of working such examples as this; but so small an angle as 33′ is not contained on the scale.

2. Extend from 33° 21′ to 65° 36′ on the line of sines, that extent will reach from 98 to 162 on the line of numbers.

PRACTICAL EXAMPLES.

1. In the plane triangle ABC,

Given $\begin{cases} AB=345 \\ AC=174\cdot07 \\ \angle A=37° 20' \end{cases}$ *Ans.* $\begin{cases} \angle B=27° 4' \\ \angle C=115° 36' \\ BC=232 \end{cases}$

Required the other parts.

2. In the plane triangle ABC,

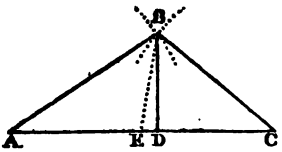
Given $\begin{cases} AB=103 \\ AC=126 \\ \angle A=56° 30' \end{cases}$ *Ans.* $\begin{cases} \angle B=72° 20' \\ \angle C=51° 10' \\ BC=110\cdot3' \end{cases}$

Required the other parts.

(138) CASE IV. *Given the three sides, to find the angles.*

$$\text{Given}\left\{\begin{array}{l}\text{The side AB}=98\\ \text{The side BC}=95{\cdot}12\\ \text{The side AC}=162{\cdot}34\end{array}\right\}\text{Required the angles, A, B, and C.}$$

BY CONSTRUCTION.



Draw the longest side AC = 162·34 from a scale of equal parts; with AB = 98 in your compasses (taken from the same scale) and one foot in A describe an arc; with BC = 95·12 in your compasses cross it in B; then ABC is the triangle required. The angles measured by a scale of chords (64) will be A = 32° 15′, B = 114° 24′, and C = 33° 21′.

BY CALCULATION. *See Rule III.*

Let E be the middle of the base AC, and BD perpendicular to AC.

To find the segments AD *and* BC *of the base.*

double AC = 324·68 - - -	2·51145
: AB + BC = 193·12 - - - -	2·28583
: : AB − BC = 2·88 - - -	0·45939
: ED = 1·713 - - - -	0·23377

Then ½ AC + ED = 81·17 + 1·713 = 82·883 = AD the greater segment, and ½ AC − ED = 81·17 − 1·713 = 79·457 = DC the less segment.

To find the angles in the right-angled triangles ADB *and* CDB.

AB = 98 - -	1·99123	BC = 95·12 - -	1·97827
: radius, sine 90° -	10·00000	: radius, sine 90° - -	10·00000
: : AD = 82·883 - -	1·91847	: : CD = 79·457 -	1·90013
: sin DBA = 57° 45′ or cos DAB = 32° 15′ -	9·92724	: sin DBC = 56° 39′ or cos DCB = 33° 21′ -	9·92186

Then DBA + DBC = 57° 45′ + 56° 39′ = 114° 24′ = ABC.

OR THUS,

$$\left.\begin{array}{rl}\tfrac{1}{2}(\text{AC}+\text{AB}+\text{BC}) & =177{\cdot}73\ \text{log.}=2{\cdot}24976\\ \tfrac{1}{2}(\text{AC}+\text{AB}+\text{BC})-\text{BC} & =\ 82{\cdot}61\ \text{log.}=1{\cdot}91703\end{array}\right\}\ \text{sum}+20=24{\cdot}16679$$

$$\left.\begin{array}{rl}\text{AB} & =\ 98{\cdot}\quad\ \text{log.}=1{\cdot}99123\\ \text{AC} & =162{\cdot}34\ \text{log.}=2{\cdot}21043\end{array}\right\}\ \text{sum}\quad=\ 4{\cdot}20166$$

$$2)19{\cdot}96513$$

$$\tfrac{1}{2}\angle\text{A}=16^\circ\ 7'\ 30''\ \ldots\ldots\ldots\text{cos}\quad 9{\cdot}98257$$

$$2$$

$$\angle\text{A}=32^\circ\ 15'\ 0''$$

OR THUS,

$$\begin{array}{l}\left.\begin{array}{l}\tfrac{1}{2}(\mathrm{AC}+\mathrm{AB}+\mathrm{BC})-\mathrm{AC}=\ \ 15{\cdot}39\ \log=1{\cdot}18724\\ \tfrac{1}{2}(\mathrm{AC}+\mathrm{AB}+\mathrm{BC})-\mathrm{AB}=\ \ 79{\cdot}73\ \log=1{\cdot}90162\end{array}\right\}\text{sum}+20=23.08886\\ \left.\begin{array}{l}\tfrac{1}{2}(\mathrm{AC}+\mathrm{AB}+\mathrm{BC})\qquad\ \ =177{\cdot}73\ \log=2{\cdot}24976\\ \tfrac{1}{2}(\mathrm{AC}+\mathrm{AB}+\mathrm{BC})-\mathrm{BC}=\ \ 82{\cdot}61\ \log=1{\cdot}91708\end{array}\right\}\text{sum}\quad=\quad 4{\cdot}16679\end{array}$$

$$2)18{\cdot}92207$$

$$\tfrac{1}{2}\angle\mathrm{A}=16^\circ\ \ 7'\ 27''\ldots\ldots\ldots\text{tan}\quad 9{\cdot}46104$$

$$\begin{array}{r}2\\ \hline 32^\circ\ 14'\ 54''\\ \hline\end{array}$$

The remaining angles may be found by Rule I.

BY GUNTER'S SCALE, *according to the first method.*

1. Extend the compasses from 324·68 to 193·12 on the line of numbers, that extent will reach from 2·88 to 1·713 the distance of a perpendicular from the middle of the base.

2. Extend from 98 to 82·883 on the line of numbers, that extent will reach from 90° to 57° 45′ on the line of sines.

3. Extend from 95·12 to 79·457 on the line of numbers, that extent will reach from 90° to 56° 39′ on the line of sines.

BY GUNTER'S SCALE, *according to the second method.*

1. Extend the compasses from half the sum of the three sides 177·73 to one of the containing sides AB = 98, that extent will reach from AC = 162·34 the other containing side, to a fourth number 89·5, on the line of numbers.

2. Extend the compasses from this fourth number 89·5 to (the difference between the half sum of the three sides and the side opposite to the angles sought) 82·61 on the line of numbers, that extent will reach from 90° on the line of sines to the required angle 32°, on the line of versed sines, immediately under the line of sines.

This is derived from the proportions in the investigation of Gunter's Rule (46 and note).

PRACTICAL EXAMPLES.

1. In the plane triangle ABC,

$$\text{Given}\left\{\begin{array}{l}\mathrm{AB}=142{\cdot}02\\ \mathrm{BC}=70\\ \mathrm{AC}=104\end{array}\right.\qquad \textit{Ans.}\left\{\begin{array}{l}\angle\mathrm{A}=\ \ 27^\circ\ 59'\\ \angle\mathrm{B}=\ \ 44^\circ\ 12'\\ \angle\mathrm{C}=107^\circ\ 49'\end{array}\right.$$

Required the angles.

2. In the plane triangle ABC,

$$\text{Given}\left\{\begin{array}{l}\mathrm{AB}=112{\cdot}65\\ \mathrm{BC}=112\\ \mathrm{AC}=120\end{array}\right.\qquad \textit{Ans.}\left\{\begin{array}{l}\angle\mathrm{A}=57^\circ\ 27'\\ \angle\mathrm{B}=64^\circ\ 34'\\ \angle\mathrm{C}=57^\circ\ 59'\end{array}\right.$$

SCHOLIUM.

(139) There are some authors and teachers of trigonometry, who make no distinction of cases between right and oblique-angled triangles, but divide the whole into three cases; because the three rules necessary for solving the problems that occur in oblique trigonometry, are sufficient for solving those which occur in right-angled trigonometry. For instance, Rule I. (131) will solve all the cases in right-angled triangles (except the 6th), and the first and second cases in oblique-angled triangles: Rule II. (133) will solve the 6th case in right-angled triangles and the 3d case in oblique; and Rule III. (134) will solve the last case in oblique triangles.

CHAP. III.

THE APPLICATION OF PLANE TRIGONOMETRY TO THE MENSURATION OF HEIGHTS, DISTANCES, ETC.

(140) THE mensuration of heights and distances depends upon the rules of plane trigonometry already explained, together with the use of certain instruments for taking angles.

(141) Horizontal and vertical angles are usually measured with a theodolite furnished with one or two telescopes, and a vertical arc; and if the horizontal and vertical arcs of the instrument be described with a radius of not less than $3\frac{1}{4}$ inches, the observed angles may be measured to half a minute, or the 120th part of a degree.

(142) Angles which are oblique to the horizon are generally taken with a sextant, which must be held in such a position, that its plane may coincide with the two objects and the eye. When vertical angles are taken with this instrument, an artificial horizon must be used, and the reflected image of the object from the glasses of the sextant must be brought to coincide with the reflected image of the same object in the artificial horizon.

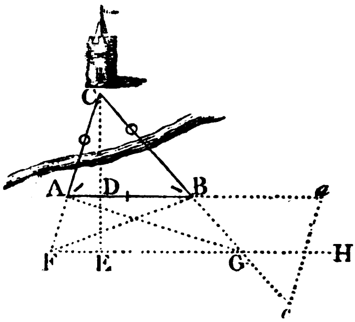
(143) Base lines are generally measured with rods, or the *four pole* Gunter's chain; but common tape of 50 or 100 feet in length is often preferred both for accuracy and expedition: especially if it be kept dry, and the ground be tolerably level.

(144) The use of instruments must be acquired under the direction of a person well skilled in their several adjustments,

as but little information can be obtained from written description*, and even the most expert observer will find it necessary, in several cases, to apply corrections to the different angles according to the situations of the objects. See Chapter IV.

EXAMPLE I.

Being on one side of a river, and wanting to know the distance of a fort, or other object, on the other side, suppose I measured 500 yards along the side of the river in a straight line AB and found the two angles between this line and the object to be CAB = 74° 14′ and CBA = 49° 23′. Required the distance between each station and the object.



∠ ACB = 180° — ∠ (CAB + CBA) = 56° 23′		sin ∠ ACB = 56° 23′ -	9·92052
sin ∠ ACB = 56° 23′ -	9·92052	: AB = 500 - -	2·69897
: AB = 500 - -	2·69897	: : sin ∠ CAB = 74° 14′	9·98334
: : sin ∠ CBA = 49° 23′ -	9·88029	: BC = 577·8 - -	2·76179
: AC = 455·8 - -	2·65874		

(145) Or, produce AB till B*a* be equal to it, and CB till the angle *a* be equal to the angle A; then will the distances B*c* and *ac* be equal to the distances BC and AC.

(146) Or, without any instrument to measure an angle, the distances AC and BC may be found. Make AF of any length in a line with AC, and BG in a line with BC; measure AF, AB, BG, FB, and AG; then the three sides of the triangles BAF and ABG will be given, with which the angles BAF and ABG may be found; their supplements will give the angles CAB and CBA, and hence the distances AC and BC may be found as in the example.

(147) The perpendicular distance DC may be readily determined, for having found BC, the angle B and BC will be given; or find AC, then AC and the angle A will be given to find DC.

The perpendicular distance DC may be found without any instrument for taking angles. With a cross-staff fixed at D observe the object at C, and set up a staff in a line with it at E, and another at B at right angles to CE, move the cross-staff from

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* Copper-plates, well executed, of theodolites, sextants, quadrants, and other instruments for taking angles, are given in Adams's Geometrical and Graphical Essay, edited and published by Mr. Wm. Jones, mathematical instrument maker in Holborn, London.

D to E, and set up a pole or staff at H perpendicular to EC. Measure the distances DB, DE, and also EG in the direction EH, so that G, B, and C make one straight line. Then

EG—DB : DE : : EG : EC from which take DE.

Or, EG—DB : DE : : DB : DC.

The four following examples are referred to the foregoing figure, and serve to exercise the above observations.

EXAMPLE II.

An engineer wanted to know the breadth of a river (over which the general intended to pass the whole army), in order to determine how many pontoons of five feet broad and six feet asunder he should have occasion to make use of. Perceiving an object (C) on the opposite bank of the river close to the edge of the water, he measured 144 paces or 360 feet along the edge of the river (as AB). The angle at A between the object and the base line was 83° 57′, and at B it was 80° 32′; required the perpendicular breadth (DC) of the river, and the number of pontoons sufficient to form a passage for the troops.

Answer. The perpendicular breadth of the river is 528 paces or 1320 feet, a military pace being $2\frac{1}{2}$ feet; and 120 pontoons will answer the purpose.

EXAMPLE III.

Two ships of war intending to cannonade a fort at C, separated from each other 500 yards, and coming as near to the shore as possible without being in danger of running aground, the officers in each ship observe the angles formed between the other ship and the fort, and find them, viz. A=38° 16′, and B=37° 9′. Are the ships at a proper distance for commencing a cannonade, the most convenient distance being about 300 yards?

Answer. A is 312 yards from the fort, and B 320.

EXAMPLE IV.

Wanting to know my distance from an object at C, I measured from E to D in a line with the object 29 poles, from D I measured DB at right angles to ED=43 poles, and coming to E I measured EG in the direction EH=54 poles, what is my distance DC from the object?

Answer. $113\frac{4}{11}$ poles.

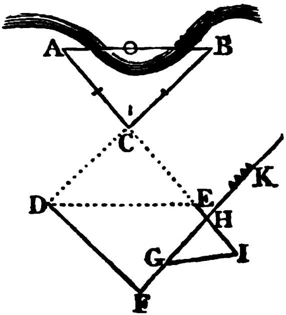
EXAMPLE V.

Wanting to know the distance of two objects A and B (which are 500 yards from each other) from a tower at C on the opposite side of a river, and, having no other instrument than a chain of 22 yards long, I measured AF in a straight line with AC, and found it to be 300 yards long. I likewise measured FB and found it to be 649·2 yards, I then measured BG=690 yards and AG=1000 yards, required the distances AC and BC.

Answer. AC=456·8 yards, and BC=578·5.

EXAMPLE VI.

Suppose I wanted to know the distance between two places A and B, accessible at both ends of the line AB, and that I measured AC=735 yards, and BC=840, also the angle ACB=55° 40′. What is the distance between A and B?



BC + AC = 1575 - - 3·19728
: BC − AC = 105 - - - 2·02119
: : tan ½ supp C = 62° 10′ - 10·27738
: tan ½ (A − C) 7° 11′ 48″ - 9·10129

∠A = 69° 21′ 48″

∠B = 54° 58′ 12″
sin ∠B = 54° 58′ 12″ - - 9·91321
: AC = 735 - - - - 2·86629
: : sin ∠ C = 55° 40′ - - 9·91686
: AB = 741 - - - 2·86994

(148) If the lines AC and BC be produced till DC be equal to CB and EC to AC, then will the distance DE be equal to the distance AB without calculation.

EXAMPLE VII.

Wanting to know the angle ACB formed by two walls AC and BC, I measured DC=840 yards, EC=735 yards, and DE=741 yards: hence the angle ACB=DCE is required.

Answer. ACB=55° 39′.

EXAMPLE VIII.

Suppose ACB to represent the flanked angle of a bastion, (which of course will be inaccessible on account of the covert-way, ditch, &c.), and it is required to determine the measure of that angle.

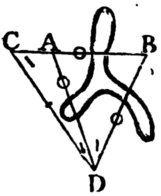
Place a pole at D, in the direction of the face BC of the bastion (out of the reach of musket shot). With a cross-staff at D direct a person to set up another pole at F perpendicular to DC. Measure DF=375 paces or $937\frac{1}{2}$ feet, make FK perpendicular to DF, and set up another pole at H in the direction of the face AC of the bastion. Prolong CH, and measure the three sides of the triangle GHI; suppose GH=10 yards, GI=$12\frac{3}{25}$ yards, and HI = 10 yards, then will the angle GHI be equal to the flanked angle ACB. Required this angle.

Answer. The angle ACB = 74° 36′, the proper flanked angle of a regular pentagon.

(149) In the same manner (except measuring the triangle GHI,) we may draw a line KF, parallel to the inaccessible face BC of the bastion, in order to place a battery at K, to produce the greatest effect; viz. at about 375 paces distance, and that the direction of the fire may be nearly perpendicular to $\frac{1}{3}$ of the face BC.

EXAMPLE IX.

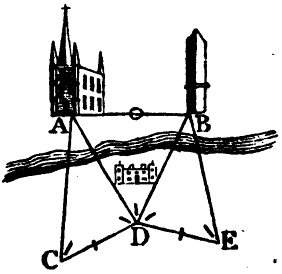
Wanting to know the distance of an object at D, from two others, A and B, and also the distance between A and B, I set up a pole at C, in a right line with AB, at C the angle ACD was 57° 0′; I then measured CD = 784·8 yards, and at D the angles CDA and ADB were 14° 0′ and 41° 30′. Required the several distances above mentioned.



Answer. AD = 696·1, DB = 712·4, and AB = 499·3 yards.

EXAMPLE X.

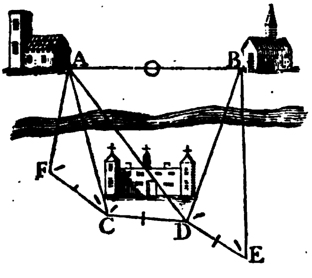
Wanting to know the distance between a church at A and an obelisk at B; both objects could only be seen from a particular place, as D; at D I took the angles ADC = 89°, ADB = 72° 30′, and BDE = 54° 30′; I then measured DE = 200 yards, and at E I took the angle BED = 88° 30′; and, lastly, I measured DC = 200 yards, and at C took the angle DCA = 50° 30′. Required the distance AB.



Answer. AD = 237·63, DB = 332·22, and AB = 345·5 yards.

EXAMPLE XI.

I wanted to know the distance between two places, A and B, but could not meet with any station from which I could see both objects. I measured a line CD = 200 yards; from C the object A was visible, and from D the object B was visible, at each of which places I set up a pole. I also measured FC = 200 yards, and DE = 200



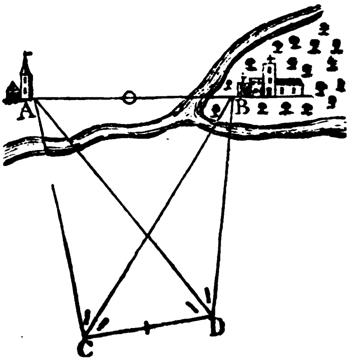
yards, and at F and E set up poles.—I then took the angle AFC =83°, ACF=54° 31′, ACD=53° 30′, BDC=156° 25′, BDE= 54° 30′, and BED=88° 30′. Required the distance AB.

Answer. DB=332·22, AD=237·63, AC=293·92; the angle ADC=83° 55′, consequently ADB=72° 30′, hence AB=345·5.

EXAMPLE XII.

Wanting to know the distance between two inaccessible objects A and B, I measured a base line CD=300 yards; at C the angle BCD was 58° 20′, and ACB 37°; at D the angle CDA was 53° 30′, and ADB 45° 15′. Required the distance AB.

Answer. First find CA= 465·98 by Example I. and also CB=761·47; and then find AB =479·8 yards by Example VI.

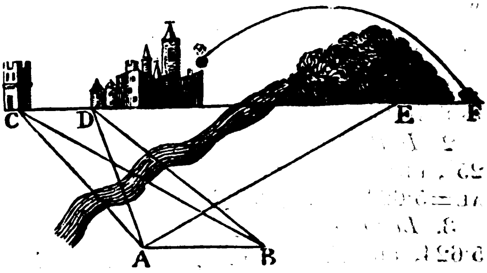


EXAMPLE XIII.

Suppose the base line in the foregoing example had been 908·36 feet, the angle ACB=14° 34′, and ACD=60° 50′; also the angle ADC=96° 44′, and BDC=115° 23′; what would have been the distance AB in this case?

Answer. 674·65 feet.

EXAMPLE XIV.



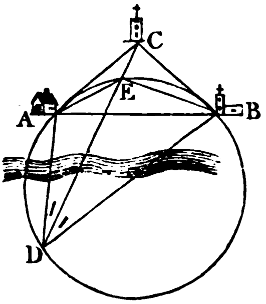
Wanting to prolong the inaccessible line CD, in order to place a mortar battery behind the obstacle E in the direction of CD, I measured a base line AB = 414·2 yards, at A I took the angles CAD=49° 13′, CAB=123° 45′, and the angle CAE= 100°, having first set up a pole at E behind the obstacle. At B, I took the angles DBC=33° 45′, and DBA=67° 30′. Hence it is required to find the angle AEF, which will determine the position of EF with respect to CD

Answer. Exactly in the same manner as in the two preceding examples, find AC = 601·3, AD = 622, and the angle ACD = 67° 30′, then ACD + CAE = AEF = 167° 30′; if, therefore, the angle AEF be made = 167° 30′, the point F will be determined in the same line with CD. The distances CD or CE are easily calculated if necessary.

EXAMPLE XV.

From a station at D I perceived three objects, A, B, C, whose distances from each other I knew to be as follows, AB = 12 miles, BC = $7\frac{1}{5}$ miles, and AC = 8 miles: at D I took the angles CDB = 25°, and ADC = 19°. Hence it is required to find my distance from the objects.

CONSTRUCTION.



Make a triangle ABC, whose three sides are 12, $7\frac{1}{5}$, and 8 (138), viz. make AB = 12, BC = $7\frac{1}{5}$, and AC = 8. At B make the angle EBA = ADE = 19°, and at A make the angle EAB = BDE = 25°; draw AE and BE, and through the point of intersection E and the points A, B, describe a circle. From C through E draw CED, join BD and AD, then will AD, CD, and BD be the distances required. From this construction it appears, that if the angles at the station D had been greater than the angles A and B of the triangle ABC, the point E would have fallen beyond C, but this would not alter the manner of construction, neither would it materially affect the method of calculation.

CALCULATION.

1. *In the triangle* ABC are given AB = 12, BC = $7\frac{1}{5}$, and AC = 8, to find the angle BAC = 35° 34′ 38″.

2. *In the triangle* AEB are given all the angles, viz. EAB = 25°, EBA = 19°, and AEB = 136°, and the side AB = 12, to find AE = 5·624, and BE = 7·301.

3. *In the triangle* CAE we have given the side AC = 8, AE = 5·624, and the angle CAE = BAC − BAE = 10° 34′ 38″, to find the angle AEC = 146° 45′ 7″; but because this is the exterior angle of the triangle ADE, if we subtract the angle ADE from it (= 19°) we shall have the angle EAD = 127° 45′ 7″, and if to this angle we add the angle CAE = 10° 34′ 38″, we shall have the angle CAD = 138° 19′ 45″.

4. *In the triangle* DAC all the angles are given, and the side AC to find AD and CD, viz. CAD=138° 19′ 45″, and ADC=ADE=19°, the sum of these taken from 180° leaves ACD=22° 40′ 15″, and AC=8. Hence AD=9·471 miles, and DC=16·337 miles.

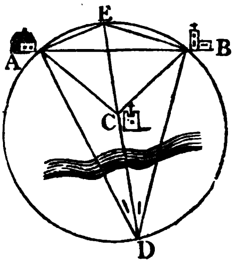
5. *In the triangle* DEB the side BE and all the angles are given. Because the opposite angles of every quadrilateral inscribed in a circle are equal to two right angles, the supplement of EAD=127° 45′ 7″ gives the angle EBD=52° 14′ 53″, and the sum of EBD and EDB subtracted from 180° leaves DEB=102° 45′ 7″, and BE has been found=7·301. Hence DB=16·85 miles.

EXAMPLE XVI.

Suppose three objects A, B, C, were seen from D, (and stood as *per* figure, Example XV.), and that their distances were, AB=12 furlongs, BC=9, and AC=6 furlongs. The angle BDC=33° 45′ and ADC=22° 30′. It is required to determine the distances AD, DC, and DB.

Answer. This example is of the same kind as the 15th, and the distances may be found in a similar manner, AD=14·01 DC=15·64, and DB=10·65.

EXAMPLE XVII.



From a station D, I perceived three objects A, B, and C (situated as in the annexed figure), whose distances were as follows, AB=12 miles, AC=8 miles, and BC=$7\frac{1}{3}$ miles. At D I took the angles CDB=25°, and ADC=19°. Hence it is required to find my distance from the objects.

CONSTRUCTION.

This is constructed exactly as Example XV. except turning the vertex of the triangle ACB towards the place of observation D.

CALCULATION.

1. *In the triangle* ABC are given AB=12, BC=$7\frac{1}{3}$, and AC=8, to find the angle BAC=35° 34′ 38″.

2. *In the triangle* AEB are given the angle EAB=25°, EBA=19°, and AEB=136°, and the side AB=12, to find AE=5·624 and BE=7·301.

3. *In the triangle* CAE there are given AE=5·624, AC=8, and the angle EAC=EAB+BAC=60° 34′ 38″, to find the angle ACE=43° 5′ 12″; if from this angle (being the exterior angle

of the triangle DCA) we subtract the angle ADC = 19°, the remainder 24° 5′ 12″ will be the angle DAC.

4. *In the triangle* ACD there are given the side AC and all the angles, viz. AC = 8, ADC = 19°, DAC = 24° 5′ 12″, and ACD = 136° 54′ 48″, to find AD = 16·786, and CD = 10·029.

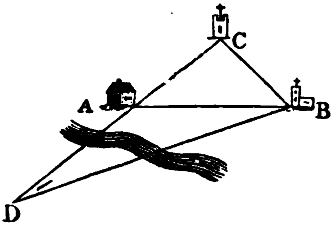
5. *In the triangle* EBD all the angles and the side EB are given, to find BD, viz. EDB = CDB = 25°, EBD = 180° − EAD = 180° − $\overline{\text{DAC} + \text{CAE}}$ = 95° 20′ 10″, and BED = 59° 39′ 50″, and EB = 7·301. Hence BD = 14·91.

EXAMPLE XVIII.

Suppose the three objects, A, B, C, were seen from D, (and stood as *per* figure to Example XVII.) and that their distances were AB = 106, AC = 65½, and BC = 53¼. The angle BDC = 13° 30′, and ADC = 29° 50′. It is required to find the distances DA, DC, and DB.

Answer. This example is of the same kind as the 17th, DA = 131·1, DC = 107·4, and DB = 151·4.

EXAMPLE XIX.

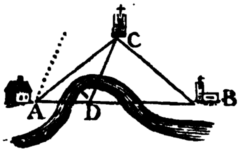


From a station at D, I perceived three objects, A, B, C, (situated as in the annexed figure,) whose distances were as follows: AB = 12 miles, AC = 8, and BC = 7⅓. At D the angle ADB was 17° 47′ 19″. Required the distances DA, DC, and DB.

Answer. With the three given sides of the triangle ABC find the angle CAB = 35° 34′ 38″, the supplement of this angle = BAD = 144° 25′ 22″; also CAB − ADB = 17° 47′ 19″ = ABD. Hence all the angles of the triangle ADB are given, and one side AB, to find the remaining sides.

Viz. AD = 12, and BD = 22·85; hence DC = 20.

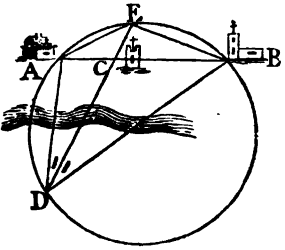
EXAMPLE XX.



Three objects, A, B, and C, whose distances were AC = 8, BC = 7⅓, and AB = 12 miles, were visible from one station at D, from whence the angle ADC was observed to be 107° 56′ 13″. Required the distances DA, DC, and DB.

Answer. With the three sides of the triangle ABC find the angle BAC = 35° 34′ 38″. Then in the triangle ADC all the angles will be given, and the side AC, to find AD = 5, and DC = 4·892; hence DB = 7 miles.

EXAMPLE XXI.



Three objects, A, B, and C, in a straight line, whose distances were AC = 3·626, AB = 12, and BC = 8·374 miles, were visible from one station at D, from whence the angle ADC was 19°, and BDC = 25°. Required the distances DA, DC, and DB.

CONSTRUCTION.

The construction to this example is almost exactly the same as the construction to Example XV.

CALCULATION.

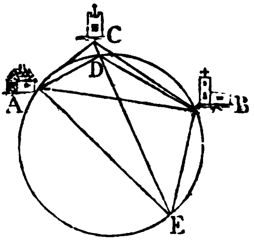
1. *In the triangle* EAB are given the angle EAB = 25°, EBA = 19°, and consequently AEB = 136°, and the side AB = 12, to find AE = 5·624, and BE = 7·301.

2. *In the triangle* AEC there are given AC = 3·626, AE = 5·624, and the contained angle EAC = 25°, to find the angles about C, viz. ACE = DCB = 121° 45′ 17″, the supplement of which is ACD = ECB = 58° 14′ 43″.

3. *In the triangle* ACD, all the angles and the side AC are given, to find AD = 9·4704 miles, and DC = 10·863.

4. *In the triangle* ADB or DCB all the angles and two sides are given, to find DB = 16·848 miles.

EXAMPLE XXII.



Three objects, A, B, and C, forming a triangle, were visible from one station at D within the triangle; the angle ADB was observed to be 123° 45′, CDB = 132° 22′, and consequently ADC = 103° 53′; the distance AB was known to be 12 miles, BC = 9, and AC = 6 miles; required the distance of each object from the station at D.

CONSTRUCTION.

Make a triangle whose three sides are AB = 12, BC = 9, and AC = 6. At B make the angle ABE = the supplement of ADC = 76° 7′ = ADE, and at A make the angle BAE = the supplement of BDC=47° 38′=BDE, through the intersection E of AE and BE, and the points A, and B, describe a circle. Then DA, DC, DB, are the distances required.

CALCULATION.

1. *In the triangle* AEB all the angles and one side are given, to find AE. Thus the angle ABE=76° 7′, and BAE=47° 38′ by construction: hence AEB=56° 15′, and AB=12: hence AE =14·01.

2. *In the triangle* ABC all the sides are given: hence may be found the angle CAB=46° 34′ 4″, this added to the angle BAE gives CAE=94° 12′ 4″.

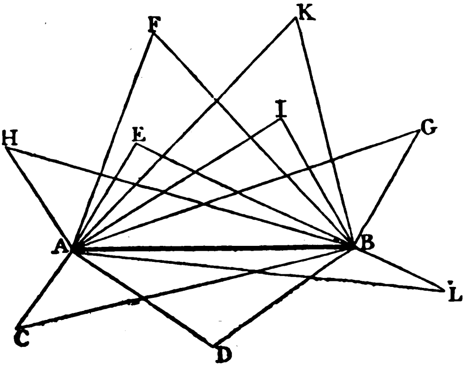
3. *In the triangle* CAE are given AC=6, AE=14·01, and the angle CAE=94° 12′ 4″, to find the angle ACE=63° 18′ 12″.

4. *In the triangle* DAC all the angles are given, viz. ACD= 63° 18′ 12″, ADC=103° 53′, and DAC=12° 48′ 48″, and the side AC=6, to find AD=5·522 and DC=1·371.

5. *In the triangle* BDC, two sides BC and CD, and an angle BDC opposite to one of them are given, to find the side BD= 8·02.

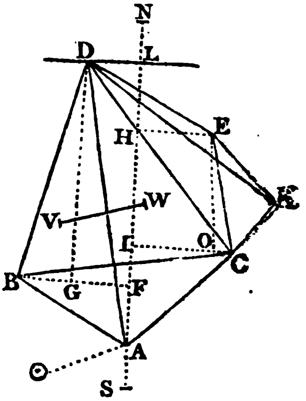
SCHOLIUM.

(150) Various other problems concerning distances might be formed, from the different situations which objects may be supposed to have, but they are only applications of the preceding examples. —For instance, the distances of the most remarkable places in a town, the distances of several villages from each other, the plan of a camp, &c., may be taken from what has already been explained.



Let A, B, C, D, E, F, G, H, I, K, L, be several remarkable objects, whose situations are to be laid down in a map. Choose a convenient situation AB for a base line, from which you can see all the objects; let this base line be as long as possible, in proportion to the most distant object. From the extremity A measure the angles GAB, IAB, KAB, EAB, FAB, HAB, &c. from B, measure the angles GBA, KBA, IBA, FBA, EBA, HBA, &c. Now it is evident from what has been already shown, that having the base AB given, and the several angles, it will be easy to find the sides, or distances, AG, AI, AK, &c. By Example I.

(151) A country may be surveyed, and the distances of remote objects may be obtained by means of a series of triangles. Thus, measure a base AB, and let C and D be two objects which can be seen from A and B. Measure the horizontal angles BAC, BAD, ABC, ABD, and calculate the distance DC, as in Example XII., and likewise the distances DB, DA, BC, AC. In like manner, if E and K be two objects visible from C and D, the distance EK may be found, and its position with respect to CD; and thus the measurement may be continued to any distance. But, in order that the conclusion may be more accurate, the mensuration from one base to another may be carried on by different sets of triangles; for instance, two objects V and W might have been chosen instead of A and B, by means of which the distance EK might have been ascertained; and, it is plain, that by taking several series of triangles which lead to the same two objects, the mean of the results will be more accurate than the measurement obtained from one triangle. When a series of triangles has been carried on for a considerable distance, the interval of two objects, whose distance has been determined by calculation, should be actually measured, in order to detect any error which may have been made in the calculated distance. This is called a *Base of verification.* The trigonometrical survey of England and Wales had its first commencement in 1784; and from a base of 27404·2 feet measured on *Hounslow Heath,* and continued by a series of triangles to *Salisbury Plain,* the medium distance between *Beacon Hill* and *Old Sarum* was 36574·3 feet, and from actual admeasurement the distance was 36574·4 feet, a difference of little more than *one* inch.*



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* Trigonometrical Survey of England and Wales, Vol. I. pages 279 and 280, published by Mr. Faden, Charing Cross, London.

(152) By the assistance of a survey, executed with such accuracy, the length of an arc of a meridian may be measured. Let NS be the meridian of any place A, and let ⊙ represent the point of the horizon where the sun sets; measure the angle BA⊙; then, by having the latitude of the place A given, the amplitude NA⊙ is easily ascertained[*], and consequently their difference BAN is given; also if from BAC there be taken BAN, the remainder CAN is given. Let BF, CI, be drawn perpendicular to the meridian, then because AB and the angle BAN are given, AF and FB are easily found: in the same manner, because AC and the angle CAN are given, the lines AI and IC may be found.

If from the angle ABC there be taken the angle ABF, the remainder FBC, added to DBC, gives DBF; by means of which, and the given side DB, DG or FL will be found, and likewise BG. By adding AF to FL we get AL, and by deducting BG from BF we shall have GF = DL.

In a similar manner the several distances from the point A to the perpendiculars, let fall from each of the angular points of the figure upon the meridian NS, may be determined, as also the perpendiculars themselves. By this method the extensive survey of the kingdom of France was carried on from the measure of nineteen bases.[†]

The measures being all reduced to the meridian NS, the difference of latitude between A and D may be determined. Find, from observation with a zenith sector[‡], the true zenith distance of a star at each place, the difference of these zenith distances will be the difference of latitude between A and D. Perhaps it may be proper to remark that the perpendicular DL does not accurately coincide with a circle[§], or parallel, of latitude, and where the utmost exactness is required the difference must be calculated and applied to AL; the true difference of latitude between A and L will then be obtained. This difference of latitude will be to 1° as the length of the meridian AL is to the length of a degree. The difference of latitude between A and each of the other places, B, C, K, &c. may be found by a similar process.

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[*] Or if the sun be above the horizon, its altitude may be taken in the direction A ⊙, and thence its azimuth ⊙ AS may be computed.

[†] Traité de Trigonométrie, par M. Cagnoli, Chap. XI. Art. 341.

[‡] Trigonometrical Survey of England and Wales, Vol. II. Part II. published by Mr. Faden.

[§] The difference is not material, except the distance DL be very great. See Chapter IV. art. (154).

(158) In small surveys, military sketches, &c. where great accuracy is not required, angles may be taken with a good pocket compass, having two sights, and being fitted to the top of a staff, sharp at one end so that it may be stuck in the ground, and allow the needle to move freely about.

Every circle is divided into 360 degrees; the compass is divided into 32 points, therefore 1 point is 11° 15′; ¼ point = 2° 48′ 45″; ½ point = 5° 37′ 30″; and ¾ point = 8° 26′ 15″. These points are reckoned from the north and south, towards the east and west, as in the following table.

North.		South.		Points.	Degrees.
N.b.E.	N.b.W.	S.b.E.	S.b.W.	1	11° 15′
N.N.E.	N.N.W.	S.S.E.	S.S.W.	2	22 30
N.E.b.N.	N.W.b.N.	S.E.b.S.	S.W.b.S.	3	33 45
N.E.	N.W.	S.E.	S.W.	4	45 0
N.E.b.E.	N.W.b.W.	S.E.b.E.	S.W.b.W.	5	56 15
E.N.E.	W.N.W.	E.S.E.	W.S.W.	6	67 30
E.b.N.	W.b.N.	E.b.S.	W.b.S.	7	78 45
E.	W.	E.	W.	8	90 0

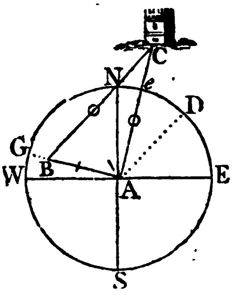
The variation of the magnetic needle ought to be allowed for, in measuring angles with a compass.

At London the north point of the needle rests in a direction of about 24° 30′ westward of the true north.

EXAMPLE XXIII.

Having occasion to travel through the counties of Kent and Surrey, I perceived the fort built by Lady James on Shooter's Hill, which bore from me N.N.E., and travelling 20 miles in a W.N.W. direction, I perceived the fort again, which bore N.E.b.E. Required my distance from it at each station.

CONSTRUCTION.



With the chord of 60° describe a circle, draw the meridian NS, and EW at right angles to it.

Set off the N.N.E. point of the compass, or 22° 30′, from N to *e*, draw AC; set off the W.N.W. point of the compass, viz. 67° 30′ from N to G, and draw AG, on which set off AB = 20 miles from a scale of equal parts. Lastly, set off the N.E.b.E. point = 56° 15′ from N to D, draw AD, and parallel thereto, through B, draw BC; then AC and BC are the distances required, and C the fort.

CALCULATION.

Because AD is parallel to BC, the angle CAD = BCA = the difference between ND and N*e* = 33° 45′. The angle BAC = the sum of NG and N*e* = 90°, hence the angle CBA = 56° 15′. By case second, right-angled triangles, AC = 29·93 and BC = 86 miles.

EXAMPLE XXIV.

I observed a church which bore from me N.E.b.N. and going $2\frac{1}{2}$ miles in a N.N.W. direction, the church then bore E.S.E. Required my distance from it at each observation.

Answer. At the first observation the distance was 1·8 mile, and at the second 2·1 miles.

EXAMPLE XXV.

From a ship at sea I observed a point of land to bear E.b.S. and after sailing N.E. 12 miles, it bore S.E.b.E. Required the distance of the last place of observation from the point of land.

Answer. 25·77 miles.

EXAMPLE XXVI.

From a station at H, an object G, the distance of which I knew to be 4 miles, bore N.E.b.N.$\frac{1}{2}$E.; and another C, distant $2\frac{1}{4}$ miles, bore S.E.b.S.$\frac{1}{2}$E. Required the distance between C and G.

Answer. CG = 4·957 miles.

EXAMPLE XXVI

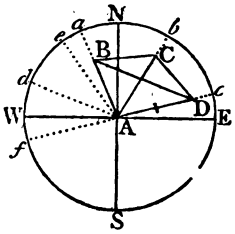
Saint Paul's church at London bore from me N.b.E. and travelling W.N.W. 15 miles, it bore N.E.$\frac{1}{2}$E. Required my distance from St. Paul's at the last place of observation.

Answer. 23·19 miles.

EXAMPLE XXVIII

Coasting along the sea-shore I observed two headlands, the first bore N.N.W. the second N.N.E.$\frac{3}{4}$E. Then steering E.N.E.$\frac{1}{2}$E. 16 miles, the first headland bore W.N.W. the second N.W.b.N.$\frac{1}{4}$W. Required the bearing and the distance of the two headlands from each other.

CONSTRUCTION.



With the chord of 60° describe a circle; draw the meridian NS and EW at right angles to it.

1. Set off the N.N.W. point of the compass, or 22° 30′ from N to *a*, and draw A*a*.

2. Set off the N.N.E.$\frac{3}{4}$E. point, or 30° 56′ 15″ from N to *b*, and draw A*b*.

3. Set off the E.N.E.$\frac{1}{2}$E. point, or 73° 7′ 30″, from N to *c*, draw A*c*, on which set off AD = 16 miles from a scale of equal parts.

4. Set off the W.N.W. point from N to *d* = 67° 30′, draw A*d*, and parallel thereto, through D, draw DB, then B is the situation of the first headland.

5. Set off the N.W. b. N.$\frac{1}{4}$W. point of the compass, or 36° 33′ 45″ from N to *e*, draw A*e* and DC parallel to it, then C is the situation of the second headland. Join BC, which will be the distance required.

CALCULATION.

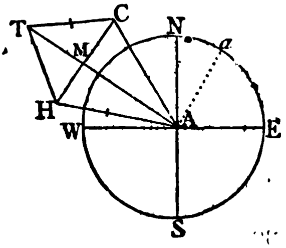
1. *In the triangle* BAD, the angle BAD = arc *ac* = 95° 37′ 30″, the angle ABD = angle *d*A*a* = arc *da* = 67° 30′ − 22° 30′ = 45°, for ABD and *d*A*a* are alternate angles. Hence the angle BDA = 39° 22′ 30″, and the side AD = 16. The distance AB may be found = 14·35.

2. *In the triangle* ADC the angle CAD = arc *bc* = 73° 7′ 30″ − 30° 56′ 15″ = 42° 11′ 15″. Because A*e* is parallel to DC, and AD is produced to *f*, the angle *e*A*f*, measured by the arc *ef*, or the supplement of *ec* = ADC = 70° 18′ 45″: hence the angle ACD = 67° 30′; hence AC may be found = 16·31 miles.

3. *In the triangle* BAC are given AB = 14·35, AC = 16·31, and the angle BAC, measured by the arc *ab* = 53° 26′ 15″; hence the angle ABC will be found = 70° 31′, and the distance BC = 13·9 miles.

4. If from the supplement 109° 29′ of ABC, the angle BAN be taken, the remainder 86° 59′ will be the bearing of the second headland C from the first B, viz. N.86° 59′E., or of the first headland B from the second C, viz. S.86° 59′W.

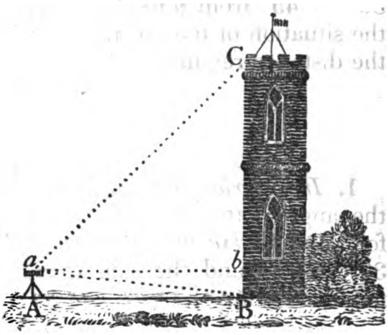
EXAMPLE XXIX.



The distance between a tower T, and a church C, was known to be 20 miles; from a ship at anchor A, I saw a windmill at M in a right line with the tower, the bearing of both was N.W.b.W. the church at the same time bore N.N.W.½W. After which we weighed anchor, and sailed upon a W.b.N. course AH = 25 miles, and the mill and the church were in a straight line HC, and bore N.E.b.N. Required the distances from the church, tower, and windmill, at each station, or place of observation A and H.

Answer.
AM = 23·10 | HM = 9·57
AC = 26·19 | HC = 21·91
AT = 38·83 | HT = 18·41

EXAMPLE XXX.



From the bottom B of a tower, I measured 200 feet in a direct line AB on an horizontal plane. I then took the angle C *a b* = 46° 30′[*], the height A*a* of my instrument was 5 feet. Required the height BC of the tower.

Answer. Here are given the base *ab* and the angle *cab* of a right-angled triangle, to find the perpendicular *bc* = 210·76, to which add the height of the instrument, then BC = 215·76 feet. For it is evident that Aa=Bb, since ab is parallel to AB and equal thereto.

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[*] In *Crakelt*'s translation of *Mauduit*'s Trigonometry, page 182, it is shown, *that the error of an altitude*, BC, *is to the error committed in taking the angle* cab, *as double the height* BC *is to the sine of double the observed angle* cab. The error, therefore, which can prevail in the determination of the said height, will be the least possible, when the *sine* of double the observed angle is the greatest possible; that is, when this angle is 45°. Hence, says he, when we want to find the height of any object, we must so contrive it that the observed angle be the nearest possible to 45°. He then shows that in observing an object at an angle of 45°, if an error of one minute be made in the determination of the angle, the error in the altitude will be $\frac{1}{1719}$ part of the altitude: also if the angle observed be greater or less than 45°, the error in the height will be increased in the ratio of radius to the sine of double the said angle.

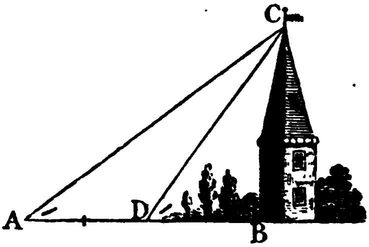
EXAMPLE XXXI.

Wanting to know the height of a church BC, I measured 45 yards from the bottom B upon a horizontal plane AB, at A I took the angle cab=48° 12′, the height of my instrument was 5 feet. Required the height of the church.

Answer. The height bc=50·33 yards, to which add Bb the height of the instrument, then BC=52 yards the height of the church.

EXAMPLE XXXII.

Wanting to know the height of a steeple BC, inaccessible on account of several small trees on one side, and uneven ground on the other sides, I took at D an angle CDB = 51° 30′, and measuring the distance AD =75 feet, I took at A the angle CAB=26° 30′. Required the height of the steeple, and the distance of the first station from its base.



SOLUTION.

Subtract the angle CAB from the angle CDB, the remainder 25°=the angle ACD. Then in the oblique triangle ADC all the angles are given to find DC=79·18; and in the right-angled triangle DBC, the hypothenuse DC, and angle CDB are given, to find BC=61·97, and DB=49·29.

EXAMPLE XXXIII.

Wanting to know the height of an inaccessible object BC, I took at A the angle CAB=28° 34′, and measuring in a straight line towards the object from A to D 30 yards, the angle CDB was found to be 50° 9′. Required the height of the object, and my distance from it at the second station.

Answer. The height BC=29·94 yards, and distance DB= 24·99.

EXAMPLE XXXIV.

Two observers at the distance of half a mile from each other, on an horizontal plane, took the altitude or angle of elevation of a cloud at the same moment of time, the same point of the cloud being observed by both, and in the same direction : the angles were 35° and 64°. Required the perpendicular height of the cloud, and its distance from each observer.

Answer. The perpendicular height was 935·75, the distance from the observers 1041·1, and 1631·4 yards.

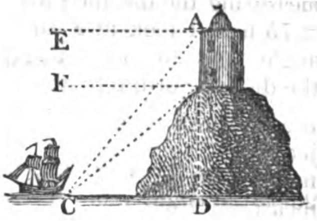
EXAMPLE XXXV

In the year 1784 two observers on Blackheath, at the exact distance of a mile from each other, observed, at the same instant, the angle of elevation of Lunardi's balloon; the angle at the nearest station to it was 36° 52′, and the other angle 30° 58′. Required the perpendicular height of the balloon, and its distance from each station.

Answer. The balloon's distance from the nearest station was 5·006 miles, from the other station 5·837 miles, and the perpendicular altitude 3·003 miles.

EXAMPLE XXXVI

From the top (A) of Flamborough-head light-house, the angle of depression (EAC) of a ship at anchor was 3° 38′; and at the bottom (B) of the light-house, the angle of depression (FBC) was 2° 43′; required the horizontal distance (DC) of the vessel, and the height (DB) of the promontory above the level of the sea; the light-house (AB) being 85 feet high.

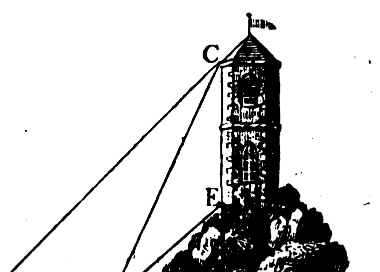


The ∠ ACD = ∠ EAC = 3° 38′, the ∠ BCD = ∠ FBC = 2° 43′; and AB = 85 feet.

Hence CD will be found = 5296·5 feet, and DB = 251·32 feet. *Answer.*

EXAMPLE XXXVII.

Wanting to know the height of a tower EC, which stood upon a hill, I took at A the angle of elevation CAB = 44°; I then measured AD = 134 yards in a straight line towards the tower, and found the angle CDB to be 67° 50′, and CDE = 16° 50′. Required



the heights of the tower and hill, the height of the instrument being five feet.

SOLUTION.

1. Subtract the angle CAB from CDB, the remainder 23° 50′ = ACD, hence all the angles in the triangle CAD are given, and one side AD, to find DC = 230·36 yards.

2. *In the triangle* CDE all the angles are given, viz. CDE = 16° 50′, DCE = 90° − 67° 50′ = 22° 10′, hence the angle CED = 141°, and CE = 106 yards, the height of the tower.

3. *In the right-angled triangle* CDB the angle CDB and CD are given, to find BC = 213·34 yards; hence BE = BC − CE = 107·34 yards, to which add the height of the instrument, and you have 109·01 yards for the height of the hill.

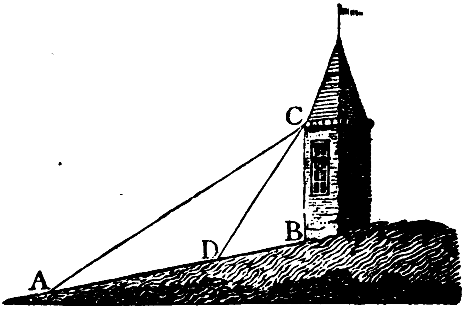
EXAMPLE XXXVIII.

Wanting to know the height of a castle standing upon an eminence, I took an angle of elevation CAD = 40°, and measuring 40 yards in a straight line towards the castle, the angle CDB was 63° 20′, and CDE 14° 30′: what was the height of the castle EC, and the distance DB?

Answer. EC = 24·69 yards, and DB = 29·13 yards.

EXAMPLE XXXIX.

Wanting to know the height of an object standing on an inclined plane AB, I measured from the bottom of the object a distance BD = 40 feet; at D, I took the angle CDB = 41°; going on in the same direction 60 feet farther, the angle CAB was 23° 45′: Required the height of the object.



SOLUTION.

1. *In the triangle* ADC are given the angle CAB = 23° 45′, ACD = CDB − CAB = ACD = 17° 15′, and AD = 60 feet, to find CD = 81·488.

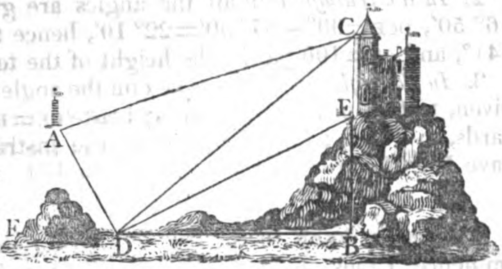
2. *In the triangle* CDB are given CD = 81·488, DB = 40, and the angle CDB = 41°, to find BC = 57·62.

EXAMPLE XL.

An object standing on a declivity whose height I wished to determine, I measured 150 yards from the base of it, and then took an angle CDB=47° 50′, and in the same direction I measured 80 yards farther, and then took the angle CAB=38° 30′. Required the height of the object.

Answer. DC=307·1, and BC the height=234·4 yards.

EXAMPLE XLI.



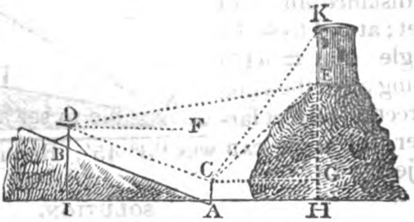
Wanting to know the height of a castle CE standing upon a hill, at D the angle CDB was 58°, CDE =25°, and ADC =72° 10′. The ground not permitting me to retreat in a straight line towards F, nor advance towards B, I therefore measured from D to A 52 yards; at A the angle CAD was 64° 30′. Required the height of the castle EC, and the height of the hill BE, above the level of the first station D.

SOLUTION.

1. *In the triangle* ADC are given the angles ADC=72° 10′, CAD=64° 30′, and consequently ACD=43° 20′, and the side AD=52 yards, to find DC=68·393 yards.

2. The remaining part of the solution is the same as in Example XXXVII.; CE will be found=34·464 and BE=23·536 yards.

EXAMPLE XLII.



Wanting to know the height of a tower EK standing upon a hill, the height HE of the hill, and the horizontal distance AH from the station A.—At A the angle GCK was 3° 38′, and the angle GCE was 2° 43′; then having set up a staff AC, whose height was equal to that of the theodolite, I measured 1809·5 feet up the sloping ground AB, as a base, in a direct line with the tower, keeping the points K, E, C, B in the same vertical plane. At B I took the following angles, viz. FDC=BAI=1° 54′, and EDF=1° 33′. Required the horizontal distance AH, the height HE of the hill, the height EK of the tower, and IB the elevation of the station B above that at A.

SOLUTION.

1. *In the triangle* EDC, the ∠EDC = 3° 27′, ECD = 175° 23′, DEC = 1° 10′, and the side DC = 1809·5 feet, hence CE = 5348·1 feet.

2. *In the triangle* GCE, the ∠GCE = 2° 43′, and CE = 5348·1 feet, hence CG = AH = 5342·1 feet, and GE = 253·5 feet.

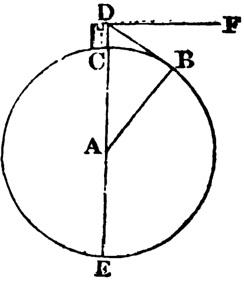
3. *In the triangle* KCE, the ∠KCE = 0° 55′, CKE = 86° 22′, and CE = 5348·1 feet, hence EK = 85·7 feet.

4. *In the triangle* BAI, AB = 1809·5 feet, and the angle BAI = 1° 54′, hence BI = 60 feet.

Lastly, if to the height GE, the height AC of the instrument be added, it will give HE, the height of the hill.

EXAMPLE XLIII.

At the distance of 25 miles from a tower its top just appeared in the horizon; required its height. The diameter of the earth being 7964 miles, and its circumference 25019·7024 miles.



SOLUTION.

25019·7024 miles : 360° : : 25 miles : 21′ 35″ = ∠DAB. In the right-angled triangle ABD, AB and the ∠DAB are given, to find AD = 3982·078446 miles, from which take AC = AB, the remainder DC = ·078446 miles = 414·19 feet, the height of the tower.

EXAMPLE XLIV.

Supposing it were possible to see a light-house or other object D, in the horizon, at the distance of 200 miles, it is required to find its height CD, the diameter EC of the earth being 7964 miles, and its circumference 25019·7024 miles.

Answer. The ∠DAB = 2° 52′ 40″, AD = 3987·028 miles, and DC = 5·028 miles = 26547·84 feet, being 5914·84 feet higher than *Chimboraço*, the highest of the *Andes*.

EXAMPLE XLV.

The Peak of Teneriffe is said to be 2½ miles above the level of the sea; at what distance can it be seen, supposing the radius of the earth to be 3982 miles?

Answer. $DB = \sqrt{ED \times DC}$ (36 Euclid III.) = 141·12 miles.

EXAMPLE XLVI.

If the Peak of Teneriffe be $2\frac{1}{2}$ miles high, and the angle CDB made by a plumb-line, and a line DB meeting the surface of the sea in the farthest visible point B, be equal to 87° 58′ 13″, it is required to find the diameter CE of the earth, supposing it to be a perfect sphere, and the utmost distance DB that can be seen from the top D of the mountain.

Answer. $\text{DB} = \frac{(\text{rad.} + \sin \text{CDB}) \times \text{DC}}{\cos \text{CDB}} = 141\cdot12$ miles, and CE $=7964$ miles.

CHAP. IV.

OBSERVATIONS ON THE ADMEASUREMENT OF A BASE LINE.

(154) Where the ground is perfectly level, the manner of measuring a straight line from one object to another appears to be simple and easy; yet, on account of the curvature of the earth, no two points on its surface can be exactly situated in the same horizontal line; the chord of the arc, and not the arc itself, being the horizontal distance. Now the radius of one circle is to the radius of any other circle, as any arc of the former is to a similar arc of the latter. If we take, for instance, the base line measured on *Hounslow-heath* (151) 27404·2 feet, the radius of the earth 3982 miles, or 21024960 feet, we shall have 21024960 feet : 27404·2 feet : : 1 : $\frac{137021}{105124800}$, the length of the measured arc in terms of radius 1. But the difference between any arc and its chord, the radius being 1, is $\frac{1}{24}$ of the cube of the length of the arc*; hence $\left(\frac{137021}{105124800}\right)^3 \times \frac{1}{24} = \cdot000000000092265$ will express the difference in terms of radius 1, which multiplied by 21024960 feet, the radius of the earth, produces ·00194, the extent in feet by which a terrestrial arc of 27404·2 feet exceeds the chord of the same arc, a difference scarcely worth notice, even where the greatest accuracy is required.

(155) When the ground on which a base line is to be measured is sloping, it will be necessary, in some cases, to reduce it to a horizontal level. Thus, after having determined the direction of the base AF†, by poles LA, I*m*, H*n*, G*o*, pointed at one end, and fixed perpendicularly in the ground by means of a plumb-

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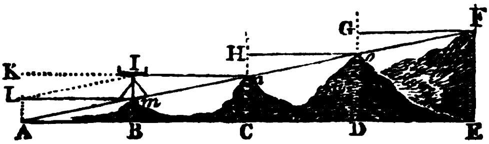
* See Chapter V. Note to art. (270).

† The point A, and the summits of the hills *m*, *n*, *o*, F, should be connected, so as to form a regular slope AF.



line; the sum of the horizontal distances L*m*, I*n*, H*o*, GF will evidently be equal to the whole horizontal distance AE; and if the heights AL, *m*I, *n*H, G*o* be successively measured, their sum will give the whole height EF.

(156) If the ground be irregular, or if it ascend and descend alternately, it is evident that the difference between the heights of the poles must be added when ascending, and subtracted when descending, in order to determine the different elevations and depressions of the ground.

(157) Surveyors generally ascertain the altitudes of irregular hills by the assistance of a spirit-level, and perpendicular poles placed at convenient distances from each other. This practice is called levelling.

(158) A base line, on a sloping ground, may likewise be measured by taking angles at its extremities with a theodolite. Thus, let I*m* represent a theodolite, AL a pole fixed perpendicular to the horizon and equal in length to the height of the instrument; also, let KI be a horizontal line (which may be ascertained by the bubble of air in the spirit-level of the telescope resting in the middle) and KIL the angle of depression between the top of the pole AL and the horizontal line KI.

Then, because KI is parallel to AB, the angle KIL is equal to the angle *m*AB; then (120), rad : A*m* : : cos ∠ KIL : AB, or $\text{AB} = \dfrac{\text{A}m.\ \cos\angle\ \text{KIL}}{\text{rad}}$.

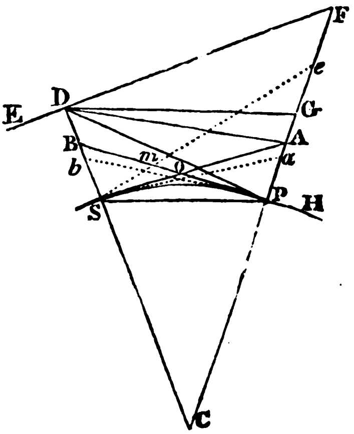
(159) If $\text{A}m = 400$ yards, and $\angle\ \text{KIL} = 4°$, AB will be 399·026 yards, hence the difference between A*m* and AB is less than 1 yard. It appears from this example, that when the measured base is inclined to the horizon in a small angle, a reduction of this kind will be unnecessary, except in cases where great accuracy is required.

OF THE ERRORS WHICH OCCUR IN TAKING ANGLES OF ELEVATION AND DEPRESSION WITH A THEODOLITE.

(160) When the observer is at a considerable distance from the object, the altitude taken with a theodolite will require correction. In the first place, the horizon of the observer and that of the object observed are not the same. Let C be the centre of the earth, D the summit of a mountain, and HPOB the horizon of the observer. Through D draw EDF perpendicular to DC, and it will be the horizon of the point D. Now SD will

be the true height of the mountain above the horizontal line PS; DPS the true angle of elevation, and EPB the observed angle, from which the height of the mountain is determined to be BD, instead of SD.

Again, the ∠ PBD is supposed to be a right angle, in ordinary calculations, but in reality it is equal to the sum of the two interior angles BPC and PCB (32 Euclid I.) $=90°+\angle$ C.



(161) The French academicians measured an inclined base at Peru, the length PD of which was found to be 6274·057 toises *, the angle of elevation DPB was 1° 5′ 43″ (the effect of refraction being deducted). Now, rad : PD : : sin DPD : BD, hence $\text{BD}=\frac{\text{PD. sin DPB}}{\text{rad}}=119{\cdot}93$ toises. In order to calculate BS, PD may be used instead of PB, and CS and CB may be considered as equal to each other without sensible error. And as $(2\text{CS}+\text{BS})\times\text{BS}=\text{PB}^2$ (36 Euclid III.), it follows that $2\text{CS}\,.\,\text{BS}=\text{PD}^2$, and hence $\text{BS}=\frac{\text{PD}^2}{2\text{CS}}$.

(162) If the diameter of the earth be taken $=6543373$ toises, as deduced from the admeasurements in Lapland, Paris, and Peru, BS will be found $=\frac{\overline{6274{\cdot}057}|^2}{6543373}=6{\cdot}0158$ toises. By adding BS to BD the true height SD of the mountain $=125{\cdot}9458$ toises. Had the height DS been determined from the triangle DPS, by the most exact calculation, it would have been $=125{\cdot}97$ toises. Hence it appears that in most cases the angle at C may be rejected, that PB may be taken $=$ PD, the ∠ PBD $=$ a right angle, and CS $=$ CB without material error.

(163) Angles of elevation or depression taken with a theodolite may be corrected thus: Let D be the place of the telescope when the theodolite stands on the vertical line CD; P the situation of the telescope on the vertical line CP. Then if the tele-

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* Bouguer, Figure de la Terre. A toise = 6 French feet, and 107 French feet = 114 English feet.

scope at P be directed to an object at D, the elevation of that object above the horizontal line HPB is the $\angle$ DPB; and when the telescope is at D, and directed to an object at P, the angle of depression, below the horizontal line EDF, is the $\angle$ FDP.

Now, because PB touches the circle, and PS cuts it, the $\angle$ BPS (measured by half the arc PS*) $= \frac{1}{2} \angle$ C; therefore $\angle$ DPS $= \frac{1}{2} \angle$ C $+ \angle$ of elevation DPB.

Through D draw DG parallel to SP, then the $\angle$ GDP $= \angle$ DPS (29 Euclid I), hence $\angle$ GDP $= \frac{1}{2} \angle$ C $+ \angle$ of elevation DPB; and because the triangles CPS and CDG are isosceles, and the $\angle$ C is common to both of them, the $\angle$ CPS $= \angle$ CDG; but the angles CPB and CDF are right angles, therefore the $\angle$ BPS $(= \frac{1}{2} \angle$ C$)$ $= \angle$ FDG.

Again, $\angle$ FDP $= \angle$ GDP $+ \angle$ FDG $= \frac{1}{2} \angle$ C $+ \angle$ of elevation DPB $+ \frac{1}{2} \angle$ C; therefore $\angle$ FDP $= \angle$ C $+ \angle$ of elevation DPB; to each of these equals add the $\angle$ DPB, then $\angle$ of depression FDP $+$ $\angle$ of elevation DPB $= \angle$ C $+$ twice $\angle$ of elevation DPB; hence

$$\frac{(\angle \text{ of depr. FDB} + \angle \text{ of elev. DPB}) - \angle \text{ C}}{2} = \text{the true } \angle \text{ of elevation.}$$

When the angles are both elevations or both depressions, their difference must be diminished by the $\angle$ C, and half the remainder will be the true $\angle$ of elevation of the higher of the two objects.

The $\angle$ C is generally very small, and where the measured base does not exceed six or seven hundred yards, it may be rejected.

EXAMPLE XLVII.

(164) Suppose D and P to be two objects fixed exactly at the same height above the ground as the height of the telescope of the theodolite; now if the $\angle$ FDP of depression be 26′, and the $\angle$ DPB of elevation 14′, what will be the error in observation? The arc PS, or distance of the stations, being 8000 feet.

The length of a degree in latitude 51° 9′ is 364950 feet†; 364950 feet : 60′ : : 8000 feet : 1′ 19″ nearly $= \angle$ C. Then

$$\frac{(26' + 14') - 1'\ 19''}{2} = 19'\ 20''\cdot5 \text{ the true } \angle \text{ of elevation DPB;}$$

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* *The angle formed between the tangent of any arc and its chord, is measured by half that arc.*

The $\angle$ TBF (Plate I. fig. 1.) is measured by half the arc BF. For the $\angle$ TBF is equal to the $\angle$ F*b*B in the alternate segment (33 Euclid III.), and the $\angle$ F*b*B $= \frac{1}{2} \angle$ FCB (20 Euclid III.), therefore the $\angle$ TBF $= \frac{1}{2} \angle$ FCB; but the $\angle$ FCB is measured by the arc BF, therefore the $\angle$ TBF is measured by half the arc BF.

† Trigonometrical Survey of England and Wales, Vol. II. Part II. page 113.

hence 19′ 20″·5 − 14′ = 5′ 20″·5 is the error of the instrument, or the quantity by which the ∠ of elevation was too small, or the ∠ of depression too large.

THE NATURE OF TERRESTRIAL REFRACTION, AND ITS EFFECTS ON ANGLES OF ELEVATION.*

(165) As terrestrial refraction arises from the gross vapours, and exhalations of various kinds, which are suspended in the air near the surface of the earth, and which are perpetually changing, it is very difficult to ascertain the exact quantity of it at any particular time.

(166) The course of a ray of light in its passage through the atmosphere is, in general, that of a curve which is concave towards the earth, and the observer views the object in the direction of a tangent to this curve; hence the apparent, or observed angle of elevation is always greater than the true angle.

(167) The altitudes of the heavenly bodies when within 5° or 6° of the horizon, should never be used where a very accurate result is required. The figures of the sun and moon, when near the horizon, are sometimes elliptical, having the minor axis perpendicular, and the major axis parallel to the horizon. This change of figure arises from the refraction of the under limb being greater than that of the upper. But a perpendicular object, situated on the surface of the earth, will not have its length altered by refraction, the refraction of the bottom being the same as that of the top.

(168) The allowances usually made for refraction are too uncertain for any reliance to be placed on them, as scarcely two writers agree on this subject. Dr. Maskelyne makes it $\frac{1}{10}$ of the intermediate arc PS between the observer and the object; Bouguer $\frac{1}{9}$; Legendre $\frac{1}{14}$; General Roy from $\frac{1}{8}$ to $\frac{1}{24}$; and in the second volume of the Trigonometrical Survey, the variation is found to be from $\frac{1}{5}$ to $\frac{1}{32}$ of the intermediate arc.† This difference does not arise from inaccuracy of observation, but from circumstances which cannot be avoided, as the evaporation of rains, dews, &c. which produce variable and partial refractions.

(169) The following method is used in the Trigonometrical Survey‡ for ascertaining the quantity of refraction:

Let C be the centre of the earth, P and S two stations on surface; PB, SA the horizontal lines at right angles to CA and CB; also, suppose A and B to be the true places of the objects observed, and a and b their apparent places. Then the $\angle b\text{PS}$ will be the refraction at P, and the $\angle a\text{SP}$ that at S.*

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* See a paper by Mr. Huddart, in the Philosophical Transactions f[illegible]97, page 29. ; and another by the Rev. S. Vince, 1799, page 13. Also the [illegible]gonometrical Survey of England and Wales, Vol. I. page 175.

† Pages 177 and 178, Part I.

‡ Volume I. page 175.

In the quadrilateral figure CSOP, the angles at P and S are right angles, therefore the $\angle \text{SOP} + \angle \text{C} =$ two right angles, but the three angles of the triangle SOP = two right angles, hence $\angle \text{OSP} + \angle \text{SOP} + \angle \text{OPS} = \angle \text{SOP} + \angle \text{C}$, consequently $\angle \text{OSP} + \angle \text{OPS} = \angle \text{C}$, which is measured by the intermediate arc PS.

Now, the sum of both refractions $\angle a\text{SP} + \angle b\text{PS} = (\angle \text{ASP} + \angle \text{BPS}) - (\angle \text{AS}a + \angle \text{BP}b) = (\angle \text{OSP} + \angle \text{OPS}) - (\angle \text{AS}a + \angle \text{BP}b) = \angle \text{C} - (\angle \text{AS}a + \angle \text{BP}b)$. Hence the following

(170) Rule. Subtract the sum of the two depressions from the contained arc, and half the remainder is the mean refraction.

(171) If one of the objects (A) instead of being depressed, be elevated, suppose to the point e, the $\angle$ of elevation being $e\text{SA}$; then the sum of the angles $m\text{SP}$, $m\text{PS}$ will be greater than $\angle \text{OPS} + \angle \text{OSP}$ (the $\angle \text{C}$, or contained arc PS) by the $\angle$ of elevation $e\text{SA}$. Hence $\angle m\text{sp} + \angle m\text{PS} = \angle \text{C} + \angle e\text{SA}$, from each of these equals take the $\angle \text{BP}b$, then the sum of the two refractions $\angle m\text{SP} + \angle b\text{PS} = \angle \text{C} + \angle e\text{SA} - \angle \text{BP}b$; that is, subtract the depression from the sum of the contained arc and elevation, and half the remainder is the mean refraction. Perhaps it may be necessary to remark, that previous to the observations the error of the instrument must be accounted for (163).

EXAMPLE XLVIII.

(172) The refraction between Dover Castle and Calais church was thus determined.†

Let C be the centre of the earth, PS the surface; D the station on Dover castle; A the top of the great balustrade of Calais steeple; EDF the horizontal line; also let PG = SD; then the $\angle \text{FDG} = \frac{1}{2} \angle \text{C}$, or half the arc PS (163). The distance from Dover to Calais is 137455 feet, hence 364950 feet : 60′ : : 137455 feet : 22′ 35″ the $\angle$ C; hence = FDG = 11′ 17″½.

The height of D above low-water spring tides = 479 feet.
The height of A (communicated from France = 140½ feet.

AG = 328½

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* In observing these angles two instruments are used, one at P and another at S; and the reciprocal observations are made at the same instant of time by means of signals, or by watches previously regulated for that purpose. The observer at P takes the depression of S, at the same moment which the observer at S takes the depression of P.

† Trigonometrical Survey, Vol. I. page 178.

The triangle DGA may be considered as isosceles, and DG or DA = 137455 feet, the distance between Dover and Calais. Hence $\frac{1}{2}$ AG : rad : : DA : sec ∠ DAG = 89° 55′ 53″, the double of which deducted from 180°, leaves 8′ 14″ for the ∠ GDA, to which add the ∠ FDG = 11′ $17\frac{1}{2}$″, and the whole angle FDA = 19′ $31\frac{1}{2}$″ supposing there was no refraction; but the ∠ FDA was determined from observation to be 17′ 59″, hence the refraction was (19′ $31\frac{1}{2}$″ − 17′ 59″ =) 1′ $32\frac{1}{2}$″, being about $\frac{1}{13}$ of the contained arc.

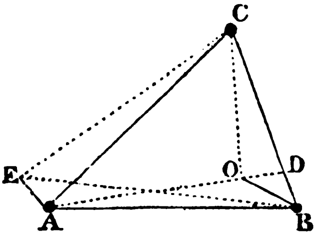
(173) Mr. Huddart is of opinion, that a true correction for the effect of terrestrial refraction cannot be obtained by taking any part of the contained arc *; for different points, though nearly at the same distance from the observer, will have various refractions.

OF THE REDUCTION OF ANGLES TO THE CENTRE OF THE STATION.

(174) In surveys of kingdoms and counties, where signals on the steeples of churches, vanes of spires, &c. are used for points of observation, the instrument cannot be placed exactly at the centre of the signal, and consequently the angle observed will be different from that which would have been found at the centre. The correction is generally very small, and is only necessary where great accuracy is required.

The observer may be considered in three different positions with respect to the centre, viz. he is either in a line with the centre and one of the objects; or a line drawn from the centre through his situation would, if produced, pass between the objects; or a line drawn from the centre to the place of the observer, when produced, would pass without the objects.

(175) FIRST, let the observer be at D, in a line between the objects B and C, viz. on one side of the triangle ABC; B being the proper centre of the station and the ∠ ABC that required. It is plain that the ∠ CDA, being the exterior ∠ of the triangle ADB, is too large by the interior ∠ DAB.



Therefore ∠ ABC = ∠ CDA − ∠ DAB.

(176) SECONDLY. Let the observer be at O, within the triangle ABC, and let B be the centre of the station, and the ∠ ABC that required. Now ∠ AOC + ∠ OAC + ∠ OCA = ∠ ABC +

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* Philosophical Transactions for 1797, page 29. et seq.

$\angle OAB + \angle OAC + \angle OCA + \angle OCB$, each of the sums being equal to two right angles; therefore the $\angle AOC = \angle ABC + \angle OAB + \angle OCB$, that is, the $\angle AOC$ is greater than the $\angle ABC$ by the sum of the angles OAB and OCB. Therefore $\angle ABC$[*] $= \angle AOC - (\angle OAB + \angle OCB)$.

(177) Thirdly. Let the observer be at E, without the triangle ABC, and let A be the centre of the station, and the $\angle CAB$ that required.

$$\text{Now } \angle CAB + \angle ACB + \angle ABE + \angle EBC = \angle CEB + \angle ECA + \angle ACB + \angle EBC,$$

each of the sums being equal to two right angles; therefore

$\angle CAB + \angle ABE = \angle CEB + \angle ECA$, and consequently

$\angle CAB = \angle CEB + \angle ECA - \angle ABE$.

EXAMPLE XLIX.

Let A and B represent the vanes on two steeples, E the situation of the theodolite upon the steeple A, and O its situation upon the steeple B. Then, suppose

$AE = 12$ feet
$\angle CEB = 74° 32'$
$\angle CEA = 139° 39'$

$BO = 10{\cdot}5$ feet
$\angle AOC = 49° 27'$
$\angle COB = 137° 55'$

It is required to find the angles CAB and ABC, the distance AB being 5000 feet.

SOLUTION.

The $\angle CEB = \angle CAB$ nearly, and $\angle AOC = \angle ABC$ nearly, with these angles and AB, find AC and BC (as in Example I. Chap. III.) $= 4581{\cdot}7$ and $5811{\cdot}5$. Then,

$AC : \sin \angle CEA :: AE : \sin \angle ECA = 5' \, 50''$
$AB : \sin \angle BEA :: AE : \sin \angle ABE = 7' \, 29''$

Hence $\angle CAB = 74° \, 30' \, 21''$ (177).

$BC : \sin \angle COB :: BO : \sin \angle OCB = 4' \, 10''$
$AB : \sin \angle AOB :: BO \cdot \sin \angle OAB = 0' \, 56''$

Hence $\angle ABC = 49° \, 21' \, 54''$ (176).

With the corrected angles CAB, ABC, and the distance AB, the sides AC and BC may be determined.

OF THE REDUCTION OF ANGLES FROM ONE PLANE TO ANOTHER.

(178) Angles which are inclined to the horizon, may be reduced to the corresponding horizontal angles, in cases where

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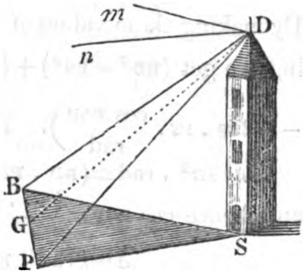
[*] If the proper centre of the station were at O, and the observer at B, it is plain that the angles OAB and OCB must be added to the $\angle ABC$ to obtain the $\angle AOC$.

very great accuracy is required. Let the lines PS, PB, BS be three chords of terrestrial arcs, that is, let the points P, B, and S, be all equally distinct from the centre of the earth, and let the point D be elevated so as to be farther from the centre of the earth than any of the points P, B, S, it is required to reduce the triangle BDP to the triangle BSP.

The line SD may be supposed to be perpendicular to each of the chords SB and SP without sensible error, though strictly speaking the angles DSB and DSP are each equal to $90° + \frac{1}{2}$ the arc which the chords SB and SP subtend, by art. (163) and Note.



Likewise the chords SB and SP may be used instead of their corresponding arcs (154). By inspection of the figure it is plain that BD is greater than BS, and PD greater than PS; but the base PB is common to the two triangles BDP and BSP, therefore the ∠ BSP is greater than the ∠ BDP. (21 EUCLID I.)

Draw DG perpendicular to BP, and conceive SG to be joined, then since DS is perpendicular to the plane BSP, the angle BGS is a right angle. (KEITH's *Geom.* Book IX, Prop. 4, cor. 3.)

Hence by art. (120) $\text{GB} = \frac{\text{BD} \cdot \cos \text{DBP}}{\text{rad}}$, and $\text{GB} = \frac{\text{BS} \cdot \cos \text{SBP}}{\text{rad}}$,

therefore $\text{BD} \cdot \cos \text{DBP} = \text{BS} \cdot \cos \text{SBP}$, or $\cos \text{SBP} = \frac{\text{BD} \cdot \cos \text{DBP}}{\text{BS}}$;

but $\text{BD} = \frac{\text{rad BS}}{\cos \text{DBS}}$; conseq. $\cos \text{SBP} = \frac{\text{rad} \cdot \cos \text{DBP}}{\cos \text{DBS}}$, or cos DBS : cos DBP : : rad : cos SPB.

(179) That is, *The cosine of the ∠ of elevation : the cosine of the ∠ inclined to the horizon : : radius : cosine of the horizontal ∠.*

Exactly in the same manner the ∠ SPB may be found.
Then 180° − (∠ SBP + ∠ SPB) = ∠ BSP.

(180) But the ∠ BDP taken on the elevated situation D, may be reduced to its corresponding ∠ BSP, by using only the observed ∠ BDP, and the angles of depression *m*DB, and *n*DP.

The $\angle m\text{DB} = \angle \text{DBS}$ and the $\angle n\text{DP} = \angle \text{DPS}$; hence we have to shew the three angles necessary for this reduction are BDP, DBS, and DPS.

From the triangle BDP we have by art. (119)

$$\cos\angle \text{BDP} = \frac{\text{BD}^2 + \text{PD}^2 - \text{PB}^2}{2\text{BD} \,.\, \text{PD}} \times \text{rad}$$

$$\therefore\ \text{PB}^2 = \text{BD}^2 + \text{PD}^2 - \left(2\text{BD} \,.\, \text{PD} \,.\, \frac{\cos \text{BDP}}{\text{rad}}\right)$$

And from the triangle SPB, we have similarly

$$\text{PB}^2 = \text{BS}^2 + \text{PS}^2 - \left(2\text{BS} \,.\, \text{PS} \,.\, \frac{\cos \text{BSP}}{\text{rad}}\right)$$

By making these values of PB^2 equal to each other, and reducing, we get $(\text{BD}^2 - \text{BS}^2) + (\text{PD}^2 - \text{PS}^2) = \left(2\text{BD} \,.\, \text{PD} \,.\, \frac{\cos \text{BDP}}{\text{rad}}\right)$ $- \left(2\text{BS} \,.\, \text{PS} \,.\, \frac{\cos \text{BSP}}{\text{rad}}\right)$. But $\text{BD}^2 - \text{BS}^2 = \text{PD}^2 - \text{PS}^2 = \text{SD}^2$,

$$\therefore\ \text{SD}^2 \,.\, \text{rad} = (\text{BD} \,.\, \text{PD} \,.\, \cos \text{BDP}) - (\text{BS} \,.\, \text{PS} \,.\, \cos \text{BSP}),$$

and hence $\cos \text{BSP} = \dfrac{(\text{BD} \,.\, \text{PD} \,.\, \cos \text{BDP}) - (\text{SD}^2 \,.\, \text{rad})}{\text{BS} \,.\, \text{PS}}$

$$\text{Also rad} : \text{BD} :: \sin \text{DBS} : \text{SD}$$
$$\text{rad} : \text{PD} :: \sin \text{DPS} : \text{SD}$$

Hence, $\text{rad}^2 : \text{BD} \,.\, \text{PD} :: \sin \text{DBS} \,.\, \sin \text{DPS} : \text{SD}^2$ (14 KEITH'S *Geom.* VII.)

And $\therefore\ \text{SD}^2 \times \text{rad} = \text{BD} \,.\, \text{PD} \,.\, \sin \text{DBS} \,.\, \sin \text{DPS} \,.\, \dfrac{1}{\text{rad}}$.

$$\text{Again, rad} : \text{BD} :: \cos \text{DBS} : \text{BS}$$
$$\text{rad} : \text{PD} :: \cos \text{DPS} : \text{PS}$$

Hence $\text{rad}^2 : \text{BD} \,.\, \text{PD} :: \cos \text{DBS} \,.\, \cos \text{DPS} : \text{BS} \,.\, \text{PS}$,

And $\therefore\ \text{BS} \,.\, \text{PS} = \text{BD} \,.\, \text{PD} \,.\, \cos \text{DBS} \,.\, \cos \text{DPS} \,.\, \dfrac{1}{\text{rad}^2}$

Whence by substitution

$$\cos\angle \text{BSP} = \frac{\cos \text{BDP} - \left(\sin \text{DBS} \,.\, \sin \text{DPS} \,.\, \frac{1}{\text{rad}}\right)}{\cos \text{DBS} \,.\, \cos \text{DPS} \,.\, \frac{1}{\text{rad}^2}}$$

$$= \frac{(\cos \text{BDP} \,.\, \text{rad}^2) - (\sin \text{DBS} \,.\, \sin \text{DPS} \,.\, \text{rad})}{\cos \text{DPS} \,.\, \cos \text{DPS}}$$

(181) M. de Lambre gives the following formula * for find-

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* This formula is demonstrated Prop. 7th, Chap. XIV. Book III.

ing the angle BSP, which he demonstrates by spherical trigonometry, viz.

$$\sin\tfrac{1}{2}\angle \mathrm{BSP} = \text{rad.}\sqrt{\frac{\sin\frac{1}{2}(\mathrm{BDP}+\mathrm{DBS}-\mathrm{DPS})\,.\,\sin\frac{1}{2}(\mathrm{BDP}+\mathrm{DPS}-\mathrm{DBS})}{\cos\mathrm{DBS}\,.\,\cos\mathrm{DPS}}}$$

The sides BS and PS of the triangle BSP are easily found, from the height DS and the angles DBS and DPS being given.

EXAMPLE L.

Wanting to know the height of an object SD (standing on the horizontal plane PSB), and also the horizontal distances SP and SB, I measured a base PB = 300 yards, at P the ∠ of elevation DPS was 8° 15′, and the ∠ DPB, inclined to the horizon, was 57° 15′; at B the ∠ of elevation DBS was 8° 45′ 50″, and the ∠ DBP, inclined to the horizon, was 63° 15′. Required the horizontal ∠ BSP, the horizontal distances SP, and SB, and the height of the object above the plane.

Answer. ∠ BPS = 56° 51′ 47″, ∠ PBS = 62° 54′ 29″ (179), BS = 289·40, PS = 307·69, and DS = 44·613 yards.

The height DS, and the distances BS and PS, may be found without reducing the angles. For, in the triangle DBP all the angles are given, and the side BP, whence BD and PD may be found; then in the triangle DPS, the ∠ DPS and the side DP are given to find PS and SD; also, in the triangle DBS, DB and the ∠ DBS are given, from which BS may be found.

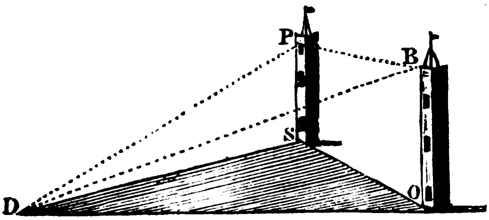
EXAMPLE LI.

From the top of a tower 44·613 yards high, the angle BDP, subtended by two distant objects B and P, was 59° 30′; and the angles of depression *m*DB = 8° 45′ 50″, and *n*DP = 8° 15′. Hence it is required to find the horizontal distances PB, BS, and PS.

Answer. The ∠ *m*DB = ∠ DBS = 8° 45′ 50″, ∠ *n*DP = ∠ DPS = 8° 15′, and ∠ BDP = 59° 30′. By the formula, art. (180), or (181), the ∠ BSP = 60° 13′ 38″; also BS = 289·40, PS = 307·69, and PB = 300. The same answer may be obtained without reducing the angles, and without the formula.

(182) The formulæ in arts. (180) and (181), may be applied to any two triangles, whether their bases be in the same horizontal plane, and their vertices elevated, as in examples L. and LI., or their vertices be in the same horizontal plane, and the extremities of the base of the one triangle be elevated above the extremities of the base of the other.

In the annexed figure, if the angle PDB be measured with a sextant, and the vertical angles BDO, PDS be measured with a theodolite,



$$\cos \angle \mathrm{SDO} = \frac{(\cos \mathrm{PDB} \,.\, \mathrm{rad}^2) - (\sin \mathrm{PDS} \,.\, \sin \mathrm{BDO} \,.\, \mathrm{rad})}{\cos \mathrm{PDS} \,.\, \cos \mathrm{BDO}}$$

or by M. de Lambre's formula *, viz.

$$\sin \tfrac{1}{2} \angle \mathrm{SDO} = \mathrm{rad} \sqrt{\frac{\sin \frac{1}{2} (\mathrm{PDB} + \mathrm{PDS} - \mathrm{BDO}) \,.\, \sin \frac{1}{2} (\mathrm{PDB} + \mathrm{BDO} - \mathrm{PDS})}{\cos \mathrm{PDS} \,.\, \cos \mathrm{BDO}}}$$

EXAMPLE LII.

From a station at D in the horizontal plane DSO, I took the angle PDB, subtended by the tops of two towers $=37° \, 53' \, 20''$, and also the angles of elevation BDO$=4° \, 23' \, 55''$, and PDS$=4° \, 17' \, 21''$. The height of the tower BO is known to be 40 yards, and that of PS 30 yards; from which it is required to find the horizontal distance of my station from each of the towers, and their horizontal distance from each other.

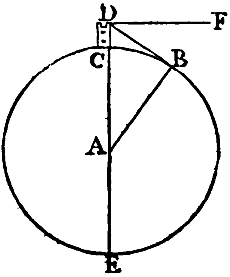
Answer. By the formula in art. (182), $\angle$ SDO$=38° \, 0'$; DS$=$ 400 yards, DO$=520$, and SO$=320$ yards.

The same answer may be found without the formula. For, with the $\angle$ BDO and height BO, find BD$=521{\cdot}56$, and DO$=520$; with the $\angle$ PDS and height PS, find DP$=401{\cdot}13$, and DS$=400$. Then with DP, DB, and the $\angle$ PDB, find PB$=$ $320{\cdot}47$; lastly, SO $= \sqrt{\mathrm{PB}^2 - (\mathrm{BO} - \mathrm{PS}^2)} = 320$ yards, the horizontal distance.

OF THE DIP, OR DEPRESSION OF THE HORIZON AT SEA.

(183) The dip or depression of the horizon at sea is the angle contained between the horizon of the observer, and the farthest visible point on the surface of the sea.

For, if an observer whose eye is situated at D, takes the altitude of a celestial object by a sextant, or Hadley's quadrant, and brings that object to the surface of the water at B, instead of the horizon DF, he evidently makes the altitude too great by the $\angle$ FDB.



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* Traité de Trigonométrie, par M. Cagnoli, Appendice, page 467.; or, Prop. 7. Chap. XIV. Book III. following.

Now the $\angle ADF$ is a right angle, and the $\angle ABD$ is likewise a right angle, therefore the $\angle FDB = \angle DAB$. Hence, in the right-angled triangle ABD, there is given AB the radius of the sphere, $AD = AC + DC$, the radius of the sphere increased by the height of the eye above the surface of the sea, to find the $\angle DAB$, or quantity to be subtracted from the observed altitude of any celestial object. Hence $\cos\angle FDB = \cos\angle DAB = \dfrac{\text{rad}\times AB}{AC+CD}$

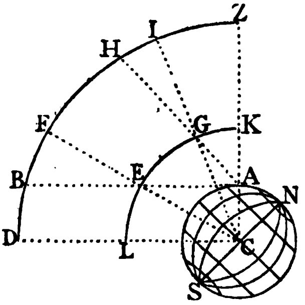
OTHERWISE,

$(EC+CD)\times DC = DB^2$ (36 Euclid III.), that is $EC\,.\,DC + DC^2 = DB^2$; or $2AC\,.\,DC + DC^2 = DB^2$, and rejecting DC^2 as being small $\sqrt{2AC\,.\,DC} = DB$, but $AB = AC : \text{rad} :: DB = \sqrt{2AC\,.\,DC} : \tan DAB = \tan\angle FDB$, therefore

$$\tan\angle FDB = \frac{\text{rad}\ \sqrt{2AC}}{AC}\times\sqrt{DC} = \text{rad}\sqrt{\frac{2}{AC}}\times\sqrt{DC}.$$

Now the first of these terms is a constant quantity, and if the diameter of the earth be 7964 miles, the logarithm of this quantity in feet will be 6·48915; hence $\log\tan\angle FDB = 6{\cdot}48915 + \frac{1}{2}\log DC$ in feet.

OF THE PARALLAX OF THE CELESTIAL BODIES.



(184) That part of the heavens in which a celestial body appears when viewed from the surface of the earth is called its *apparent place:* and the point in which it would be seen at the same instant from the centre of the earth, is called its *true place*, the difference between the *true* and *apparent* place is called the parallax in altitude.

Let C be the centre of the earth, A the place of an observer on its surface, whose visible horizon is AB, true horizon CD, and zenith Z. Let ZIFD be a portion of a great circle in the heavens, and E the absolute place of any object in the visible horizon; join CE and produce it to F; then F is the true place of the object, and B its apparent place in the heavens; and the angle $BEF = AEC$ is the parallax.

The parallax is the greatest when the object is in the horizon; for the angle AEC is greater than AGC, hence the more elevated an object is (its distance from the earth's centre continuing the same) the less is the parallax. When the object is in the zenith the parallax vanishes, for then AC and AZ, are in the same straight line CZ.

Since the apparent place (H) of a celestial body is more distant from the zenith (Z) then the true place (I), it therefore follows, that the parallax in altitude must be added to the observed altitude, in order to obtain the true place as seen from the earth's centre. The stars on account of their immense distance from the earth have no sensible parallax, and the sun's mean parallax is only 8″·6. The moon's greatest horizontal parallax is 61′ 32″, least 52′ 50″.

(185) *The horizontal parallax being given, to find the parallax at any given altitude.*

In the right-angled triangle EAC

rad : EC : : sin AEC : AC

And in the triangle GAC

GC=EC : sin GAC : : AC : sin AGC

And by comparing these two proportions

rad : sin GAC : : sin AEC : sin AGC.

But the sine of an arc is equal to the sine of its supplement, therefore sin GAC=sin GAK, and GAK is the complement of GAE, therefore sin of GAK=cos GAE, hence

rad : cos GAE : : sin AEC : sin AGC.

The last two terms being small, the arcs may be substituted for their sines without sensible error.

Hence the following rule:

Radius
: *Cosine of the apparent altitude,*
: : *The horizontal parallax in seconds,*
: *The parallax in altitude in seconds.*

EXAMPLE LIII.

The apparent altitude of the moon's centre is 24° 29′ 44″, the horizontal parallax 55′ 2″. Required the parallax in altitude.

Rad=sin 90° . . . . 10·00000
: cos moon's altitude 24° 29′ 44″. . . 9·95904
: : horizontal parallax 55′ 2″=3302″ log=3·51878
: parallax in altitude 3005″ log= . . 3·47782

Hence the parallax in altitude is 3005″=50′ 5″.

OF THE ADMEASUREMENT OF ALTITUDES BY THE BAROMETER AND THERMOMETER.

(186) One of the most simple and easy practical rules for measuring the elevations and depressions of objects by means of

the barometer and thermometer, is that given in the Encyclopædia Brit. Article Pneumatics, or in Dr. Rees' New Cyclopædia, under the word Barometer. The directions are as follow: — Let the observers be provided with two portable barometers, each of the same construction, with a nonius properly adapted to the scale, and a thermometer attached to each, having their bulbs each of the same diameter, nearly, as the diameters of the barometric tubes.

Place one of these barometers in the shade at the top of the eminence, with a detached thermometer near it; and let the other barometer be placed below in like manner, with a detached thermometer near it. When the thermometers have acquired the temperature of the air, that is, when the fluid in each becomes stationary, the observers must note down the temperatures shown by the thermometers, and the heights of the mercurial columns in the barometers.

Then the elevation of the higher barometer above the lower may be determined by the formula, which is deduced from the following observations.

(187) 1. The height through which we must rise in order to produce any fall of the mercury in the barometer, is inversely proportional to the density of the air; that is to the height of the mercury in the barometer.

2. When the barometer stands at 30 inches, and the air and quicksilver are of the temperature 32°, we must rise through 87 feet, in order to produce a depression of $\frac{1}{10}$th of an inch.

3. But if the air be of a different temperature, the 87 must be increased or diminished by $\frac{21}{100}$ of a foot for every degree of difference of the temperature from 32°.

4. Every degree of difference of the temperatures of the mercury at the two stations makes a change of 2·833 feet, or 2 feet 10 inches in the elevation. Hence,

If d be the difference between 32° and the mean temperature of the air, D the difference between the barometric heights in tenths of an inch, δ the difference between the mercurial temperatures, m the mean barometric height, and E the correct elevation. Then $\text{E} = \frac{30\text{D}\,(87 \pm \cdot 21d)}{m} \pm 2{\cdot}833\,\delta$.

This formula may be given in words, thus:

(188) 1. Multiply the difference between 32° and the mean temperature of the air by ·21, and add the product to 87, if the mean temperature be above 32°, but subtract it if below.

2. Multiply the sum or difference, found above, by thirty times the difference, between the barometric heights, in tenths

of inches, and divide the product by the mean of the barometric heights, the quotient will give the approximated elevation.

3. Multiply the difference between the mercurial temperatures by 2·833 feet, and add this product to the approximated elevation, if that of the upper barometer be the greatest, otherwise subtract it, the result will be the corrected elevation in feet.

EXAMPLE LIV.

Suppose that the mercury in the barometer at the lower station was at 29·4 inches, that its temperature was 50° of Fahrenheit's thermometer, and the temperature of the air 45°; and let the height of the mercury at the upper station be 25·19 inches, its temperature 46°, and the temperature of the air 39°, what would be the elevation of the higher barometer above the lower?

Solution by the formula.

$$\text{Here}\begin{cases} d = \frac{1}{2}\,(45^\circ + 39) - 32^\circ = 10^\circ \\ \text{D} = (29{\cdot}4 - 25{\cdot}19) \times 10 = 42{\cdot}1 \\ \delta = 50 - 46 \quad - \; - \; - \quad = 4^\circ \\ m = \frac{1}{2}\,(29{\cdot}4 + 25{\cdot}19) \quad = 27{\cdot}295 \end{cases}$$

$$\text{And } \text{E} = \frac{30 \times 42{\cdot}1 \times (87 + \overline{{\cdot}21 \times 10})}{27{\cdot}295} + (4 \times 2{\cdot}833)$$

$=4111{\cdot}522$ feet, the correct elevation.

(189) If several sets of observations be made at each station, after short intervals of time, and the mean of the separate results be taken, the conclusion will probably be more accurate than that derived from a single observation.

CHAP. V.

OF THE SIGNS OF TRIGONOMETRICAL QUANTITIES, ETC.

(190) In the investigation of some theorems in spherical trigonometry, and in the application of trigonometry to several astronomical and analytical problems, the changes of the signs of the tangents, sines, &c. from affirmative or +, to negative or −, require particular attention. They likewise serve to illustrate the use of the negative sign in the application of algebra to geometry.

(191) The tangents, sines, and versed sines are counted

from, or have their origin at B, the beginning of the arc (Plate I. fig. 1.). The cosines, secants, and cosecants, originate at the centre C.

(192) The cotangents and coversed sines originate at the end of the first quadrant, as at A.

(193) The *sine* increases from nothing at B till it becomes equal to the radius, at the end of the first quadrant AB; decreases along the second quadrant from A to *b*, and vanishes at the point *b*, showing the sine of a semicircle to be nothing. After this, the sine will lie on the contrary side of the diameter; therefore being reckoned affirmative before, must now be counted negative. Through the third quadrant *b*D, the sine I*h* increases negatively till it becomes equal to the radius; after which it decreases negatively till it arrives at the point B, where it is nothing as before.

(194) The *cosine* is equal to the radius when the arc is nothing; but decreases through the first quadrant BA, at the end of which it is nothing: during the second quadrant it is negative; for the cosine CI will lie in an opposite direction to the cosine CG. Through the third quadrant *b*D the cosine decreases negatively, at D the negation is destroyed; and in the fourth quadrant DB it again becomes affirmative.

(195) The *tangent* at the beginning of the arc is nothing, and increases to infinity during the first quadrant BA, at the point A there is no tangent, for CA produced can never meet BT (99). In the second quadrant A*b* the tangent B*t* is negative: for the tangents being reckoned from the point B, the tangent B*t* will lie in an opposite direction from BT. Through this second quadrant the tangent decreases negatively, from infinity to nothing. In the third quadrant the tangent is again affirmative, and increases from nothing to infinity. In the fourth quadrant it decreases from negative infinity to nothing, just the same as in the second quadrant.

(196) The *cotangent* at the beginning B, of the arc, that is when the arc is very small, is affirmative infinity, but in the second quadrant it will fall on the contrary side of AD, and consequently will be negative. In the third quadrant it will again be affirmative, and in the fourth negative, exactly the same as the tangent.

(197) The *secant* at the commencement of the first quadrant, when the arc is nothing, is equal to the radius, and increases affirmatively to the end of the first quadrant BA, where it ceases to exist (100). In the second quadrant it is negative, for the *revolving radius* falls on the contrary side of AD, and continues so through the third quadrant; but in the fourth

quadrant it is again affirmative, or falls on the same side of AD as in the first quadrant.

(198) The *cosecant* will agree with the sine, for the same reason that the secant agrees with the cosine.

(199) The *versed sine* increases from nothing during the first two quadrants, till it becomes equal to the diameter its utmost limit. It decreases for the last two quadrants, till it becomes nothing, but being always counted in the same direction from B to *b* is always affirmative.

(200) We may hence lay down the following synopsis of algebraic signs for all the four quadrants: —

	From 0° to 90°	From 90° to 180°	From 180° to 270°	From 270° to 360°
Sine . .	+	+	—	—
Cosine .	+	—	—	+
Tangent	+	—	+	—
Cotangent	+	—	+	—
Secant .	+	—	—	+
Cosecant	+	+	—	—
Versed sine	+	+	+	+

(201) In analytical enquiries, arcs of all magnitudes*, whether greater or less than 180°, are frequently used; but in trigonometry, every arc or angle made use of must be less than 180°. If any arc be considered as affirmative, an equal arc in a contrary direction will be negative.

Let BF and BH be two equal arcs (*Plate I. Fig.* 1.); now if the arc BF be considered as positive, the arc BH in a contrary direction will be negative. The arcs BF and BH are equal and have the same cosine GC, and the same versed sine BG: but the sine HG, lying in a contrary direction to the sine FG, becomes negative, or contrary to the sine FG of the positive arc BF.

It is shown in the XIth proposition following, that if $A =$ any arc, $\tan A = \frac{\sin A}{\cos A} \times \text{rad}$; $\cot A = \frac{\cos A}{\sin A} \times \text{rad}$; and $\text{cosec}\, A = \frac{\text{rad}^2}{\sin A}$. These quantities will consequently have a different sign for a negative arc, to that which they will have for a positive arc; but as sec $\text{A} = \frac{\text{rad}^2}{\cos \text{A}}$, the secant will have the same sign for a negative arc as it will have for an affirmative arc.

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* For if to any arc BF, there be added one or more circumferences of the circle, they will terminate exactly in the point F, and the augmented arc will have the same positive or negative sine, cosine, &c. with the arc BF. Thus if C denote an entire circumference, or 360°, $\text{sine}\ x = \text{sine}\ (c + x) = \text{sine}\ (2c + x) = \text{sine}\ (3c + x)$, &c. and the same will take place with respect to the cosine, tangent, &c. *Legendre's* Geometry, 6th edit. page 336.

(202) If the side, tangent, cotangent, or cosecant of A − B be found at the conclusion of a problem, and the arc A be less than the arc B, the signs of the sine, tangent, cotangent, or cosecant of A − B, will be contrary to those which are given in the foregoing table. It may also be remarked, that in trigonometrical calculations, when a required quantity comes out a sine, the case is frequently ambiguous; because the sine of an arc and the sine of its supplement are equal, and have the same sign. But when the required quantity is expressed by a cosine, tangent, or a cotangent, there is no ambiguity, for a positive cosine, tangent, or cotangent, shows the arc to be less than 90°; and a negative cosine, tangent, or cotangent, denotes an arc between 90° and 180°.

(203) Since the sine of an arc = sine of its supplement, tangent of an arc = tangent of its supplement, &c.; if A represent any arc less than 90°, we shall have,

$$\begin{aligned}
\sin\ (90^\circ + \text{A}) &= \sin\ (90^\circ - \text{A}) = \cos\ \text{A} \\
\tan\ (90^\circ + \text{A}) &= \tan\ (90^\circ - \text{A}) = \cot\ \text{A} \\
\sec\ (90^\circ + \text{A}) &= \sec\ (90^\circ - \text{A}) = \text{cosec}\ \text{A} \\
\cos\ (90^\circ + \text{A}) &= \cos\ (90^\circ - \text{A}) = \sin\ \text{A} \\
\cot\ (90^\circ + \text{A}) &= \cot\ (90^\circ - \text{A}) = \tan\ \text{A} \\
\text{cosec}\ (90^\circ + \text{A}) &= \text{cosec}\ (90^\circ - \text{A}) = \sec\ \text{A}
\end{aligned}$$

$$\begin{aligned}
\text{Also } \sin\ (180^\circ - \text{A}) &= +\sin\ \text{A} \\
\cos\ (180^\circ - \text{A}) &= -\cos\ \text{A} \\
\tan\ (180^\circ - \text{A}) &= -\tan\ \text{A} \\
\cot\ (180^\circ - \text{A}) &= -\cot\ \text{A} \\
\sec\ (180^\circ - \text{A}) &= -\sec\ \text{A} \\
\text{cosec}\ (130^\circ - \text{A}) &= +\text{cosec}\ \text{A}
\end{aligned}$$

$\text{vers}\ (180^\circ - \text{A}) = \text{rad} - \cos\ (180^\circ - \text{A}) = \text{rad} + \cos\ \text{A}$, which is sometimes called the *suversed* sine of A, and is written suvers A.

$\text{covers}\ (180^\circ - \text{A}) = \text{rad} - \sin\ (180^\circ - \text{A}) = \text{rad} - \sin\ \text{A}$.

(204) The cosine of half an arc is equal to the sine of half the supplement of that arc, the sine of half an arc is equal to the cosine of half the supplement of that arc, &c.

For if $\frac{1}{2}$A be substituted for A we shall have,

$$\cos \tfrac{1}{2}\text{A} = \sin\ (90^\circ - \tfrac{1}{2}\text{A}) = \sin \frac{180^\circ - \text{A}}{2}$$

$$\sin \tfrac{1}{2}\text{A} = \cos\ (90^\circ - \tfrac{1}{2}\text{A}) = \cos \frac{180^\circ - \text{A}}{2}$$

$$\cot \tfrac{1}{2}A = \tan (90^\circ - \tfrac{1}{2}A) = \tan \frac{180^\circ - A}{2}$$

$$\tan \tfrac{1}{2}A = \cot (90^\circ - \tfrac{1}{2}A) = \cot \frac{180^\circ - A}{2}$$

$$\operatorname{cosec} \tfrac{1}{2}A = \sec (90^\circ - \tfrac{1}{2}A) = \sec \frac{180^\circ - A}{2}$$

$$\sec \tfrac{1}{2}A = \operatorname{cosec} (90^\circ - \tfrac{1}{2}A) = \operatorname{cosec} \frac{180^\circ - A}{2}$$

And as A, in trigonometry, must always be less than 180°, $\frac{1}{2}$A will always be less than 90°, and consequently its sine, tangent, &c. will have the same sign as in the table in art (200).

GENERAL PROPERTIES OF THE SINES, TANGENTS, CHORDS, ETC. OF SINGLE ARCS.

PROPOSITION VIII. (*Plate I. Fig.* 1.)

(205) *The chord of any arc is a mean proportional between the diameter and the versed sine of that arc.*

The angle *b*FB being in a semicircle, is a right angle (31 Euclid III). And FG, by the definition of a sine, is perpendicular to *b*B, therefore the triangles *b*GF, and BGF, are similar to each other, and to the whole triangle *b*FB (8 Euclid VI). Hence,

*b*B : BF : : BF : BG, where *b*B is the diameter, and BF the chord, and BG is the versed sine of the arc B*i*F.

(206) COROL. *The sine of any arc is a mean proportional between the versed sine of that arc, and the versed sine of its supplement.*

For the ∠ F*b*G = ∠ BFG, consequently *b*G : GF : : GF : BG.

PROPOSITION IX. (*Plate I. Fig.* 1.)

(207) *The square of the sine of any arc added to the square of its versed sine, is equal to the square of the chord of that arc; or to four times the square of the sine of half that arc.*

For $GF^2 + GB^2 = BF^2$ (47 Euclid I).

Draw *ci* parallel to *b*F, and it will cut BF at right angles in Z; but if it cut it at right angles, it will bisect it (3 Euclid III), therefore BZ = ZF, and BF = 2BZ, and consequently $BF^2 = 4BZ^2$.

PROPOSITION X. (*Plate I. Fig.* 1.)

(208) *The tangent of any arc is a mean proportional between the sum and difference of the radius and the secant; viz.*

sec + rad : tan : : tan : sec − rad

Join FC, and produce it to *h* and to T, so as to meet the tangent BT of the arc BF. Then $hT \times FT = TB^2$ (36 Euclid III), that is $hT : TB :: TB : FT$. (17 Euclid VI) But $hT = \text{sec } CT + \text{rad}$, and $FT = \text{sec } CT - \text{rad}$; therefore $\text{sec} + \text{rad} : \text{tan} :: \text{tan} : \text{sec} - \text{rad}$.

PROPOSITION XI. (*Plate I. Fig.* 1.)

(209) *The right angled triangles* FGC, TBC, CAK, CEF, *are equiangular and similar.*

For FGC, TBC are right angles, and ∠ FCB = ∠ TCB, therefore the remaining ∠ CFG = ∠ CTB.

The triangle CEF = triangle CGF, for the ∠ CFG = ∠ FCE (29 Euclid I), and ∠ FEC = ∠ CGF, being each of them right angles, and the side CF is common to both the triangles, therefore they are equal (26 Euclid I). Again, AK is parallel to EF, by the definition of a tangent and sine: therefore the triangle CAK is equiangular with the triangle CEF; and consequently with CGF and CBT.

In the triangles CGF *and* CBT.

$$(\text{A}) \quad \left\{ \begin{array}{ccccccc} CG & : & GF & :: & CB & : & BT \\ \text{cos} & : & \text{sin} & :: & \text{rad} & : & \text{tan} \end{array} \right.$$

$$(\text{B}) \quad \left\{ \begin{array}{ccccccc} CF & : & FG & :: & CT & : & TB \\ \text{rad} & : & \text{sin} & :: & \text{sec} & : & \text{tan} \end{array} \right.$$

$$(\text{C}) \quad \left\{ \begin{array}{ccccccc} CB & : & CG & :: & CT & : & CF \\ \text{rad} & : & \text{cos} & :: & \text{sec} & : & \text{rad} \end{array} \right.$$

In the triangle CEF = CGF *and* CKA.

$$(\text{D}) \quad \left\{ \begin{array}{ccccccc} CF & : & CE = GF & :: & CK & : & CA \\ \text{rad} & : & \text{sin} & :: & \text{cosec} & : & \text{rad} \end{array} \right.$$

In the triangles CTB *and* CKA.

$$(\text{E}) \quad \left\{ \begin{array}{ccccccc} CB & : & BT & :: & AK & : & CA \\ \text{rad} & : & \text{tan} & :: & \text{cot} & : & \text{rad} \end{array} \right.$$

$$(\text{F}) \quad \left\{ \begin{array}{ccccccc} AK & : & EF = CG & :: & CK & : & CF \\ \text{cot} & : & \text{cos} & :: & \text{cosec} & : & \text{rad} \end{array} \right.$$

(210) By comparing these analogies
cos : sin :: cot : rad (A) and (E).
sec : tan :: cosec : rad (B) and (D).
sec : rad :: cosec : cot (C) and (F).

(211) From (E) *tan. cot* $= rad^2$.

Therefore the tangent of any arc × its cotangent = the tangent of any other arc × its cotangent.

(212) From (D) *sin . cosec* $= rad^2$.

Therefore the sine of any arc × its cosecant = sine of any other arc × its cosecant.

(213) From (C) $cos . sec = rad^2$.

Therefore the cosine of any arc × its secant = cosine of any other arc × its secant.

(214) Also $rad^2 = sin^2 + cos^2$.

For, $\mathrm{CF}^2 = \mathrm{GF}^2 + \mathrm{CG}^2$ (47 Euclid I).

(215) $rad^2 = sec^2 - tan^2 = cosec^2 - cot^2$

For, $\mathrm{CB}^2 = \mathrm{CT}^2 - \mathrm{TB}^2$ (47 Euclid I), and $\mathrm{AC}^2 = \mathrm{CK}^2 - \mathrm{AK}^2$.

(216) $rad^2 = cos . sec$ (by C art 209) $= sin . cosec$ (by D) $= tan . cot$ (by E).

(217) $sin = \frac{tan . cos}{rad}$ (by A) $= \frac{tan . rad}{sec}$ (by B) $= \frac{rad}{cosec}$ (by D) $= \frac{cos . rad}{cot}$ (210), &c.

(218) Let A denote any arc*; then adopting the notation in arts (107) and (203), we have

I. $$sin\ \mathrm{A} = \sqrt{\mathrm{rad}^2 - \cos^2 \mathrm{A}} = \frac{\cos \mathrm{A} . \tan \mathrm{A}}{\mathrm{rad}} = \frac{\mathrm{rad} . \tan \mathrm{A}}{\sec} = \frac{\mathrm{rad} . \cos \mathrm{A}}{\cot \mathrm{A}} = \frac{\mathrm{rad}^2}{\sqrt{\mathrm{rad}^2 + \cot^2 \mathrm{A}}} = \frac{\mathrm{rad} . \tan \mathrm{A}}{\sqrt{\mathrm{rad}^2 + \tan^2 \mathrm{A}}} = \frac{\mathrm{rad}^2}{\mathrm{cosec}\ \mathrm{A}} = \frac{\sec \mathrm{A} . \cos \mathrm{A}}{\mathrm{cosec}\ \mathrm{A}} = \frac{\tan \mathrm{A} . \cot \mathrm{A}}{\mathrm{cosec}\ \mathrm{A}} = \frac{\mathrm{rad}}{\mathrm{see}\ \mathrm{A}} \sqrt{\sec^2 \mathrm{A} - \mathrm{rad}^2} = \sqrt{\mathrm{suvers}\ \mathrm{A}. \mathrm{vers}\ \mathrm{A}} = \sqrt{2\ \mathrm{rad} . \mathrm{vers}\ \mathrm{A} - \mathrm{vers}^2 \mathrm{A}}.$$

II. $$cos\ \mathrm{A} = \mathrm{rad} - \mathrm{vers}\ \mathrm{A} = \mathrm{suvers}\ \mathrm{A} - \mathrm{rad} = \sqrt{\mathrm{rad}^2 - \sin^2 \mathrm{A}} = \frac{\mathrm{rad} . \sin \mathrm{A}}{\tan \mathrm{A}} = \frac{\mathrm{rad}^2}{\sqrt{\mathrm{rad}^2 + \tan^2 \mathrm{A}}} = \frac{\mathrm{rad}^2}{\sec \mathrm{A}} = \frac{\tan \mathrm{A} . \cot \mathrm{A}}{\sec \mathrm{A}} = \frac{\sin \mathrm{A} . \mathrm{cosec}\ \mathrm{A}}{\sec \mathrm{A}} = \frac{\sin \mathrm{A} . \cot \mathrm{A}}{\mathrm{rad}} = \frac{\mathrm{rad} . \cot \mathrm{A}}{\mathrm{cosec}\ \mathrm{A}} = \frac{\mathrm{rad} . \cot \mathrm{A}}{\sqrt{\mathrm{rad}^2 + \cot^2 \mathrm{A}}} = \frac{\mathrm{rad}}{\mathrm{cosec}\ \mathrm{A}} \sqrt{\mathrm{cosec}^2 \mathrm{A} - \mathrm{rad}^2}.$$

III. $$tan\ \mathrm{A} = \frac{\mathrm{rad} . \sin \mathrm{A}}{\cos \mathrm{A}} = \frac{\mathrm{rad} . \sin \mathrm{A}}{\sqrt{\mathrm{rad}^2 - \sin^2 \mathrm{A}}} = \frac{\mathrm{rad}}{\cos \mathrm{A}} . \sqrt{\mathrm{rad}^2 - \cos^2 \mathrm{A}} = \frac{\mathrm{rad}^2}{\cot \mathrm{A}} = \frac{\mathrm{rad}^2}{\sqrt{\mathrm{cosec}^2 \mathrm{A} - \mathrm{rad}^2}} = \frac{\cos \mathrm{A} . \sec \mathrm{A}}{\cot \mathrm{A}} = \sqrt{\sec^2 \mathrm{A} - \mathrm{rad}^2} = \frac{\sin \mathrm{A} . \mathrm{cosec}\ \mathrm{A}}{\cot \mathrm{A}} = \frac{\mathrm{rad} . \sec \mathrm{A}}{\mathrm{cosec}\ \mathrm{A}} = \frac{\mathrm{rad}^3 . \cos \mathrm{A}}{\sin \mathrm{A} . \cot^2 \mathrm{A}} = \frac{\mathrm{rad}\sqrt{2\,\mathrm{rad} . \mathrm{vers}\ \mathrm{A} - \mathrm{vers}^2 \mathrm{A}}}{\mathrm{rad} - \mathrm{vers}\ \mathrm{A}} = \frac{\mathrm{rad}\sqrt{(2\,\mathrm{rad} . \mathrm{suvers}\ \mathrm{A}) - \mathrm{suvers}^2 \mathrm{A}}}{\mathrm{suvers}\ \mathrm{A} - \mathrm{rad}}.$$

IV. $$cot\ \mathrm{A} = \sqrt{\mathrm{cosec}^2 \mathrm{A} - \mathrm{rad}^2} = \frac{\mathrm{rad}^2}{\tan \mathrm{A}} = \frac{\sin \mathrm{A} . \mathrm{cosec}\ \mathrm{A}}{\tan \mathrm{A}}$$

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* Emerson's Trigonometry, 2d Edit. Prop. I. Scholium.

$$=\frac{\cos A \,.\, \sec A}{\tan A}=\frac{\text{rad} \,.\, \text{cosec } A}{\sec A}=\frac{\text{rad} \,.\, \cos A}{\sin A}=\frac{\text{rad}}{\sin A}\sqrt{\text{rad}^2-\sin^2 A}$$

$$=\frac{\text{rad} \,.\, \cos A}{\sqrt{\text{rad}^2-\cos\ A}}=\frac{\text{rad}^2}{\sqrt{\sec^2 A-\text{rad}^2}}=\frac{\text{rad}^3 \,.\, \sin A}{\cos A \,.\, \tan^2 A}=$$

$$\frac{(\text{rad}-\text{vers } A) \,.\, \text{rad}}{\sqrt{(2\ \text{rad} \,.\, \text{vers } A)-\text{vers}^2 A}}=\frac{(\text{suvers } A-\text{rad}) \,.\, \text{rad}}{\sqrt{(2\ \text{rad} \,.\, \text{suvers } A)-\text{suvers}^2 A}}$$

V. *sec* $A=\sqrt{\text{rad}^2+\tan^2 A}=\dfrac{\text{rad}^2}{\cos A}=\dfrac{\text{rad} \,.\, \tan A}{\sin A}=$

$$\frac{\text{rad}^3}{\sin A \,.\, \cot A}=\frac{\text{rad} \,.\, \text{cosec } A}{\cot A}=\frac{\tan A \,.\, \cot A}{\cos A}=\frac{\sin A \,.\, \text{cosec } A}{\cos A}=$$

$$\frac{\text{rad}^2}{\sqrt{\text{rad}^2-\sin^2 A}}=\frac{\text{rad}}{\cot A}\sqrt{\text{rad}^2+\cot^2 A}=\frac{\text{rad} \,.\, \text{cosec } A}{\sqrt{\text{cosec}^2 A-\text{rad}^2}}=$$

$$\frac{\tan A \,.\, \text{cosec } A}{\text{rad}}=\frac{\text{rad}^2}{\text{rad}-\text{vers } A}=\frac{\text{rad}^2}{\text{suvers } A-\text{rad}}.$$

VI. *cosec* $A=\sqrt{\text{rad}^2+\cot^2 A}=\dfrac{\text{rad}^2}{\sin A}=\dfrac{\text{rad} \,.\, \sec A}{\tan A}=$

$$\frac{\text{rad} \,.\, \cot A}{\cos A}=\frac{\text{rad}^3}{\cos A \,.\, \tan A}=\frac{\tan A \,.\, \cot A}{\sin A}=\frac{\cos A \,.\, \sec A}{\sin A}=$$

$$\frac{\text{rad}^2}{\sqrt{\text{rad}^2-\cos^2 A}}=\frac{\text{rad}}{\tan A}\sqrt{\text{rad}^2+\tan^2 A}=\frac{\text{rad} \,.\, \sec A}{\sqrt{\sec^2 A-\text{rad}^2}}=$$

$$\frac{\cot A \,.\, \sec A}{\text{rad}}=\frac{\text{rad}^2}{\sqrt{2\ \text{rad} \,.\, \text{vers } A-\text{vers}^2 A}}$$

VII. *vers* $A=2\ \text{rad}-\text{suvers } A=\text{rad}-\cos A=\text{rad}-$

$$\sqrt{\text{rad}^2-\sin^2 A}=\text{rad}-\frac{\text{rad}^2}{\sqrt{\text{rad}^2+\tan^2 A}}=\text{rad}-\frac{\text{rad} \,.\, \cot A}{\sqrt{\text{rad}^2+\cot^2 A}}$$

$$=\text{rad}-\frac{\text{rad}^2}{\sec A}=\text{rad}-\frac{\text{rad}}{\text{cosec } A}\sqrt{\text{cosec}^2 A-\text{rad}^2}$$

VIII. *suvers* $A=2\ \text{rad}-\text{vers } A=\text{rad}+\sqrt{\text{rad}^2-\sin^2 A}=$

$$\text{rad}+\cos A=\text{rad}+\frac{\text{rad}^2}{\sqrt{\text{rad}^2+\tan^2 A}}=\text{rad}+\frac{\text{rad} \,.\, \cot A}{\sqrt{\text{rad}^2+\cot^2 A}}=$$

$$\text{rad}+\frac{\text{rad}}{\text{cosec } A}\sqrt{\text{cosec}^2 A-\text{rad}^2}=\frac{(\sin A+\tan A) \,.\, \text{rad}}{\tan A}$$

$$=\text{rad}+\frac{\sin A \,.\, \cot A}{\text{rad}}=\text{rad}+\frac{\text{rad}^2}{\sec A}=\frac{(\cot A+\text{cosec } A)\ \text{rad}}{\text{cosec } A}.$$

(219) The coversed sine may be found by subtracting the sine from radius; that is, $\text{covers } A=\text{rad}-\sin A$; also the chord $=\sqrt{\text{vers}^2 A+\sin^2 A}=\sqrt{(\text{rad}-\cos A)^2+(\text{rad}^2-\cos^2 A)}=\sqrt{2\ \text{rad} \cdot (\text{rad}-\cos A)}$.

(220) Besides the preceding formulæ, others may be de-

duced, thus, $\sin^2 A+\cos^2 A=\text{rad}^2$ (214), $\text{rad}^2+\tan^2 A=\sec^2 A$, and $\text{rad}^2+\cot^2 A=\text{cosec}^2 A$ (215). Now because $\tan A=\frac{\text{rad} . \sin A}{\cos A}$ (218), $\tan^2 A=\frac{\text{rad}^2 . \sin^2 A}{\cos^2 A}$, hence $\text{rad}^2+\tan^2 A$

$$=\text{rad}^2+\frac{\text{rad}^2 . \sin^2 A}{\cos^2 A}=\frac{\text{rad}^2 . (\cos^2 A+\sin^2 A)}{\cos^2 A}=\frac{\text{rad}^4}{\cos^2 A} \text{ \&c.}$$

GENERAL PROPERTIES OF SINES, TANGENTS, ETC. OF DOUBLE ARCS AND OF HALF ARCS.

PROPOSITION XII. (*Plate I. Fig.* 1.)

(221) *The right-angled triangles* *b*GF, BGF, *b*FB, *and* CZB, *are equiangular and similar; and* CZ, *the cosine of the arc* B*i*, *is equal to the half of* *b*F, *the chord of the supplement of double the arc* B*i*.

For *b*GF, BGF, and *b*FB have already been shown to be equiangular (205), and the triangles CZB, FGB, have the angle at B common to both of them; also the angle *b*FB, being an angle in a semicircle, is a right angle; and since CZ is parallel to *b*F, by construction (207), the angle CZB is likewise a right angle.

Now BZ : CZ : : BF : F*b*, but BZ = ½BF (207), therefore CZ = ½*b*F.

Hence the following proportions.

(A) CB : BZ (sine arc B*i*) : : BF(=2BZ) : BG
radius : sine of an arc : : double that sine : versed sine of double that arc.

(B) CB : BZ : : *b*F (=2CZ) : GF
radius : sine of an arc : : double its cosine : sine double the arc.

(C) CB : CZ : : *b*F (=2CZ) : *b*G
radius : cosine of an arc : : double its cosine : suversed sine of double the arc.

(D) CB : CZ : : BF (=2BZ) : GF
radius : cosine of an arc : : double the sine : sine of double the arc.

(E) BF (=2BZ) : BG : : *b*F (=2CZ) : GF
double the sine of an arc : versed sine double arc : : double cosine : sine of double the arc.

(F) BF (=2BZ) : GF : : *b*F (=2CZ) : *b*G
double the sine of an arc : sine double the arc : : double the cosine : suversed sine of double the arc.

(G) BZ : CZ : : GB : GF
sine of an arc : its cosine : : versed sine double arc : sine double arc.

(H) $\left\{\begin{array}{l} BZ : CZ :: GF : bG \\ \text{sine of an arc : its cosine :: sine double arc : su-} \\ \text{versed sine of double the arc.} \end{array}\right.$

(R) $\left\{\begin{array}{l} BG : GF :: GF : bG \\ \text{versed sine double arc : sine double arc :: sine double} \\ \text{arc : suversed sine of double the arc.} \end{array}\right.$

(222) By comparing these proportions, with the proportions Prop. XI. and casting out the equal terms, various other proportions may be formed. Thus by (B) in art (209)

radius : sine : : secant : tangent, and by (A) in art (221)

radius : sine : : double sine : versed sine double arc, therefore secant : tangent : : double the sine : versed sine double the arc, &c. for others.

(223) COROL. I. *The rectangle of the radius and the sine of any arc, is equal to double the rectangle of the sine and cosine of half that arc.*

For by (D) art (221) $CB : CZ :: 2BZ : GF$

Therefore $CB \times GF = 2BZ \times CZ$.

(224) COROL. II. *The rectangle of radius and half the versed sine of an arc, is equal to the square of the sine of half that arc.*

For by (A) art (221) $CB : BZ :: BF : BG$

And $CB : BZ :: \frac{1}{2}BF : \frac{1}{2}BG$

That is $CB : BZ :: BZ : \frac{1}{2}BG$

Therefore $CB \times \frac{1}{2}BG = BZ^2$; now CB is the radius, BG the versed sine of BF, and BZ is the sine of Bi, which is half the arc BF.

(225) COROL. III. *The rectangle of radius and half the suversed sine of an arc, is equal to the square of the cosine of half that arc.*

For by (C) art (221) $CB : CZ :: bF : bG$

$CB : CZ :: \frac{1}{2}bF : \frac{1}{2}bG$

$CB : CZ :: CZ : \frac{1}{2}bG$

Therefore $CB \times \frac{1}{2}bG = CZ^2$, where CB is the radius, bG the suversed sine of the arc BF, and CZ the cosine of half the arc BF = cosine of the arc Bi.

(226) COROL. IV. *The diameter of the circle is to the versed sine of any arc, as the square of the radius is to the square of the sine of half that arc.*

For, (224) $CB \times \frac{1}{2}BG = BZ^2$, multiply by the radius CB,

then $CB^2 \times \frac{1}{2}BG = CB \times BZ^2$

therefore $CB : \frac{1}{2}BG :: CB^2 : BZ^2$

or $2CB : BG :: CB^2 : BZ^2$

(227) COROL. V. *The rectangle of the sine of any arc, and of the cotangent of its half, is equal to double the square of the*

cosine of half that arc. And the same is true, writing tangent and sine, for cotangent and cosine,

For, if A = any arc, we have by (218) $\sin \mathrm{A} = \dfrac{\mathrm{rad} \,.\, \cos \mathrm{A}}{\cot \mathrm{A}}$, and $\cos \mathrm{A} = \dfrac{\mathrm{rad} \,.\, \sin \mathrm{A}}{\tan \mathrm{A}}$.

And $\mathrm{rad} : \cos \mathrm{A} :: 2 \sin \mathrm{A} : \sin 2\mathrm{A}$ by (A) art (221), hence $\mathrm{rad} : \cos \mathrm{A} :: \dfrac{2\, \mathrm{rad} \,.\, \cos \mathrm{A}}{\cot \mathrm{A}} : \sin 2\mathrm{A}$; therefore $\sin 2\mathrm{A} = \dfrac{2 \cos^2 \mathrm{A}}{\cot \mathrm{A}}$; or $\sin 2\mathrm{A} \,.\, \cot \mathrm{A} = 2 \cos^2 \mathrm{A}$, that is $\sin \mathrm{A} \,.\, \cot \tfrac{1}{2}\mathrm{A} = 2 \cos^2 \tfrac{1}{2}\mathrm{A}$.

Again, by substituting the value of the cosine of A in the second term of the first proportion, we shall obtain $\sin 2\mathrm{A} \,.\, \tan \mathrm{A} = 2 \sin^2 \mathrm{A}$, that is $\sin \mathrm{A} \,.\, \tan \tfrac{1}{2}\mathrm{A} = 2 \sin^2 \tfrac{1}{2}\mathrm{A}$.

(228) Using the same notation as before, the following formulæ may be easily obtained, by simple algebraical reductions.*

$$\text{I. } \textit{sin}\ 2\mathrm{A} = \frac{2 \cos \mathrm{A} \,.\, \sin \mathrm{A}}{\mathrm{rad}} = \frac{2 \sin^2 \mathrm{A}}{\tan \mathrm{A}} = \frac{2 \cos^2 \mathrm{A}}{\cot \mathrm{A}} =$$
$$\frac{2\, \mathrm{rad} \,.\, \sin \mathrm{A}}{\sec \mathrm{A}} = \frac{2\, \mathrm{rad} \,.\, \cos \mathrm{A}}{\mathrm{cosec}\, \mathrm{A}} = \frac{2\, \mathrm{rad}^2 \,.\, \tan \mathrm{A}}{\mathrm{rad}^2 + \tan^2 \mathrm{A}} = \frac{2\, \mathrm{rad}^2 \,.\, \tan \mathrm{A}}{\sec^2 \mathrm{A}} =$$
$$\frac{2\, \mathrm{rad}^2}{\cot \mathrm{A} + \tan \mathrm{A}} = \frac{2\, \mathrm{rad}^2 \,.\, \cot \mathrm{A}}{\mathrm{rad}^2 + \cot^2 \mathrm{A}} = \frac{2\, \mathrm{rad}^2 \,.\, \cot \mathrm{A}}{\mathrm{cosec}^2\, \mathrm{A}}.$$

$$\text{II. } \textit{cos}\ 2\mathrm{A} = \frac{\cos^2 \mathrm{A} - \sin^2 \mathrm{A}}{\mathrm{rad}} = \frac{\mathrm{rad}^2 - 2 \sin^2 \mathrm{A}}{\mathrm{rad}} =$$
$$\frac{2 \cos^2 \mathrm{A} - \mathrm{rad}^2}{\mathrm{rad}} = \frac{\mathrm{rad}^2 - \tan^2 \mathrm{A}}{\mathrm{rad}^2 + \tan^2 \mathrm{A}} \,.\, \mathrm{rad} = \frac{\cot \mathrm{A} - \tan \mathrm{A}}{\cot \mathrm{A} + \tan \mathrm{A}} \,.\, \mathrm{rad} =$$
$$\frac{\cot^2 \mathrm{A} - \mathrm{rad}^2}{\cot^2 \mathrm{A} + \mathrm{rad}^2} \,.\, \mathrm{rad} = \frac{2\, \mathrm{rad}^2 - \sec^2 \mathrm{A}}{\sec^2 \mathrm{A}} \,.\, \mathrm{rad} = \frac{2 \cos \mathrm{A} - \sec \mathrm{A}}{\sec \mathrm{A}} \,.\, \mathrm{rad} =$$
$$\frac{\mathrm{cosec}^2\, \mathrm{A} - 2\, \mathrm{rad}^2}{\mathrm{cosec}^2\, \mathrm{A}} \,.\, \mathrm{rad} = \frac{\mathrm{cosec}\, \mathrm{A} - 2 \sin \mathrm{A}}{\mathrm{cosec}\, \mathrm{A}} \,.\, \mathrm{rad}.$$

$$\text{III. } \textit{tan}\ 2\mathrm{A} = \frac{2\, \mathrm{rad}^2 \,.\, \tan \mathrm{A}}{\mathrm{rad}^2 - \tan^2 \mathrm{A}} = \frac{2\, \mathrm{rad}^2}{\cot \mathrm{A} - \tan \mathrm{A}} =$$
$$\frac{2\, \mathrm{rad} \,.\, \cos \mathrm{A} \,.\, \sin \mathrm{A}}{\mathrm{rad}^2 - 2 \sin^2 \mathrm{A}} = \frac{2\, \mathrm{rad} \,.\, \cos \mathrm{A} \,.\, \sin \mathrm{A}}{2 \cos^2 \mathrm{A} - \mathrm{rad}^2} = \frac{2\, \mathrm{rad}^2 \,.\, \cot \mathrm{A}}{\cot^2 \mathrm{A} - \mathrm{rad}^2} =$$
$$\frac{2\, \mathrm{rad}^2 \,.\, \tan \mathrm{A}}{2\, \mathrm{rad}^2 - \sec^2 \mathrm{A}} = \frac{2\, \mathrm{rad}^2 \,.\, \cot \mathrm{A}}{\mathrm{cosec}^2\, \mathrm{A} - 2\, \mathrm{rad}^2}.$$

$$\text{IV. } \textit{cot}\ 2\mathrm{A} = \frac{\mathrm{rad}^2 - \tan^2 \mathrm{A}}{2 \tan \mathrm{A}} = \frac{\cot^2 \mathrm{A} - \mathrm{rad}^2}{2 \cot \mathrm{A}} = \frac{\cot \mathrm{A} - \tan \mathrm{A}}{2} =$$

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* Emerson's Trigonometry, 2d edit. Prop. II. Scholium.

$$\frac{\text{rad}^2-2\sin^2 A}{2\cos A\,.\,\sin A}\,.\,\text{rad}=\frac{2\cos^2 A-\text{rad}^2}{2\cos A\,.\,\sin A}\,.\,\text{rad}=\frac{2\,\text{rad}^2-\sec^2 A}{2\tan A}=\frac{\text{cosec}^2 A-2\,\text{rad}^2}{2\cot A}.$$

V. *sec* $2A=\dfrac{\text{rad}\,.\,\sec^2 A}{2\,\text{rad}^2-\sec^2 A}=\dfrac{\text{rad}\,.\,\sec A}{2\cos A-\sec A}=$

$$\frac{\text{rad}^3}{2\cos^2 A-\text{rad}^2}=\frac{\text{rad}^3}{\text{rad}^2-2\sin^2 A}=\frac{\text{rad}^2+\tan^2 A}{\text{rad}^2-\tan^2 A}\,.\,\text{rad}=\frac{\cot A+\tan A}{\cot A-\tan A}\,.\,\text{rad}=\frac{\cot^2 A+\text{rad}^2}{\cot^2 A-\text{rad}^2}\,.\,\text{rad}=\frac{\text{rad}\,.\,\text{cosec}^2 A}{\text{cosec}^2 A-2\,\text{rad}^2}.$$

VI. *cosec* $2A=\dfrac{\sec A\,.\,\text{cosec}\,A}{2\,\text{rad}}=\dfrac{\text{rad}\,.\,\sec A}{2\sin A}=\dfrac{\text{rad}\,.\,\text{cosec}\,A}{2\cos A}=$

$$\frac{\text{rad}^3}{2\cos A\,.\,\sin A}=\frac{\text{rad}^2+\tan^2 A}{2\tan A}=\frac{\sec^2 A}{2\tan A}=\frac{\cot A+\tan A}{2}=\frac{\text{rad}^2+\cot^2 A}{2\cot A}=\frac{\sec^2 A}{2\tan A}=\frac{\text{cosec}^2 A}{2\cot A}.$$

VII. *vers* $2A=\dfrac{2\sin^2 A}{\text{rad}}=\dfrac{2\,\text{rad}^2-2\cos^2 A}{\text{rad}}=$

$$\frac{2\,\text{vers}\,A\,.\,\text{suvers}\,A}{\text{rad}}=\frac{2\,\text{vers}\,A\,.\,(2\,\text{rad}-\text{vers}\,A)}{\text{rad}}=\frac{2\,\text{rad}\,.\,\tan^2 A}{\text{rad}^2+\tan^2 A}=\frac{2\,\text{rad}\,.\,\tan^2 A}{\sec^2 A}=\frac{2\,\text{rad}\,.\,\tan A}{\cot A+\tan A}=\frac{2\,\text{rad}^3}{\text{rad}^2+\cot^2 A}=\frac{2\,\text{rad}^3}{\text{cosec}^2 A}=\frac{2\,\text{rad}\,.\,\sin A}{\text{cosec}\,A}=\frac{\sec^2 A-\text{rad}^2}{\sec^2 A}\,.\,2\,\text{rad}=\frac{\sec A-\cos A}{\sec A}\,.\,2\,\text{rad}.$$

VIII. *suvers* $2A=\dfrac{2\,\text{rad}^2-2\sin^2 A}{\text{rad}}=\dfrac{2\cos^2 A}{\text{rad}}=$

$$\frac{2\,\text{rad}^2}{\text{rad}^2+\tan^2 A}\,.\,\text{rad}=\frac{2\cot^2 A}{\text{rad}^2+\cot^2 A}\,.\,\text{rad}=\frac{2\,\text{rad}\,.\,\cot A}{\tan A+\cot A}=\frac{2\,\text{rad}^3}{\sec^2 A}=\frac{2\,\text{rad}\,.\,\cos A}{\sec A}=\frac{2\,\text{cosec}^2 A-2\,\text{rad}^2}{\text{cosec}^2 A}\,.\,\text{rad}=\frac{2\,\text{cosec}\,A-2\sin A}{\text{cosec}\,A}\,.\,\text{rad}=\frac{2(\text{suvers}\,A-\text{rad})^2}{\text{rad}}=\frac{2\,(\text{rad}-\text{vers}\,A)^2}{\text{rad}}.$$

The coversed sine may be expressed in terms of the rest by substituting its value in any of the above forms.

$$\text{covers}\,2A=\text{rad}-\sin 2A=\text{rad}-\frac{2\cos A\,.\,\sin A}{\text{rad}}=\frac{\text{rad}^2-2\cos A\,.\,\sin A}{\text{rad}},$$

and chord $2A=2\sin A$.

(229) If $\frac{1}{2}A$ be substituted for A in each of the foregoing expressions, we shall have,

$$\text{I. } \textit{sin}\ \text{A} = \frac{2\cos\frac{1}{2}\text{A}\ .\ \sin\frac{1}{2}\text{A}}{\text{rad}} = \frac{2\sin^2\frac{1}{2}\text{A}}{\tan\frac{1}{2}\text{A}}, \ \&\text{c.}$$

$$\text{II. } \textit{cos}\ \text{A} = \frac{\cos^2\frac{1}{2}\text{A} - \sin^2\frac{1}{2}\text{A}}{\text{rad}} = \frac{\text{rad}^2 - 2\sin^2\frac{1}{2}\text{A}}{\text{rad}}, \ \&\text{c.}$$

$$\text{III. } \textit{tan}\ \text{A} = \frac{2\,\text{rad}^2\ .\ \tan\frac{1}{2}\text{A}}{\text{rad}^2 - \tan^2\frac{1}{2}\text{A}} = \frac{2\,\text{rad}^2}{\cot\frac{1}{2}\text{A} - \tan\frac{1}{2}\text{A}} \ \&\text{c.}$$

$$\text{IV. } \textit{cot}\ \text{A} = \frac{\text{rad}^2 - \tan^2\frac{1}{2}\text{A}}{2\tan\frac{1}{2}\text{A}} = \frac{\cot^2\frac{1}{2}\text{A} - \text{rad}^2}{2\cot\frac{1}{2}\text{A}}, \ \&\text{c.}$$

$$\text{V. } \textit{sec}\ \text{A} = \frac{\text{rad}\ .\ \sec^2\frac{1}{2}\text{A}}{2\,\text{rad}^2 - \sec^2\frac{1}{2}\text{A}} = \frac{\text{rad}\ .\ \sec\frac{1}{2}\text{A}}{2\cos\frac{1}{2}\text{A} - \sec\frac{1}{2}\text{A}}, \ \&\text{c.}$$

$$\text{VI. } \textit{cosec}\ \text{A} = \frac{\sec\frac{1}{2}\text{A}\ .\ \text{cosec}\,\frac{1}{2}\text{A}}{2\,\text{rad}} = \frac{\text{rad}\ .\ \sec\frac{1}{2}\text{A}}{2\sin\frac{1}{2}\text{A}}, \ \&\text{c.}$$

$$\text{VII. } \textit{vers}\ \text{A} = \frac{2\sin^2\frac{1}{2}\text{A}}{\text{rad}} = \frac{2\,\text{rad}^2 - 2\cos^2\frac{1}{2}\text{A}}{\text{rad}}, \ \&\text{c.}$$

$$\text{VIII. } \textit{suvers}\ \text{A} = \frac{2\,\text{rad}^2 - 2\sin^2\frac{1}{2}\text{A}}{\text{rad}} = \frac{2\cos^2\frac{1}{2}\text{A}}{\text{rad}}, \ \&\text{c.}$$

(230) By finding the values of $\sin\frac{1}{2}$A, $\cos\frac{1}{2}$A, $\tan\frac{1}{2}$A, &c. from the most convenient of the foregoing equations, the sine, cosine, &c. of the half arc will be obtained in terms of the sine, cosine, &c. of the whole arc, by easy algebraic reductions.

$$\text{I. } \textit{sin}\ \tfrac{1}{2}\text{A} = \text{rad}\sqrt{\frac{\text{rad} - \cos\text{A}}{2\,\text{rad}}} = \text{rad}\sqrt{\frac{\sec\text{A} - \text{rad}}{2\sec\text{A}}} = \sqrt{\frac{\text{rad}^2 - \text{rad}\ .\ \cos\text{A}}{2}} = \tfrac{1}{2}\left\{\sqrt{\text{rad}^2 + \text{rad}\ .\ \sin\text{A}} - \sqrt{\text{rad}^2 - \text{rad}\ .\ \sin\text{A}}\right\} = \sqrt{\frac{\text{rad}\ .\ \text{vers}\,\text{A}}{2}} = \text{rad}\sqrt{\frac{2\,\text{rad} - \text{suvers}\,\text{A}}{2\,\text{rad}}} = \tfrac{1}{2}\,\text{chord}\ \text{A}.$$

$$\text{II. } \textit{cos}\ \tfrac{1}{2}\text{A} = \text{rad}\sqrt{\frac{\text{rad} + \cos\text{A}}{2\,\text{rad}}} = \text{rad}\sqrt{\frac{\sec\text{A} + \text{rad}}{2\sec\text{A}}} = \sqrt{\frac{\text{rad}^2 + \text{rad}.\cos\text{A}}{2}} = \tfrac{1}{2}\left\{\sqrt{\text{rad}^2 + \text{rad}.\sin\text{A}} + \sqrt{\text{rad}^2 - \text{rad}.\sin\text{A}}\right\} = \sqrt{\frac{2\,\text{rad}^2 - \text{rad}\ .\ \text{vers}\,\text{A}}{2}} = \sqrt{\frac{\text{rad}\ .\ \text{suvers}\,\text{A}}{2}}.$$

$$\text{III. } \textit{tan}\ \tfrac{1}{2}\text{A} = \text{rad}\sqrt{\frac{\text{rad} - \cos\text{A}}{\text{rad} + \cos\text{A}}} = \text{rad}\sqrt{\frac{\sec\text{A} - \text{rad}}{\sec\text{A} + \text{rad}}} = \text{cosec}\ \text{A} - \cot\text{A} = \frac{\text{rad}\ .\ \sin\text{A}}{\text{rad} + \cos\text{A}} = \frac{\text{rad}^2 - \text{rad}\ .\ \cos\text{A}}{\sin\text{A}} = \text{rad}\sqrt{\frac{\text{vers}\,\text{A}}{2\,\text{rad} - \text{vers}\,\text{A}}} = \text{rad}\sqrt{\frac{2\,\text{rad} - \text{suvers}\,\text{A}}{\text{suvers}\,\text{A}}}.$$

IV. $\cot \frac{1}{2}\mathrm{A} = \mathrm{rad}\sqrt{\frac{\mathrm{rad} + \cos \mathrm{A}}{\mathrm{rad} - \cos \mathrm{A}}} = \mathrm{rad}\sqrt{\frac{\sec \mathrm{A} + \mathrm{rad}}{\sec \mathrm{A} - \mathrm{rad}}}$
$= \mathrm{cosec}\ \mathrm{A} + \cot \mathrm{A} = \frac{\mathrm{rad}\ .\ \sin \mathrm{A}}{\mathrm{rad} - \cos \mathrm{A}} = \frac{\mathrm{rad}^2 + \mathrm{rad}\ .\ \cos \mathrm{A}}{\sin \mathrm{A}} =$
$\mathrm{rad}\sqrt{\frac{2\ \mathrm{rad} - \mathrm{vers}\ \mathrm{A}}{\mathrm{vers}\ \mathrm{A}}} = \mathrm{rad}\sqrt{\frac{\mathrm{suvers}\ \mathrm{A}}{2\ \mathrm{rad} - \mathrm{suvers}\ \mathrm{A}}}.$

V. $\sec \frac{1}{2}\mathrm{A} = \mathrm{rad}\sqrt{\frac{2\ \mathrm{rad}}{\mathrm{rad} + \cos \mathrm{A}}} = \mathrm{rad}\sqrt{\frac{2 \sec \mathrm{A}}{\mathrm{rad} + \sec \mathrm{A}}} =$
$\mathrm{rad}\sqrt{\frac{2 \tan \mathrm{A}}{\sin \mathrm{A} + \tan \mathrm{A}}} = \mathrm{rad}\sqrt{\frac{2\ \mathrm{rad}}{2\,\mathrm{rad} - \mathrm{vers}\ \mathrm{A}}} = \mathrm{rad}\sqrt{\frac{2\ \mathrm{rad}}{\mathrm{suvers}\ \mathrm{A}}}.$

VI. $\mathrm{cosec}\ \frac{1}{2}\mathrm{A} = \mathrm{rad}\sqrt{\frac{2\ \mathrm{rad}}{\mathrm{rad} - \cos \mathrm{A}}} = \mathrm{rad}\sqrt{\frac{2 \sec \mathrm{A}}{\sec \mathrm{A} - \mathrm{rad}}} =$
$\mathrm{rad}\sqrt{\frac{2 \tan \mathrm{A}}{\tan \mathrm{A} - \sin \mathrm{A}}} = \mathrm{rad}\sqrt{\frac{2\ \mathrm{rad}}{\mathrm{vers}\ \mathrm{A}}} = \mathrm{rad}\sqrt{\frac{2\ \mathrm{rad}}{2\,\mathrm{rad} - \mathrm{suvers}\ \mathrm{A}}}.$

And in the same manner the versed sines, coversed sines, chords, &c. of the half arcs may be found.

GENERAL PROPERTIES OF SINES, TANGENTS, ETC. OF THE SUMS, AND OF THE DIFFERENCES OF ARCS.

PROPOSITION XIII. (*Plate I. Fig.* 2.)

(231) *The sum of the sines of two arcs is to their difference, as the tangent of half the sum of those arcs is to the tangent of half their difference.*

Let BA and BO be the two arcs; draw the diameter BX, and OD and AG perpendicular to it. Produce OD to meet the circumference in F, and draw HFN parallel to the diameter; join AO, and produce AG and AO to H and N; from the centre C draw C*n*E perpendicular to AO, and OP to AG; and at E draw EIK perpendicular to C*n*E, meeting CO and CB (produced) in I and K.

Because OF is perpendicular to BC it is bisected in D (3 Euclid III): hence PH is bisected in G; therefore AH is the sum of the sines AG and OD, and AP their difference. The arcs AO and OF are bisected in E and B (30 Euclid III), therefore AF is the sum of the arcs BA and BO, and AO their difference; hence BE is the half sum, and OE the half difference; to which EK and EI are tangents.

Now PH being bisected in G, ON will be bisected in M; also since AO is bisected in *n*, and ON in M, *n*M will be the half of AN; therefore

AN : AO : : *n*M : *n*O and
AN : AO : : AH : AP per similar triangles.

Hence $nM : nO :: AH : AP$;
but $nM : nO :: EK : EI$ per similar triangles;
therefore $AH : AP :: EK : EI$; or
$\sin AB + \sin OB : \sin AB - \sin OB :: \tan \frac{1}{2}(BA + BO) : \tan \frac{1}{2}(BA - BO)$.

(232) Hence, if the two arcs BA and BO be represented by A and B, we shall have

$$\sin A + \sin B : \sin A - \sin B :: \tan \tfrac{1}{2}(A+B) : \tan \tfrac{1}{2}(A-B).$$

$$\text{or, } \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

PROPOSITION XIV. (*Plate I. Fig.* 2.)

(233) *The sum of the cosines of two arcs is to their difference, as the cotangent of half the sum of those arcs is to the tangent of half their difference.*

Let BA and BO be the two arcs, as in the preceding proposition, BE their half sum, and OE their half difference.

Draw nQ and CS parallel to AG, and nit to BC; also produce CA to T, and nA, ET to r, S. Then because AO is bisected in n, GD ($=$PO) will be bisected in Q, viz. $GQ = QD$.

Hence $2CQ = CD + CG$ the sum of the cosines,
and $2GQ = CD - CG$ the difference of the cosines.
Now $nt\ (=CQ) : ni\ (=GQ) :: nr : nA$ by similar triangles,
or $2CQ : 2GQ :: nr : nA$
But $nr : nA :: ES : ET$
therefore $2CQ : 2GQ :: ES : ET$, where ES is the tangent of the complement of the arc BE, or its cotangent, and $ET = EI$ the tangent of the arc OE.
$\therefore \cos BO + \cos BA : \cos BO - \cos BA :: \cot \frac{1}{2}(BA + BO) : \tan \frac{1}{2}(BA - BO)$.

(234) Let the two arcs BA and BO be represented by A and B,

$$\text{then } \frac{\cos B + \cos A}{\cos B - \cos A} = \frac{\cot \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

PROPOSITION XV. (*Plate I. Fig.* 2.)

(235) *The sum of the tangents of two arcs is to their difference, as the sine of the sum of the arcs is to the sine of their difference.*

Let BE and OE be the two arcs, the construction remaining as in Props. XIII. and XIV.

Because AO is bisected in n, IT will be bisected in E, so that TK is equal to the sum of the tangents EI and EK, and IK to their difference. Since the arc AO is bisected in E, the arc BA is equal to the sum of the arcs BE and OE, and BO to their difference; also AG and OD are the sines of the arcs BA and BO.

The lines TK and AM being parallel, we have

$$TK : IK :: AM : OM;$$

$$\text{but } AM : OM :: AG : PG = OD,$$

$$\therefore TK : IK :: AG : OD.$$

(236) If the two arcs BE and OE be represented by A and B

$$\text{then } \frac{\tan A + \tan B}{\tan A - \tan B} = \frac{\sin (A+B)}{\sin (A-B)}.$$

PROPOSITION XVI. (*Plate I. Fig.* 3.)

(237) *The difference between the rectangle of the sines and the rectangle of the cosines of two arcs, is equal to the rectangle of the radius and the cosine of the sum of these arcs.*

Let BO and OA be the two arcs, OD and An their sines, CD and Cn their cosines; then AG will be the sine of the sum of the arcs, and CG will be the cosine.

Draw ni parallel to CB, and nQ parallel to OD; then the triangles COD, CnQ, and Ani, are equiangular and similar; for the $\angle AnC = \angle inQ$, are each right angles, and if the common $\angle inC$ be taken away, there will remain the $\angle Ani = \angle CnQ$.

Then if the arc BO be represented by A, and the arc AO by B, we shall have

$$CO : Cn :: CD : CQ, \textit{ viz. } \text{rad} : \cos B :: \cos A : CQ = \frac{\cos A \,.\, \cos B}{\text{rad}}$$

$$CO : OD :: An : ni = GQ, \textit{ viz. } \text{rad} : \sin A :: \sin B : GQ = \frac{\sin A \,.\, \sin B}{\text{rad}}.$$

But CQ − GQ = CG (the cosine of the arc BA) = cos (A + B)

$$\text{therefore } \frac{\cos A \,.\, \cos B}{\text{rad}} - \frac{\sin A \,.\, \sin B}{\text{rad}} = \cos (A+B)$$

$$\text{that is } \cos A \,.\, \cos B - \sin A \,.\, \sin B = \text{rad} \,.\, \cos (A+B).$$

(238) The same construction remaining, we shall have

$$CO : Cn :: OD : nQ, \textit{ viz. } \text{rad} : \cos B :: \sin A : nQ = \frac{\sin A \,.\, \cos B}{\text{rad}}$$

$$CO : CD :: An : Ai, \textit{ viz. } \text{rad} : \cos A :: \sin B : Ai = \frac{\cos A \,.\, \sin B}{\text{rad}}$$

But $nQ + Ai = AG$ (the sine of the arc BA) = sin (A + B)

$$\text{therefore } \frac{\sin A \,.\, \cos B}{\text{rad}} + \frac{\cos A \,.\, \sin B}{\text{rad}} = \sin (A+B)$$

$$\text{that is } \sin A \,.\, \cos B + \cos A \,.\, \sin B = \text{rad} \,.\, \sin (A+B).$$

PROPOSITION XVII. (*Plate I. Fig.* 3.)

(239) *The sum of the rectangle of the sines and the rectangle of the cosines of two arcs, is equal to the rectangle of the radius and the cosine of the difference of these arcs.*

Let BO and OA be the two arcs, make $OR = AO$, then the chord AR will be bisected in *n*, and BR will be the difference between the arcs BO and AO; through R draw RP parallel to GB, and RH parallel to AG, then will AG be the sine of the sum of the arcs BO and AO, and RH the sine of their difference. By similar triangles $An : AR :: Ai : AP$; but An is the half of AR, therefore Ai is the half of AP, that is, $Ai = iP$.

Now $CQ + GQ = CQ + QH = CH = \cos (A - B)$; and from the investigation of the preceding proposition, we have

$$CQ + GQ = \frac{\cos A \,.\, \cos B}{\text{rad}} + \frac{\sin A \,.\, \sin B}{\text{rad}};$$

$$\therefore \frac{\cos A \,.\, \cos B}{\text{rad}} + \frac{\sin A \,.\, \sin B}{\text{rad}} = \cos (A - B),$$

$$\therefore \cos A \,.\, \cos B + \sin A \,.\, \sin B = \text{rad.} \cos (A - B).$$

(240) Again, $nQ - Ai = Gi - Pi = PG = RH = \sin (A - B)$; and it may be shown, as in art. (238) that

$$nQ - Ai = \frac{\sin A \,.\, \cos B}{\text{rad}} - \frac{\cos A \,.\, \sin B}{\text{rad}};$$

$$\therefore \sin A \,.\, \cos B - \cos A \,.\, \sin B = \text{rad} \,.\, \sin (A - B).$$

PROPOSITION XVIII. (*Plate I. Fig.* 3.)

(241) *The square of the radius is to the square of the radius diminished by the rectangle of the tangents of two arcs, as the tangents of the sum of these arcs is to the sum of their tangents.*

The same construction remaining, let BO and AO be the arcs. Draw the tangent B*t*, and produce CO, CA to meet it in T, *t*; through A draw AI perpendicular to C*t*, and through T draw ET parallel to AI: when the arc AO is greater than the arc BO, the line ET will fall within the circle, but that will not affect the investigation.

The triangles CAI and CET are equiangular and similar; also the triangles *t*BC and *t*ET are equiangular and similar; for the triangles CBT and *t*ET are right angled at B and E, and have the angle at *t* common.

Hence $tT\ (= tB - TB) : tE\ (= tC - EC) :: tC : tB$,

or $tB^2 - tB \,.\, TB = tC^2 - tC \,.\, EC$,

$\therefore tC \,.\, EC = tC^2 - tB^2 + tB \,.\, TB = CB^2 + tB \,.\, TB.$

$$\text{By similar triangles} \left\{ \begin{array}{l} tT : ET :: tC : CB \\ ET : AI :: EC : AC = CB \end{array} \right.$$

$$\text{Hence } t\text{T} : \text{AI} :: t\text{C} . \text{EC} : \text{CB}^2$$
$$\text{or } t\text{B} - \text{TB} : \text{AI} :: \text{CB}^2 + t\text{B} . \text{TB} : \text{CB}^2$$
$$\therefore t\text{B} . \text{CB}^2 - \text{TB} . \text{CB}^2 = \text{AI} . \text{CB}^2 + \text{AI} . t\text{B} . \text{TB},$$
$$\text{or } t\text{B} (\text{CB}^2 - \text{AI} . t\text{B}) = \text{CB}^2 (\text{AI} + \text{TB}),$$
$$\text{hence } \text{CB}^2 : \text{CB}^2 - \text{AI} . \text{TB} :: t\text{B} : \text{AI} + \text{TB}.$$

(242) *The square of the radius is to the square of the radius increased by the rectangle of the tangents of two arcs, as the tangent of the difference of these arcs is to the difference of their tangents.*

Let BA and BO be the two arcs; then the last analogy but one in the investigation of preceding article is the property here enunciated.

(243) If two arcs be represented by A and B,

$$\text{then } \frac{\text{rad}^2}{\text{rad}^2 - \tan \text{A} . \tan \text{B}} = \frac{\tan (\text{A}+\text{B})}{\tan \text{A} + \tan \text{B}}$$

$$\text{and } \frac{\text{rad}^2}{\text{rad}^2 + \tan \text{A} . \tan \text{B}} = \frac{\tan (\text{A}-\text{B})}{\tan \text{A} - \tan \text{B}}.$$

(244) From the propositions already given, a great variety of formulæ for the sums and differences of arcs may be deduced, some of the most important of which will here be pointed out.

By Props. XVI. and XVII. it is shown,

$$\sin (\text{A}+\text{B}) = \frac{\sin \text{A} . \cos \text{B} + \cos \text{A} . \sin \text{B}}{\text{rad}} \quad . \; . \; . \quad (1)$$

$$\cos (\text{A}+\text{B}) = \frac{\cos \text{A} . \cos \text{B} - \sin \text{A} . \sin \text{B}}{\text{rad}} \quad . \; . \; . \quad (2)$$

$$\sin (\text{A}-\text{B}) = \frac{\sin \text{A} . \cos \text{B} - \cos \text{A} . \sin \text{B}}{\text{rad}} \quad . \; . \; . \quad (3)$$

$$\cos (\text{A}-\text{B}) = \frac{\cos \text{A} . \cos \text{B} + \sin \text{A} . \sin \text{B}}{\text{rad}} \quad . \; . \; . \quad (4)$$

(245) By adding the first and third equations together, and dividing by 2,

$$\frac{\sin \text{A} . \cos \text{B}}{\text{rad}} = \tfrac{1}{2} \sin (\text{A}+\text{B}) + \tfrac{1}{2} \sin (\text{A}-\text{B}) \quad . \; . \quad (5)$$

By subtracting the third equation from the first, and dividing by 2,

$$\frac{\cos \text{A} . \sin \text{B}}{\text{rad}} = \tfrac{1}{2} \sin (\text{A}+\text{B}) - \tfrac{1}{2} \sin (\text{A}-\text{B}) \quad . \; . \; . \quad (6)$$

By adding the second and fourth equations together, and dividing by 2,

$$\frac{\cos \text{A} . \cos \text{B}}{\text{rad}} = \tfrac{1}{2} \cos (\text{A}-\text{B}) + \tfrac{1}{2} \cos (\text{A}+\text{B}) \quad . \; . \; . \quad (7)$$

By subtracting the second equation from the fourth, and dividing by 2,

$$\frac{\sin A \cdot \sin B}{\text{rad}} = \tfrac{1}{2} \cos (A-B) - \tfrac{1}{2} \cos (A+B) \quad . \; . \; . \; . \quad (8)$$

(246) If in these formulæ there be substituted P for $A+B$, and Q for $A-B$, then $A=\frac{1}{2}(P+Q)$, $B=\frac{1}{2}(P-Q)$, and we shall obtain

$$\sin P + \sin Q = \frac{2}{\text{rad}} \cdot \sin \tfrac{1}{2}(P+Q) \cdot \cos \tfrac{1}{2}(P-Q) \quad . \; . \quad (9)$$

$$\sin P - \sin Q = \frac{2}{\text{rad}} \cdot \sin \tfrac{1}{2}(P-Q) \cdot \cos \tfrac{1}{2}(P+Q) \quad . \; . \quad (10)$$

$$\cos P + \cos Q = \frac{2}{\text{rad}} \cdot \cos \tfrac{1}{2}(P+Q) \cdot \cos \tfrac{1}{2}(P-Q) \quad . \; . \quad (11)$$

$$\cos Q - \cos P = \frac{2}{\text{rad}} \cdot \sin \tfrac{1}{2}(P+Q) \cdot \sin \tfrac{1}{2}(P-Q) \quad . \; . \quad (12)$$

(247) And because $\frac{\sin A}{\cos A} = \frac{\tan A}{\text{rad}} = \frac{\text{rad}}{\cot A}$, we shall obtain by division and reduction,

$$\frac{\sin P + \sin Q}{\sin P - \sin Q} = \frac{\sin \frac{1}{2}(P+Q) \cdot \cos \frac{1}{2}(P-Q)}{\cos \frac{1}{2}(P+Q) \cdot \sin \frac{1}{2}(P-Q)} = \frac{\tan \frac{1}{2}(P+Q)}{\tan \frac{1}{2}(P-Q)} \quad (13)$$

$$\frac{\sin P + \sin Q}{\cos P + \cos Q} = \frac{\sin \frac{1}{2}(P+Q)}{\cos \frac{1}{2}(P+Q)} = \frac{\tan \frac{1}{2}(P+Q)}{\text{rad}} \quad . \; . \; . \; . \quad (14)$$

$$\frac{\sin P + \sin Q}{\cos Q - \cos P} = \frac{\cos \frac{1}{2}(P-Q)}{\sin \frac{1}{2}(P-Q)} = \frac{\cot \frac{1}{2}(P-Q)}{\text{rad}} \quad . \; . \; . \; . \quad (15)$$

$$\frac{\sin P - \sin Q}{\cos P + \cos Q} = \frac{\sin \frac{1}{2}(P-Q)}{\cos \frac{1}{2}(P-Q)} = \frac{\tan \frac{1}{2}(P-Q)}{\text{rad}} \quad . \; . \; . \; . \quad (16)$$

$$\frac{\sin P - \sin Q}{\cos Q - \cos P} = \frac{\cot \frac{1}{2}(P+Q)}{\sin \frac{1}{2}(P+Q)} = \frac{\cot \frac{1}{2}(P+Q)}{\text{rad}} \quad . \; . \; . \; . \quad (17)$$

$$\frac{\cos P + \cos Q}{\cos Q - \cos P} = \frac{\cos \frac{1}{2}(P+Q) \cdot \cos \frac{1}{2}(P-Q)}{\sin \frac{1}{2}(P+Q) \cdot \sin \frac{1}{2}(P-Q)} = \frac{\cot \frac{1}{2}(P+Q)}{\tan \frac{1}{2}(P-Q)} \quad (18)$$

$$\frac{\sin (P+Q)}{\sin P + \sin Q} = \frac{2 \sin \frac{1}{2}(P+Q) \cdot \cos \frac{1}{2}(P+Q)}{2 \sin \frac{1}{2}(P+Q) \cdot \cos \frac{1}{2}(P-Q)} = \frac{\cos \frac{1}{2}(P+Q)}{\cos \frac{1}{2}(P-Q)} \quad (19)$$

$$\frac{\sin (P+Q)}{\sin P - \sin Q} = \frac{2 \sin \frac{1}{2}(P+Q) \cdot \cos \frac{1}{2}(P+Q)}{2 \sin \frac{1}{2}(P-Q) \cdot \cos \frac{1}{2}(P+Q)} = \frac{\sin \frac{1}{2}(P+Q)}{\sin \frac{1}{2}(P-Q)} \quad (20)$$

Now if two fractions be equal to each other, the numerator of the one is to its denominator, as the numerator of the other is to its denominator. Equation (13) is therefore the same as Prop. XIII., and Equation (18) is the same as Prop. XIV.

(248) If $B=A$, we shall have, from the equations (7) and (8),

$$\cos^2 A = \tfrac{1}{2} \text{rad}^2 + \tfrac{1}{2} \text{rad} \cdot \cos 2A$$
$$\sin^2 A = \tfrac{1}{2} \text{rad}^2 - \tfrac{1}{2} \text{rad} \cdot \cos 2A.$$

(249) Also when $B=A$, $Q=o$ (246) and $\cos 0°=\text{rad}$; from equations in arts. (246) and (247), we have

$$\text{rad}+\cos P=\frac{2\cos^2\frac{1}{2}P}{\text{rad}}$$

$$\text{rad}-\cos P=\frac{2\sin^2\frac{1}{2}P}{\text{rad}}$$

$$\sin P=\frac{2\sin\frac{1}{2}P\,.\cos\frac{1}{2}P}{\text{rad}}$$

$$\frac{\sin P}{\text{rad}+\cos P}=\frac{\tan\frac{1}{2}P}{\text{rad}}=\frac{\text{rad}}{\cot\frac{1}{2}P}$$

$$\frac{\sin P}{\text{rad}-\cos P}=\frac{\cot\frac{1}{2}P}{\text{rad}}=\frac{\text{rad}}{\tan\frac{1}{2}P}$$

$$\frac{\text{rad}+\cos P}{\text{rad}-\cos P}=\frac{\cot\frac{1}{2}P}{\tan\frac{1}{2}P}=\frac{\cot^2\frac{1}{2}P}{\text{rad}^2}=\frac{\text{rad}^2}{\tan^2\frac{1}{2}P}.$$

(250) Because $\tan(A+B)=\frac{\text{rad}\,.\sin(A+B)}{\cos(A+B)}$ we have from the first and second equations (244),

$$\tan(A+B)=\frac{\sin A\,.\cos B+\sin B\,.\cos A}{\cos A\,.\cos B-\sin B\,.\sin A}\,.\,\text{rad},$$

and since $\sin A=\frac{\cos A\,.\tan A}{\text{rad}}$, and $\sin B=\frac{\cos B\,.\tan B}{\text{rad}}$, by substituting these values and dividing the numerator and denominator by $\cos A\,.\cos B$, we obtain

$$\tan(A+B)=\frac{\tan A+\tan B}{\text{rad}^2-\tan A\,.\tan B}\,.\,\text{rad}^2,$$

which has been already obtained by another process in art. (243).

Also $\tan(A-B)=\frac{\tan A-\tan B}{\text{rad}^2+\tan A\,.\tan B}\,.\,\text{rad}^2$ by art (243)

and $\therefore\cot(A+B)=\frac{\text{rad}^2-\tan A\,.\tan B}{\tan A+\tan B}$ by art. (211)

and $\cot(A-B)=\frac{\text{rad}^2+\tan A\,.\tan B}{\tan A-\tan B}$ by art. (211).

(251) Let A, B, C, be any three arcs, and consider A+B as one, then by equations (1) and (2) of art (244),

$$\sin(A+B+C)=\frac{\sin(A+B)\,.\cos C+\cos(A+B)\,.\sin C}{\text{rad}}$$

$$=\frac{(\sin A\,.\cos B+\cos A\,.\sin B)\,.\cos C}{\text{rad}^2}$$

$$+\frac{(\cos A\,.\cos B-\sin A\,.\sin B)\,.\sin C}{\text{rad}^2}$$

$$\cos(A+B+C)=\frac{\cos(A+B)\cdot\cos C-\sin(A+B)\cdot\sin C}{\text{rad}}$$
$$=\frac{(\cos A\cdot\cos B-\sin A\cdot\sin B)\cdot\cos C}{\text{rad}^2}$$
$$-\frac{(\sin A\cdot\cos B+\cos A\cdot\sin B)\cdot\sin C}{\text{rad}^2}.$$

Let now the sum of the three arcs be 180°, or any multiple of 180°, then the sine of this sum will be 0, so that the first of these equations gives

$$\sin A\cdot\sin B\cdot\sin C=\sin A\cdot\cos B\cdot\cos C+\cos A\cdot\sin B\cdot\cos C+\cos A\cdot\cos B\cdot\sin C;$$

dividing both sides of this equation by $\cos A\cdot\cos B\cdot\cos C$, we have

$$\frac{\sin A}{\cos A}\cdot\frac{\sin B}{\cos B}\cdot\frac{\sin C}{\cos C}=\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}+\frac{\sin C}{\cos C};$$

that is,

$$\tan A\cdot\tan B\cdot\tan C=\tan A+\tan B+\tan C;$$

a remarkable property of the angles of a plane triangle.

Again, let the sum of the three arcs be 90°, or any multiple thereof, then the cosine of this sum will be 0, so that the second general equation above becomes,

$$\cos A\cdot\cos B\cdot\cos C=\cos A\cdot\sin B\cdot\sin C+\sin A\cdot\cos B\cdot\sin C+\sin A\cdot\sin B\cdot\cos C;$$

Dividing both sides by $\sin A\cdot\sin B\cdot\sin C$, we have

$$\cot A\cdot\cot B\cdot\cot C=\cot A+\cot B+\cot C.$$

GENERAL PROPERTIES OF SINES, TANGENTS, ETC. OF ARCS, IN ARITHMETICAL PROGRESSION.

PROPOSITION XIX. (*Plate I. Fig.* 2.)

(252) Let the three arcs BO, BE, and BA, be in arithmetical progression; viz. let AO be bisected in E, then AE=OE is half the difference between the arcs BO and BA, and therefore BE is half the sum of the arcs BO and BA (116).

Also BO+BA=BO+AE+BE=BO+OE+BE=2BE, that is the sum of the extreme arcs, BO and BA, is equal to double the mean arc BE.

Draw the diameter ACL, and join OL; draw BZ parallel to OA, join XZ and draw ZV perpendicular to BX; also from L, draw LW perpendicular, and OW parallel to BX.

Then the following triangles are equiangular and similar, *viz.* CBR, ZBV, XZV, AOP, and OLW. For CBR and ZBV are right angled at R and V, and have the angle at B common. Also XZV is right angled at V, and the angle XZV is equal to the angle VBZ, for each is the complement of VZB.

The triangle AOP having the sides AO and OP parallel to the two sides ZB and BV of the triangle ZBV, is equiangular with it; the angle AOL is a right angle, being contained in a semicircle, therefore WOL, the complement of AOP, is equal to the angle OAP: hence the angle OLW is equal to the angle AOP.

The following proportions, &c. are naturally derived from these similar triangles; but first we must observe that AG + OD = AH = WL the sum of the sines of the extreme arcs; for AH is parallel to WL, and the angles WLC and HAC being alternate angles are equal (29 Euclid I), therefore the right-angled triangles CGA, CYL having one side AC = LC, and the angles equal, are equal in all respects; hence LY = AG, and WY = OD, consequently LW = AG + OD. Again CG + CD is the sum of the cosines of the extreme arcs; for on account of the equality of the triangles LYC and AGC, YC = CG, therefore CG + CD = YC + CD = YD = WO, is the sum of the cosines of the extreme arcs.

(A) CB : BR : : BZ : BV
radius : sine of the mean arc : : double the sine of the mean arc : versed sine of double the mean arc.

(B) CB : BR : : ZX (= 2RC) : ZV
radius : sine of the mean arc : : double the cosine of the mean arc : sine of twice the mean arc.

(C) CB : BR : : AO (= 2O*n*) : OP (= CD − CG)
radius : sine mean arc : : double the sine of the common difference between the arcs : difference between the cosines, or versed sines of the extreme arcs.

(D) CB : BR : : OL (= 2C*n*) : LW
radius : sine mean arc : : double the cosine of the common difference between the arcs : the sum of the sines of the extreme arcs.

(E) CB : CR : : BZ (= 2BR) : ZV
radius : cosine of the mean arc : : double the sine of the mean arc : sine of double the mean arc.

(F) CB : CR : : ZX (= 2CR) : XV
radius : cosine mean arc : : double the cosine of the mean arc : suversed sine of double the mean arc.

(G) CB : CR : : AO (= 2O*n*) : AP (= AG − OD)
radius : cosine of mean arc : : double the sine of the common difference between the arcs : difference between the sines of the extreme arcs.

(H) CB : CR : : OL (=2C*n*) : WO (=CG+CD)
radius : cosine of the mean arc : : double the cosine of the common difference between the arcs : sum of the cosines of the extreme arcs.

(K) BR : CR : : BV : ZV
sine of the mean arc : cosine of the mean arc : : versed sine of double the mean arc : sine of double the mean arc.

(L) BR : CR : : ZV : XV
sine of the mean arc : cosine of the mean arc : : sine of double the mean arc : suversed sine of double the mean arc.

(M) BR : CR : : PO (=CD—CG) : AP (=AG—OD)
sine of the mean arc : cosine of the mean arc : : difference between the cosines (=difference between the versed sines) of the extreme arcs : the difference between the sines of the extreme arcs.

(N) BR : CR : : LW (=AG+OD) : WO (=CG+CD)
sine of the mean arc : cosine of the mean arc : : the sum of the sines of the extreme arcs : the sum of the cosines of the extreme arcs.

By comparing the above proportions, various others may be formed. Thus BZ : BV : : ZX : ZV by comparing (A) with (B), that is double the sine of the mean arc : versed sine of double the mean arc : : double the cosine of the mean arc : sine of double the mean arc, &c. for the rest.

(253) By what has been already shown, *two* arcs may be compared, and their sums and differences formed into a proportion. The arcs being in arithmetical progression, the mean arc BE is half the sum of the extreme arcs BO and BA. Therefore, in all the above proportions, and their variations, whereever the words *mean arc* occur, read *half the sum of the arcs.* Also the sine O*n*, of the arc EO, the common difference between the arcs, is the sine of half the difference between the arcs BO and BA; hence in the above proportions for *common difference* between the arcs, read *half the difference* between the arcs. Thus for example (C) will run thus; radius : sine of half the sum of the arcs : : double the sine of half the difference of the arcs : difference between the cosines of the arcs.

(254) Hence it follows that the difference between the cosines of two arcs × radius =2 × sine of half the difference × sine of half the sum of these arcs.

But this has been obtained in another manner, see the fourth equation (246).

Indeed from this proposition, the greater part of the formulæ noted in Prop. XVIII. may be deduced, and it will be no bad exercise for the young student to trace out the coincidence.

PROPOSITION XX. (*Plate I. Fig.* 2.)

(255) *If three arcs be in arithmetical progression, the sum of the sines of the extreme arcs is to their difference, as the tangent of the mean arc is to the tangent of the common difference between the arcs.*

The same construction remaining as in the preceding proposition, AH = AG + OD = sum of the sines of the extreme arcs BA and BO, and AP = AG − OD = the difference of the sines of the extreme arcs.

By Prop. XIII. AH : AP : : EK : EI.

Now EK is the tangent of the mean arc BE, and EI is the tangent of EO, the common difference between the arcs. Hence the proposition is evident.

(256) If the extreme arcs be represented by A and B, of which A is greater than B, then

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}. \quad \text{See (253).}$$

PROPOSITION XXI. (*Plate I. Fig.* 2.)

(257) *If three arcs be in arithmetical progression, radius is to double the cosine of the middle arc, as the sine of the common difference of the arcs is to the difference between the sines of the extreme arcs.*

Let the three arcs be BO, BE, and BA, as in the preceding propositions, and OE = AE the common difference between the arcs. Then An is the sine of AE, and AP = difference between the sines of BA and BO.

The triangles Ain, CnQ, are equiangular and similar; for AnC and inQ are right angles, and taking inC away from both, there will remain the ∠ Ani = ∠ CnQ. The triangle CnQ is also equiangular with the triangle CEm.

Therefore CE : Cm : : An : Ai; but by (239) Ai = iP.

Hence CE : Cm : : An : iP and doubling the consequents,

CE : 2Cm : : An : AP.

(258) If A and B represent the extreme arcs, A being the greater,

$$\frac{\text{rad}}{2\cos\frac{1}{2}(A+B)} = \frac{\sin\frac{1}{2}(A-B)}{\sin A - \sin B}.$$

PROPOSITION XXII. (*Plate I. Fig.* 2.)

(259) *If three arcs be in arithmetical progression, the rectangle of radius and the sum of the sines of the extreme arcs, is equal to the rectangle of double the cosine of the common difference of the arcs and the sine of the mean arc.*

The same construction remaining as in the foregoing propositions, $\text{CE} : \text{C}n :: \text{E}m : n\text{Q} = \dfrac{\text{C}n \times \text{E}m}{\text{CE}}$.

As $\text{GH} = \text{PG}$ and $\text{A}i = i\text{P}$, we have $\text{A}i + \text{GH} = i\text{P} + \text{PG} = n\text{Q}$, therefore $\text{AG} + \text{OD} = \text{AH} = \text{A}i + i\text{P} + \text{PG} + \text{GH} = 2n\text{Q}$.

$$\text{Hence } \text{AG} + \text{OD} = \frac{2\text{C}n \times \text{E}m}{\text{CE}}, \text{ that is}$$

$$(\text{AG} + \text{OD}) \cdot \text{CE} = 2\text{C}n \cdot \text{E}m.$$

(260) Let the extreme arcs be represented by A and B, then the mean arc will be $\frac{1}{2}(\text{A}+\text{B})$, and the common difference of the arcs $\frac{1}{2}(\text{A}-\text{B})$. Hence,

$\text{rad} (\sin \text{A} + \sin \text{B}) = 2 \cos \frac{1}{2}(\text{A}-\text{B}) \cdot \sin \frac{1}{2}(\text{A}+\text{B})$;

and if $\text{rad} = 1$,

$\sin \text{A} + \sin \text{B} = 2 \cos \frac{1}{2}(\text{A}-\text{B}) \cdot \sin \frac{1}{2}(\text{A}+\text{B})$.

Consequently, *if the sine of the mean of three equidifferent arcs be multiplied by twice the cosine of the common difference, and the sine of either of the extreme arcs be deducted from the product, the remainder will be the sine of the other extreme arc, the radius being* 1.

OF THE SINES, COSINES, TANGENTS, ETC. OF THE MULTIPLES OF ARCS.

(261) Taking $\text{rad} = 1$, we have by (244)

$\sin (\text{A}+\text{B}) = \sin \text{A} \cdot \cos \text{B} + \sin \text{B} \cdot \cos \text{A}$;

$\therefore \sin 2\text{A} = \sin (\text{A}+\text{A}) = 2 \sin \text{A} \cdot \cos \text{A}$.

Again, $\sin 3\text{A} = \sin (\text{A}+2\text{A}) = \sin \text{A} \cdot \cos 2\text{A} + \sin 2\text{A} \cdot \cos \text{A}$,

but by (231) $\cos 2\text{A} = 2 \cos^2 \text{A} - 1$,

hence $\sin \text{A} \cdot \cos 2\text{A} = \sin \text{A}\, (2 \cos^2 \text{A} - 1) =$

$2 \sin \text{A} \cdot \cos^2 \text{A} - \sin \text{A} = \sin 2\text{A} \cdot \cos \text{A} - \sin \text{A}$,

and $\therefore \sin 3\text{A} = \sin 2\text{A} \cdot \cos \text{A} - \sin \text{A} + \sin 2\text{A} \cdot \cos \text{A}$

$= 2 \sin 2\text{A} \cdot \cos \text{A} - \sin \text{A}$.

By making $\text{B} = 3\text{A}$, and following the same method,

$\sin 4\text{A} = 2 \cos \text{A} \cdot \sin 3\text{A} - \sin 2\text{A}$. Hence we obtain

$\sin \text{A} = \sin \text{A}$

$\sin 2\text{A} = 2 \cos \text{A} \cdot \sin \text{A}$

$\sin 3\text{A} = 2 \cos \text{A} \cdot \sin 2\text{A} - \sin \text{A}$

$\sin 4\text{A} = 2 \cos \text{A} \cdot \sin 3\text{A} - \sin 2\text{A}$

$$\begin{aligned}
\sin 5A &= 2\cos A \,.\, \sin 4A - \sin 3A \\
\sin 6A &= 2\cos A \,.\, \sin 5A - \sin 4A
\end{aligned}$$

where the law of continuation is evident.

By substituting the value of $\cos A = \sqrt{1 - \sin^2 A}$, &c.

$$\begin{aligned}
\sin A &= \sin A \\
\sin 2A &= 2\sin A \sqrt{1 - \sin^2 A} \\
\sin 3A &= 3\sin A - 4\sin^3 A \\
\sin 4A &= (4\sin A - 8\sin^3 A)\sqrt{1 - \sin^2 A} \\
\sin 5A &= 5\sin A - 20\sin^8 A + 16\sin^5 A \\
\sin 6A &= (6\sin A - 32\sin^5 A + 32\sin^5 A)\sqrt{1 - \sin^2 A}
\end{aligned}$$

$$\begin{aligned}
\sin A &= \sqrt{1 - \cos^2 A} \\
\sin 2A &= 2\cos A \sqrt{1 - \cos^2 A} \\
\sin 3A &= (4\cos^2 A - 1)\sqrt{1 - \cos^2 A} \\
\sin 4A &= (8\cos^3 A - 4\cos A)\sqrt{1 - \cos^2 A} \\
\sin 5A &= (16\cos^4 A - 12\cos^2 A + 1)\sqrt{1 - \cos^2 A} \\
\sin 6A &= (32\cos^5 A - 32\cos^3 A + 6\cos A)\sqrt{1 - \cos^2 A}
\end{aligned}$$

(262) Let rad $= 1$, then by art. (244)

$$\begin{aligned}
&\cos(A + B) = \cos A \,.\, \cos B - \sin A \,.\, \sin B; \\
&\therefore \cos 2A = \cos(A + A) = \cos^2 A - \sin^2 A \\
&\qquad = 2\cos^2 A - 1 = 1 - 2\sin^2 A.
\end{aligned}$$

Again, $\cos 3A = \cos(A + 2A) = \cos A \,.\, \cos 2A - \sin A \,.\, \sin 2A$, but by (261) $\sin A \,.\, \sin 2A = 2\sin^2 A \,.\, \cos A =$

$$\cos A\,(1 - \cos 2A) = \cos A - \cos A \,.\, \cos 2A;$$

hence $\cos 3A = \cos A \,.\, \cos 2A - \cos A + \cos A \,.\, \cos 2A$

$$= 2\cos A \,.\, \cos 2A - \cos A.$$

By making $B = 3A$, and pursuing the same method, we shall find $\cos 4A = 2\cos A \,.\, \cos 3A - \cos 2A$

$$\begin{aligned}
\cos A &= \cos A \\
\cos 2A &= 2\cos A \,.\, \cos A - 1 \\
\cos 3A &= 2\cos A \,.\, \cos 2A - \cos A \\
\cos 4A &= 2\cos A \,.\, \cos 3A - \cos 2A \\
\cos 5A &= 2\cos A \,.\, \cos 4A - \cos 3A \\
\cos 6A &= 2\cos A \,.\, \cos 5A - \cos 4A
\end{aligned}$$

By substituting the value of $\cos A = \sqrt{1 - \sin^2 A}$,

$$\begin{aligned}
\cos A &= \sqrt{1 - \sin^2 A} \\
\cos 2A &= 1 - 2\sin^2 A \\
\cos 3A &= (1 - 4\sin^2 A)\sqrt{1 - \sin^2 A} \\
\cos 4A &= 8\sin^4 A - 8\sin^2 A + 1 \\
\cos 5A &= (16\sin^4 A - 12\sin^2 A + 1)\sqrt{1 - \sin^2 A} \\
\cos 6A &= 1 - 18\sin^2 A + 48\sin^4 A - 32\sin^6 A
\end{aligned}$$

$$\begin{aligned}
\cos A &= \cos A \\
\cos 2A &= 2\cos^2 A - 1 \\
\cos 3A &= 4\cos^3 A - 3\cos A \\
\cos 4A &= 8\cos^4 A - 8\cos^2 A + 1 \\
\cos 5A &= 16\cos^5 A - 20\cos^3 A + 5\cos A \\
\cos 6A &= 32\cos^6 A - 48\cos^4 A + 18\cos^2 A - 1
\end{aligned}$$

(263) Making rad$=1$, we have by art. (250)

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

$$\therefore \tan 2A = \tan(A+A) = \frac{2\tan A}{1-\tan^2 A}.$$

Also $\tan 3A = \tan(A+2A) = \dfrac{\tan A + \tan 2A}{1 - \tan A \cdot \tan 2A}$

$$= \frac{\tan A + \dfrac{2\tan A}{1-\tan^2 A}}{1 - \tan A\left(\dfrac{2\tan A}{1-\tan^2 A}\right)} = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

Similarly for $\tan 4A$, &c. Hence

$$\begin{aligned}
\tan A &= \tan A \\
\tan 2A &= \frac{2\tan A}{1-\tan^2 A} \\
\tan 3A &= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} \\
\tan 4A &= \frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A} \\
\tan 5A &= \frac{5\tan A - 10\tan^3 A + \tan^5 A}{1 - 10\tan^2 A + 5\tan^4 A}
\end{aligned}$$

(264) And since $\tan A = \dfrac{1}{\cot A}$, if $\dfrac{1}{\cot A}$ be substituted for $\tan A$, we shall obtain.

$$\begin{aligned}
\cot A &= \cot A \\
\cot 2A &= \frac{\cot^2 A - 1}{2\cot A} \\
\cot 3A &= \frac{\cot^8 A - 3\cot A}{3\cot^2 A - 1} \\
\cot 4A &= \frac{\cot^4 A - 6\cot^2 A + 1}{4\cot^3 A - 4\cot A} \\
\cot 5A &= \frac{\cot^5 A - 10\cot^3 A + 5\cot A}{5\cot^4 A - 10\cot^2 A + 1}
\end{aligned}$$

Also if $\dfrac{1}{\sec A}$ be substituted for $\cos A$, the secant of the multiple of any arc may be obtained.

OF THE SINES AND COSINES OF THE POWERS OF ARCS.

(265) By art. (262) $\cos 2A = 1 - 2\sin^2 A$;
therefore $2\sin^2 A = 1 - \cos 2A$.
By art. (261) $\sin 3A = 3\sin A - 4\sin^3 A$;
hence $4\sin^3 A = 3\sin A - \sin 3A$.
By art. (262) $\cos 4A = 8\sin^4 A - 8\sin^2 A + 1$
$\therefore 8\sin^4 A = \cos 4A + 8\sin^2 A - 1$
$= \cos 4A - 4\cos 2A + 3$, by substituting for $8\sin^2 A$ its value $4 - 4\cos 2A$; and by following the same method we shall obtain

$$\sin A = \sin A$$
$$\sin^2 A = \tfrac{1}{2} - \tfrac{1}{2}\cos 2A$$
$$\sin^3 A = \tfrac{3}{4}\sin A - \tfrac{1}{4}\sin 3A$$
$$\sin^4 A = \tfrac{3}{8} - \tfrac{4}{8}\cos 2A + \tfrac{1}{8}\cos 4A$$
$$\sin^5 A = \tfrac{10}{16}\sin A - \tfrac{5}{16}\sin 3A + \tfrac{1}{16}\sin 5A$$
$$\sin^6 A = \tfrac{10}{32} - \tfrac{15}{32}\cos 2A + \tfrac{6}{32}\cos 4A - \tfrac{1}{32}\cos 6A$$
$$\sin^7 A = \tfrac{35}{64}\sin A - \tfrac{21}{64}\sin 3A + \tfrac{7}{64}\sin 5A - \tfrac{1}{64}\sin 7A$$

The law of continuation is obvious, for the odd powers are expressed in terms of the sines, and the even powers in terms of the cosines of the multiples of A; and the sines are alternately + and —.

The numerators of the co-efficients (reckoning from the right hand towards the left), are the co-efficients of a binomial whose power is the same as that of the sine of A; except in the even powers, where the term in which A is not found, has the numerator of its co-efficient only one half of the corresponding co-efficient of the binomial, and the denominators are 2 involved to the power of the sine — 1.

(266) To deduce formulæ for the successive powers of the cosine of any arc, we must apply the last set of cosines in art. (262) in the same manner as above; then

$$\cos A = \cos A$$
$$\cos^2 A = \tfrac{1}{2} + \tfrac{1}{2}\cos 2A$$
$$\cos^3 A = \tfrac{3}{4}\cos A + \tfrac{1}{4}\cos 3A$$
$$\cos^4 A = \tfrac{3}{8} + \tfrac{4}{8}\cos 2A + \tfrac{1}{8}\cos 4A$$
$$\cos^5 A = \tfrac{10}{16}\cos A + \tfrac{5}{16}\cos 3A + \tfrac{1}{16}\cos 5A$$
$$\cos^6 A = \tfrac{10}{32} + \tfrac{15}{32}\cos 2A + \tfrac{6}{32}\cos 4A + \tfrac{1}{32}\cos 6A$$
$$\cos^7 A = \tfrac{35}{64}\cos A + \tfrac{21}{64}\cos 3A + \tfrac{7}{64}\cos 5A + \tfrac{1}{64}\cos 7A$$

The law of continuation is the same as in the last article, only all the terms here are positive.

(267) The sine and cosine of any arc, or the sine and cosine of any multiple of that arc, may also be derived by substituting

the imaginary quantities $x\sqrt{-1}$ and $-x\sqrt{-1}$, successively for z, in the exponential expression $e^{z} = 1 + \frac{z}{1} + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} +$, &c.

Where e is the number whose hyperbolic logarithm is 1. The addition of the two new equations, obtained by substitution, if divided by 2, will give a series expressing the sine; and the subtraction, if divided by $2\sqrt{-1}$ will give a series for the cosine of any arc.

THE DETERMINATION OF THE VALUE OF THE SINE AND OF THE COSINE, ETC. OF ANY ARC, IN TERMS OF THAT ARC.

PROPOSITION XXIII. (*Plate IV. Fig.* 1.)

(268) *To determine the increment of an arc, in terms of the increment of the sine, tangent, secant, &c.; and thence to deduce several useful formulæ.*

Let A*m* be any arc; P*m* its sine, CP its cosine, AT its tangent, CT its secant, &c. Take the arc *mo* indefinitely small, draw *on*V parallel to P*m*, and *mn* parallel to AC; also, from the centre C, with the radius CT, describe the arc *s*T.

Then *mo* is the increment of the arc A*m*; *mn* is the decrement of the cosine, or the increment of the versed sine; *on* is the increment of the sine; T*t* the increment of the tangent, and *st* the increment of the secant.

The triangles *mno*, and CP*m*, are equiangular and similar; for the arc *mo*, being extremely small, may be considered as a straight line. Likewise the triangles CAT and T*st* are equiangular and similar, for the lines C*t* and CT are supposed nearly to coincide, so that T*s* may be considered as a straight line, the ∠ T*st* a right angle, and the ∠ *st*T = ∠ CTA. Lastly, the sectors C*om* and C*s*T are similar. Hence we deduce the following proportions:

$\mathrm{C}m : \mathrm{CP} :: mo : on$, hence $mo = \mathrm{C}m.on \div \mathrm{CP}$

$\mathrm{C}m : \mathrm{CT} :: mo : \mathrm{T}s$, or $\mathrm{T}s = \mathrm{CT}.mo \div \mathrm{C}m$

$\mathrm{CA} = \mathrm{C}m : \mathrm{CT} :: \mathrm{T}s : \mathrm{T}t$, hence $mo = \mathrm{C}m^2.\mathrm{T}t \div \mathrm{CT}^2$

$\mathrm{CA} : \mathrm{AT} :: \mathrm{T}s : st$, hence $mo = \mathrm{C}m^2.st \div \mathrm{AT}.\mathrm{CT}$

$\mathrm{P}m : \mathrm{C}m :: mn : mo$, hence $mo = \mathrm{C}m.mn \div \mathrm{P}m$

(269) Now it is shown by writers on the differential calculus, that the limiting ratio of the cotemporary increments or decrements of any two quantities, will be the ratio of the

differentials of those quantities. If, therefore, we put $z=$ the arc $\text{A}m$, om will be represented by dz; and if $\text{AT}=t$, $\text{P}m=x$, $\text{AP}=v$, and $\text{CT}=s$, then $\text{T}t$ will be represented by dt, no by dx, $\text{vP}(=nm)$ by dv, and st by ds; hence by the preceding art.

$$\text{I.}\quad dz=\frac{rdx}{\text{CP}}=\frac{rdx}{\sqrt{r^2-x^2}}$$

$$\text{II.}\quad dz=\frac{r^2dt}{\text{CT}^2}=\frac{r^2dt}{r^2+t^2}$$

$$\text{III.}\quad dz=\frac{r^2ds}{\text{AT}\,.\,\text{CT}}=\frac{r^2ds}{s\sqrt{s^2-r^2}}$$

$$\text{IV.}\quad dz=\frac{rdv}{\text{P}m}=\frac{rdv}{\sqrt{2rv-v^2}}$$

(270) If these formulæ be expanded, and the integrals of each term be taken, we shall obtain the common series for the arc, in terms of its sine, tangent, &c.

I. Thus $dz=\frac{rdx}{\sqrt{r-x^2}}=rdx\left(\frac{1}{r}+\frac{x^2}{2r^3}+\frac{3x^4}{2.4\,r^5}+\frac{3.5\,x^6}{2.4.6\,r^7}+\frac{3.5.7\,x^8}{2.4.6.8r^9}, \&c.\right)=dx+\frac{x^2\,dx}{2r^2}+\frac{3x^4\,dx}{2.4\,r^4}+\frac{3.5\,x^6\,dx}{2.4.6\,r^6}+\frac{3.5.7\,x^8\,dx}{2.4.6.8\,r^8}$,& the integral of which is $z=x+\frac{x^3}{2.3\,r^2}+\frac{3x^5}{2.4.5\,r^4}+\frac{3.5\,x^7}{2.4.6.7\,r^6}+\frac{3.5.7\,x^9}{2.4.6.8.9\,r^8}$, &c. and by reversion of the series $x=z-\frac{z^3}{2.3\,r^2}+\frac{z^5}{2.3.4.5\,r^4}-\frac{z^7}{2.3.4.5.6.7\,r^6}$, &c.

II. $dz=\frac{r^2dt}{r^2+t^2}=r^2dt\left(\frac{1}{r^2}-\frac{t^2}{r^4}+\frac{t^4}{r^6}-\frac{t^6}{r^8}, \&c.\right)=dt-\frac{t^2\,dt}{r^2}-\frac{t^4\,dt}{r^4}+\frac{t^6\,dt}{r^6}$, &c. the fluent of which is $z=t-\frac{t^3}{3r^2}+\frac{t^5}{5r^4}-\frac{t^7}{7r^6}$, &c. and by reverting the series $t=z+\frac{z^3}{3r^2}+\frac{2z^5}{15r^4}+\frac{17z^7}{315r^6}+\frac{62z^9}{2835r^8}$, &c.

In a similar manner the rest may be found.

But $\sqrt{r^2-x^2}=\cos z$; $\text{rad}-\cos z=\text{vers } z$; $\frac{r^2}{\tan z}=\cot z$; $\frac{r^2}{\cos z}=\sec z$, $\frac{r^2}{\sin z}=\text{cosec } z$.

Hence, if $A =$ any arc, and $r =$ radius, then

$$A = \sin A + \frac{\sin^3 A}{2.3\, r^2} + \frac{3 \sin^5 A}{2.4.5\, r^4} + \frac{3.5 \sin^7 A}{2.4.6.7\, r^6} + \frac{3.5.7 \sin^9 A}{2.4.6.8.9\, r^8}, \text{ \&c.}$$

$$\sin A = A - \frac{A^3}{2.3\, r^2} + \frac{A^5}{2.3.4.5\, r^4} - \frac{A^7}{2.3.4.5.6.7\, r^6} + \frac{A^9}{2.3.4.5.6.7.8.9\, r^8}, \text{ \&c.}$$

The same series will apply to the chord of any arc by substituting the diameter *d*, or $2r$ instead of r.*

(271) If $90° - A$ be substituted for A, then we shall obtain

$$90° - A = \sin (90° - A) + \frac{\sin^3 (90° - A)}{2.3\, r^2} + \frac{3 \sin^5 (90° - A)}{2.4.5\, r^4}, \text{ \&c.};$$

hence $A = 90° - \cos A - \frac{\cos^3 A}{2.3\, r^2} - \frac{3 \cos^5 A}{2.4.5\, r^4} - \frac{3.5 \cos^7 A}{2.4.6.7\, r^6}$, &c.; or

$$A = \frac{r - \cos A}{1} + \frac{r^3 - \cos^3 A}{2.3\, r^2} + \frac{3}{2.4.5}\left(\frac{r^5 - \cos^5 A}{r^4}\right) + \frac{3.5}{2.4.6.7}\left(\frac{r^7 - \cos^7 A}{r^6}\right) \text{ \&c.}$$

$$\cos A = r - \frac{A^2}{2r} + \frac{A^4}{2.3.4\, r^3} - \frac{A^6}{2.3.4.5.6\, r^5} + \frac{A^8}{2.3.4.5.6.7\, r^7}, \text{ \&c.}$$

(272) $A = \tan A - \frac{\tan^3 A}{3r^2} + \frac{\tan^5 A}{5r^4} - \frac{\tan^7 A}{7r^6} + \frac{\tan^9 A}{9r^8}$, &c. (270)

Hence $\tan A = A + \frac{A^3}{3r^2} + \frac{2A^5}{15r^4} + \frac{17A^7}{315r^6} + \frac{62A^9}{2835r^8} + \frac{1382A^{11}}{155925r^{10}} + \frac{21844\, A^{13}}{6081075r^{12}}$, &c. †

$$A = \frac{r^2}{\cot A} - \frac{r^4}{3 \cot^3 A} + \frac{r^6}{5 \cot^5 A} - \frac{r^8}{7 \cot^7 A} + \frac{r^{10}}{9 \cot^9 A}, \text{ \&c.}$$

$$\cot A = \frac{r^2}{A} - \frac{A}{3} - \frac{A^3}{45r^2} - \frac{2A^5}{945r^4} - \frac{A^7}{4725r^6} - \frac{2A^9}{93555r^8} - \frac{1382A^{11}}{638512875r^{10}}, \text{ \&c.}$$

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* Viz. chord $A = A - \frac{A^3}{2.3.4r^2}$, hence the difference between an arc and its chord, the radius being 1, is $\frac{A^3}{24}$, the 24th part of the cube of the length of the arc nearly. See Note, Chap. IV. (154.)

† Emerson's Trigonometry, 2d edit. page 32. &c.

(273) $\text{A} = 90^\circ - \frac{r^2}{\sec \text{A}} - \frac{r^4}{2.3 \sec^3 \text{A}} - \frac{3r^6}{2.4.5 \sec^5 \text{A}} - \frac{3.5r^8}{2.4.6.7 \sec^7 \text{A}}$, &c. or,

$$\text{A} = \frac{r^2}{\text{cosec A}} + \frac{r^4}{2.3 \text{ cosec}^3 \text{A}} + \frac{3r^6}{2.4.5 \text{ cosec}^5 \text{A}} + \frac{3.5r^8}{2.4.6.7 \text{ cosec}^7 \text{A}},$$ &c.

$$\sec \text{A} = r + \frac{\text{A}^2}{2r} + \frac{5\text{A}^4}{24r^3} + \frac{61\text{A}^6}{720r^5} + \frac{277\text{A}^8}{8064r^7} + \frac{50521\text{A}^{10}}{3628800r^9} + \frac{540553\text{A}^{12}}{95800320r^{11}}, \text{ \&c.}$$

$$\text{cosec A} = \frac{r^2}{\text{A}} + \frac{\text{A}}{6} + \frac{7\text{A}^3}{360r^2} + \frac{31\text{A}^5}{15120r^4} + \frac{127\text{A}^7}{604800r^6} + \frac{73\text{A}^9}{3421440r^8}, \text{ \&c.}$$

(274) $\text{A} = \sqrt{2r.\text{vers A}}\left(1 + \frac{\text{vers A}}{3.2^2 r} + \frac{3 \text{vers}^2 \text{A}}{4.5.2^3 r^2} + \frac{3.5 \text{vers}^3 \text{A}}{4.6.7.2^4 r^3}, \text{\&c.}\right)$

$$\text{vers A} = \frac{\text{A}^2}{2r} - \frac{\text{A}^4}{2.3.4\, r^3} - \frac{\text{A}^6}{2.3.4.5.6\, r^5} - \frac{\text{A}^8}{2.3.4.5.6.7.8\, r^7}, \text{ \&c.}$$

$$\text{covers A} = r - \text{A} + \frac{\text{A}^3}{2.3\, r^2} - \frac{\text{A}^5}{2.3.4.5\, r^4} + \frac{\text{A}^7}{2.3.4.5.6.7\, r^6} - \frac{\text{A}^9}{2.3.4.5.6.7.8.9\, r^8}, \text{ \&c.}$$

THE CONSTRUCTION OF A TABLE OF SINES, ETC.

(275) There are various methods of constructing a table of sines, but the following, though not the least laborious, is the most simple.

$\sin 30^\circ = \frac{1}{2} \text{ rad} = \cos 60^\circ$ (98)

$\sin 15^\circ = \frac{1}{2} \text{ chord } 30^\circ = \frac{\text{rad}}{2}\sqrt{2 - \sqrt{3}}$

$\sin 7\frac{1}{2}^\circ = \frac{1}{2} \text{ chord } 15^\circ = \frac{\text{rad}}{2}\sqrt{\frac{4 - \sqrt{2} - \sqrt{6}}{2}}$, &c.

These sines are found thus. From the square of the radius subtract the square of the sine, the square root of the difference is the cosine. The radius diminished by the cosine leaves the versed sine; and $\sqrt{\text{vers}^2 + \sin^2} = \text{chord}$, the half of which is the sine of the next arc, &c.

From the sine of 7° 30′, find the sine of 3° 45′, and so on continually, till the sines are as the arcs, which will be found at the end of the twelfth division from 30°: that is, at the end of the twelfth division, the arc will be $52'' \, 44''' \, 3'''' \, 45'''''$, and

its sine ·0002556634; but at the end of the 11th division the arc is $1' \, 45'' \, 28''' \, 7'''' \, 30'''''$, and its sine ·0005113269; therefore the sine of the 12th division is just half the sine of the eleventh, in the same manner as the twelfth arc is half the eleventh. Now in indefinitely small arcs, the arcs will be to each other as their corresponding sines, hence
$52'' \, 44''' \, 3'''' \, 45''''' : 1' :: \cdot 0002556634 : \cdot 0002908882$ the sine of one minute.

The cosine of $1' =$ the square root of the radius 1, diminished by the square of the sine of $1'$, viz. cosine $1' = 9999999577$.

And it is shown (261) that

$2 \cos 1' . \sin 1' - \sin 0' = \cdot 0005817764 = \sin 2'$, or $\cos 89° \, 58'$
$2 \cos 1' . \sin 2' - \sin 1' = \cdot 0008726645 = \sin 3'$, or $\cos 89° \, 57'$
$2 \cos 1' . \sin 3' - \sin 2' = \cdot 0011635526 = \sin 4'$, or $\cos 89° \, 56'$
$2 \cos 1' . \sin 4' - \sin 3' = \cdot 0014544406 = \sin 5'$, &c. &c.

This method may likewise be extended to the cosines by means of the formulæ in art. (262).

$2 \cos 1' . \cos 1' - \cos 0' = \cdot 9999998308 = \cos 2'$, or $\sin 89° \, 58'$
$2 \cos 1' . \cos 2' - \cos 1' = \cdot 9999996192 = \cos 3'$, or $\sin 89° \, 57'$
$2 \cos 1' . \cos 3' - \cos 2' = \cdot 9999993231 = \cos 4'$, or $\sin 89° \, 56'$
$2 \cos 1' . \cos 4' - \cos 3' = \cdot 9999989423 = \cos 5'$, or $\sin 89° \, 55'$

Proceed thus to find the sine and cosine of every minute of the arc as far as 30°.

(276) Let A and B be any two arcs, then (245) $\frac{\sin A . \cos B}{\text{rad}} = \frac{1}{2} \sin (A + B) + \frac{1}{2} \sin (A - B)$; let $A = 30°$, then $\sin A = \frac{1}{2}$ rad, and $\cos B = \sin (30° + B) + \sin (30° - B)$; hence

$$\sin (30° + B) = \cos B - \sin (30° - B).$$

If, therefore, $B = 1', 2', 3', 4'$, &c. successively, we shall have

$\sin 30° \, 1' = \cos 1' - \sin 29° \, 59'$
$\sin 30° \, 2' = \cos 2' - \sin 29° \, 58'$
$\sin 30° \, 3' = \cos 3' - \sin 29° \, 57'$, &c.

And in this manner find all the sines, and thence all the cosines, as far as 45°. By these means all the sines and cosines from 0° to 90° will be obtained; for $\sin (45° + A) = \cos (45° - A)$, and $\cos (45° + A) = \sin (45° - A)$.

(277) The sines and cosines being constructed, we have by (209)

cos : sin :: rad : tan; hence the tangents are found.
tan : rad :: rad : cot; hence the cotangents are found.
cos : rad :: rad : sec; hence the secants are found.
sin : rad :: rad : cosec; hence the cosecants are found.

The versed sines are found by subtracting the cosine from radius for an arc less than a quadrant, and adding the cosine to the radius for an arc greater than a quadrant.

(278) ARTIFICIAL, or LOGARITHMIC SINES, &c. are only the logarithms of the natural sines, &c.

The natural sines are generally calculated to the radius 1, and being in all cases, except when the arc is 90°, less than radius or 1, they must of course be decimals.

Now the logarithm of a whole number and the logarithm of a decimal is the same, only the index or whole number prefixed to the former is affirmative, and that to the latter negative. Hence if logarithmic sines, &c. were immediately formed from natural sines which are calculated to the radius 1, their indices would all be negative. To avoid this, the logarithmic radius instead of being taken 1, as in the natural sines, is generally considered as ten thousand millions.

(279) *To find the logarithmic sine of* 1′ *to* 7 *places of figures, without the index.*

rad = 1 : rad = 10,000,000,000 :: ·0002908882 (the natural sine of 1′, the radius being 1) : 2908882 (the natural sine of 1′, the radius being 10,000,000,000.) The logarithm of 2908882 = 6·4637261 the logarithmic sine of 1′. The indices to the logarithmic sines from 0′ to 4′, will be 6; from 4′ to 35′ the indices will be 7; from 35′ to 5° 45′ the indices will be 8; from thence upwards to 89° they will be 9; and at 90° the index will be 10 = the logarithm of the radius 10,000,000,000.

(280) *Hence if we take the natural sine of any arc from a table of natural sines (where the radius is unity), and multiply it by* 10,000,000,000; *the logarithm of the product will give the logarithmic sine of that arc to as many places of figures as the natural sines are carried to.* It may be proper to inform the learner, that this method will not be exactly true for the first five degrees; because the natural sines in the tables are not carried to a sufficient number of places.

(281) In logarithms the operation of multiplication is performed by addition, and division by subtraction. The logarithmic sines being constructed, the tangents, &c. are formed from art. (209) thus:

$$\begin{aligned} 10+\sin-\cos &= \tan \\ 20-\tan &= \cot \\ 20-\cos &= \sec \\ 20-\sin &= \text{cosec} \end{aligned}$$

If you double the logarithmic sine of half an arc, and subtract 9·6989700 *from the product, the remainder will be the logarithmic versed sine of that arc.*

K 2

For $\frac{1}{2}$ rad . vers $=\sin^2 \frac{1}{2}$ arc (224),

$\therefore$ log 5,000,000,000 + log vers $=2\times$ log sin $\frac{1}{2}$ arc,

or 9·6989700 + log vers $=2\times$ log sin $\frac{1}{2}$ arc,

$\therefore$ log vers $=2\times$ log sin $\frac{1}{2}$ arc $-$ 9·6989700.

(282) The following logarithmic formulæ for the sine, cosine, &c. of any arc, may in some cases be useful, *viz.*

sin = cos + tan − 10 = tan + 10 − sec = cos + 10 − cot = 20 − cosec

cos = sin + 10 − tan = 20 − sec = sin + cot − 10 = cot + 10 − cosec

tan = sin + 10 − cos = 20 − cot

cot = cos + 10 − sin = 20 − tan

sec = tan + 10 − sin = 30 − sin − cot = cosec + 10 − cot = 20 − cos

cosec = cot + 10 − cos = 30 − cos − tan = sec + 10 − tan = 20 − sin

THE END OF PLANE TRIGONOMETRY.

BOOK III.

SPHERICAL TRIGONOMETRY.

CHAPTER I.

DEFINITIONS, ETC. OF SPHERICAL ANGLES, ARCS, AND TRIANGLES.

(283) SPHERICAL TRIGONOMETRY treats on the properties of spherical triangles, or the position and magnitudes of arcs of circles described on the surface of a sphere or globe.

(284) A sphere, or globe, is a round solid body, each part of its surface being equidistant from its centre. This centre is common to every circle described on the surface of the sphere, wherein spherical trigonometry is concerned, and such circles are called great circles.

(285) All great circles divide the globe into two equal parts, and consequently bisect each other at the distance of a semicircle or 180°. Hence two arcs cannot inclose a space, unless they are both semicircles.

(286) A spherical angle is the inclination of the planes of two great circles to each other, which circles intersect or meet each other on the surface of the sphere, in a point called the angular point. The inclination of these planes must always be measured on the arc of a great circle, 90° from the angular point.

(287) A spherical triangle is formed on the surface of the sphere by the intersection of three great circles, and consists of three sides and three angles; any three of which parts being given the rest may be found.

(288) All the sides of a spherical triangle are arcs of equal circles, and the angles of a spherical triangle are measured by arcs of circles, having the same radii as the sides. Hence the sides and angles of spherical triangles are always expressed in degrees, and parts of degrees.

(289) If one angle of a triangle be 90°, it is called a right-angled triangle. — If one side be 90°, a quadrantal triangle. —If no angle or side be 90°, it is called an oblique-angled triangle.

(290) The pole of a circle is a point on the surface of the sphere equidistant from every part of that circle of which it is the pole. Consequently every circle has two poles diametrically opposite to each other, and the arc of a circle comprehended between each of these poles, and the circumference of such a circle, is a quadrant. No two circles can have the same, or a common pole. If a straight line be drawn from the pole of any circle to the centre of the sphere, it will cut the diameter of that circle at right angles.

(291) All great circles passing through the pole of another great circle, cut that circle at right angles; and if two circles cut each other at right angles, in the poles of a third circle, the four points of intersection with that third circle, will be the four poles of the cutting circles; viz. the two opposite points will be the poles of that circle which is described between them.

(292) Sides and angles of spherical triangles are said to be of the same *species*, *kind*, or *affection*, when by comparing any two sides, any two angles, or any angle and a side together, you discover each to be greater or less than a right angle, or equal to a right angle.

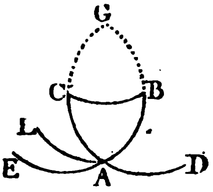
But when by comparing a side with a side, an angle with an angle, or a side with an angle, you discover one to be less and another greater than a right angle, such sides and angles are said to be of different species.

(293) Spherical triangles are equilateral, isosceles, or scalene, according as they have three equal sides, two equal sides, or three unequal sides.

N.B. In any of the following propositions, wherever the word circle, or arc of a circle occurs, it must always be understood to be a great circle, or the arc of a great circle. And, that all circles concerned in spherical trigonometry are equal to each other.

PROPOSITION I.

(294) *If one great circle intersect another great circle in any point* A, *all the angles at the point* A, *on the same side of any arc*, CAD *or* EAB, *are equal to two right angles; and all the angles about the point* A, *are equal to four right angles.*



Let CA be perpendicular to EB, then will the angles EAC and CAB be each of them a right angle. If AL be drawn, then LAB and EAL are equal to two right angles, for the angle CAB is increased by the angle LAC, and CAE is diminished by the same, therefore LAB + EAL = EAC + CAB.

In the same manner it may be proved that the angles EAD and DAB are together equal to two right angles; for if AC be perpendicular to EB, AD will likewise be perpendicular to it, therefore all the angles about the point A are equal to four right angles.

(295) COROLLARY I. *If two arcs of circles intersect each other, the vertical or opposite angles will be equal.*

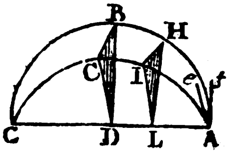
For the angles EAC and CAB are equal to two right angles, also CAB and BAD are equal to two right angles; from each of these equals take the angle CAB, then EAC is equal to BAD. In the same manner it may be shown that the angle CAB is equal to the angle EAD.

(296) COROLLARY II. *All the exterior and interior angles of a spherical triangle are together equal to six right angles.*

For the interior angle CAB and exterior angle BAD are equal to two right angles; likewise ACB and BCG are equal to two right angles; and ABC and CBG are also equal to two right angles.

PROPOSITION II.

(297) *If a great circle* BC *be described meeting two great circles* ABG *and* ACG, *which pass through the pole* A *of the circle* BC; *the angle* CDB *at the centre of the sphere, upon the circumference* BC, *is the same with the spherical angle* BAC, *and the arc* BC *is called the measure of the spherical angle* BAC.



Since A is the pole of BC, AB and AC are quadrants (290), and the angles ABC and ACB are right angles (291).

Let D be the centre of the sphere, and join DC and DB; then because the arcs AC and AB are each of them quadrants, the angles ADC and ADB are right angles: therefore the angle CDB is the inclination of the planes of the circles ABG and ACG to each other, and consequently (286), it is the measure of the angle CAB.

(298) COROLLARY. *If two circles cut each other in two points, the angles at these points are equal to each other, and to the distance between the poles of these circles.*

The angle BAC is equal to the angle BGC, for the arc BC is the measure of them both; and the distance between the point B and the pole of ABG is a quadrant, also the distance between C and the pole of ACG is a quadrant (290), therefore the difference of these distances will be equal to BC.

PROPOSITION III.

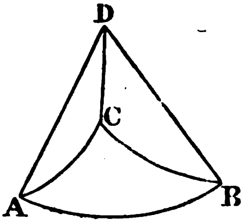
(299) *If to the point of intersection* A *of two great circles, two tangents* A*e and* A*f be drawn, the angle e*A*f will be the measure of the spherical angle* BAC.

For AG is the diameter of the sphere, and ADC and ADB are right angles; but *f*AD and *e*AD are also right angles, being tangents to the circles ABG and ACG; hence *f*A and *e*A are parallel to DB and DC, and the angle *e*A*f* is therefore equal to the angle CDB; but CDB is the measure of the spherical angle BAC, therefore *e*A*f* is the measure thereof.

In the same manner it may be shown, that the angle ILH is equal to CDB, for it is likewise the inclination of the planes ABG and ACG. All spherical angles must be measured on the arc of a great circle 90° distant from the angular point, for CB, and not IH, is the measure of the angle BAC.

PROPOSITION IV.

(300) *Any two sides of a spherical triangle are together greater than the third side.*



DEMONSTRATION. Let ABC be a spherical triangle, any two sides AC and BC, are together greater than AB. Let D be the centre of the sphere, and join DA, DC, DB.

The solid angle at D is contained by the three plane angles BDC, BDA, CDA, any two of which BDC, CDA, are together greater than the third BDA. (*Keith's Geom.* Prop. 1, Book X.)

Now the arc BC is the measure of the angle BDC, the arc AC of CDA, and the arc AB of BDA.

Therefore AC and BC together are greater than AB.

(301) COROL. *The three sides of a spherical triangle are together less than the circumference of a circle.*

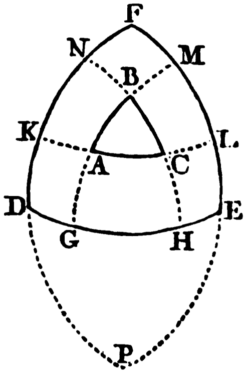
For every solid angle is contained by plane angles which are together less than four right angles (*Keith's Geom.* Prop. 2, Book X.), and the three sides of a spherical triangle are the measures of the three plane angles which contain the solid angle, and are therefore less than four quadrants, or the circumference of a circle.

PROPOSITION V.

(302) *If from the angular points* A, B, C, *of a spherical triangle* ABC *as poles, there be described three arcs of circles* EF, DE, *and* DF, *forming a new spherical triangle* DFE; *each side of this new triangle is the supplement of the measure of the angle at its pole,* viz. *the side* FE *is the supplement of the measure of the angle* A, DE *of the angle* B, *and* DF *of the angle* C. — *Likewise each angle of this new triangle is the supplement of the measure of that side of the original triangle* ABC, *to which it is opposite,* viz. *the angle* D *is the supplement of the measure of* BC, *the angle* F *of* AC, *and the angle* E *is the supplement of the measure of the side* AB.

Since B is the pole of DE, the arc of a great circle drawn from B to any point in DE, and therefore to the point E is a quadrant (290); similarly, since A is the pole of EF, the arc of a great circle from A to any point E in EF is a quadrant: hence the point E is distant the length of a quadrant from each of the points A and B, and is therefore the pole of AB. In the same manner it may be proved that F is the pole of AC, and that D is the pole of BC.

Produce the sides of the triangle BAC, if necessary, till they meet the sides of the triangle FDE in the points G, H, L, M, N, K. Then EM, EG, FL, FK, DN, DH, are quadrants; likewise AM, AL, BH, BG, CN, CK are quadrants; consequently (286) LM is the measure of the angle A, and NH the measure of the angle D.



Now EM and FL are together equal to a semicircle, or FL, LE, and LM are together equal to a semicircle; but FL and LE are equal to FE, therefore FE and LM are together equal to a semicircle; that is, they are supplements of each other; but LM is the measure of the angle A; therefore the measure of the angle A in the triangle ABC, and the side EF in the triange DFE, are supplements of each other. In the same manner it may be proved that DE is the supplement of the measure of the angle B, and that DF and the angle C are supplements of each other.

It remains now to prove that the angles in the new triangle DFE, and the sides of the original triangle ABC, are supplements of each other.

Since CN and BH are quadrants, they are together equal to a semicircle, or CN, CH, and BC, are together equal to a semicircle; but NC and CH are equal to NH, therefore NH and BC together make a semicircle; now NH is the measure of the angle D; therefore the angle D in the triangle DFE is the supplement of the side BC in the triangle ABC. In the same manner it may be shown, that the measure of the angle F is the supplement of AC, and the angle E of AB.

(303) COROLLARY I. *The three angles of any spherical triangle* ABC *are together greater than two right angles, and less than six.*

The angle A and the side FE are equal to two right angles. The angle B and the side DE are equal to two right angles. The angle C and the side DF are equal to two right angles. Therefore the three angles A, B, C, together with the three sides FE, DE, and DF, are equal to six right angles; but the three sides FE, DE, and DF, are together *less* than four right angles (301), therefore the three angles A, B, C, are together greater than two right angles.

Every spherical angle is less than two right angles (294); therefore the sum of any three spherical angles is less than six right angles.

(304) COROLLARY II. *The sides of spherical triangles may be changed into angles, and the contrary.*

Thus if three angles of a triangle are given to find the sides; subtract each of the angles from 180°, and the three remainders will be the three sides of a new triangle, with which sides find the angles of the new triangle; then subtract each of these angles from 180°, and the three remainders will be the respective sides of the original triangle, whose angles were given. And the contrary when the sides are given to find the angles.

(305) COROLLARY III. *When the three angles* (A, B, C,) *of a triangle* (ABC) *are given to find a side* (AC), *take the angle* (B) *opposite to the side required from* 180°, *and use the remainder and the other two angles* (A *and* C) *as sides in a new triangle* (DPE): *In this new triangle find the angle* (P) *opposite that side* (DE) *where the supplement is used. Subtract this angle from* 180°, *and the remainder will be the side* (AC) *required.*

For if FD and FE be continued till they meet in P; DP and EP being the supplements of FD and FE, are equal to the angles C and A; and the side DE is the supplement of the angle B opposite to AC the side required.

Now find the angle P which is equal to the angle F (298), but the supplement of the angle F is equal to the side AC, therefore the supplement of the angle P is equal to the side AC. Also the angle PDE will be equal to the side BC, and PED will be equal to the side AB. For PDE and EDF are supplements of each other, and EDF and BC are supplements of each other, therefore the angle PDE is equal to the side BC.

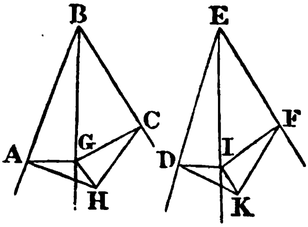
(306) Corollary IV. *A quadrantal triangle may be changed into a right-angled triangle, by calling the supplement of the angle opposite to the quadrantal side, the hypothenuse, and the other angles, the legs.*

For if AC be a quadrant, the angle at P is a right angle, being the supplement of AC (305), and DE is the hypothenuse, being the supplement of the angle B (302).

PROPOSITION VI. LEMMA.

(307) *If two solid angles be composed of three plane angles, such, that the plane angles of the one are equal to the plane angles of the other, each to each; the planes in which these equal angles are situated, shall have the same inclination to each other.*

Let B and E be two solid angles, and let the plane angle ABC be equal to the plane angle DEF, ABG to DEI, and GBC to IEF; the angle formed by the planes ABC and ABG is equal to the angle formed by the planes DEF and DEI.



For, suppose the straight line BG to be elevated above the plane ABC, then from any point G in that line let fall the perpendicular GH upon the plane ABC (*Keith's Geom.*, Prop. 11. Bk. IX.), in which plane draw HA and HC perpendiculars to AB and BC, and join AG, GC.

Make EI=BG, and from the point I draw IK perpendicular to the plane DEF, in which plane draw KD and KF at right angles to ED and EF, and join ID, IF.

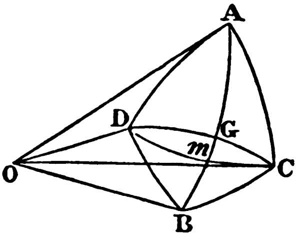
Then because GH is perpendicular to the plane ABC, and that HA is at right angles to AB a straight line in that plane, the angle BAG is a right angle (*Keith's Geom.*, Prop. 4. Bk. IX. Corol. 3.): for the same reason EDI is a right angle, therefore the triangle ABG is equal to the triangle DEI; hence AB is equal to DE, and AG to DI. In the same manner it may be shown that BC is equal to EF, and CG to FI. That granted, the quadrilateral figure ABCH, situated in the plane ABC, is equal to the quadrilateral figure DEFK, situated in the plane DEF. For if the angle ABC be applied to the angle DEF, AB, BC will coincide with DE and EF; and because all right angles are equal to each other, AH, which is at right angles to AB, will fall upon DK, which is at right angles to ED, and for the same reason CH will fall upon FK, therefore the point H will fall upon the point K, and AH will be equal to DK.

Now because the triangles AHG and DKI are right-angled at H and K, the hypothenuse AG equal to the hypothenuse DI, the side AH to DK, these triangles are equal (*Keith's Geom.*, Prop. 14. Bk. III. note 11.), therefore the angle HAG is equal to the angle KDI. But the angle HAG is the inclination of the two planes ABC, ABG; and the angle KDI is the inclination of the two planes DEF, DEI; therefore these planes have equal inclinations to each other.

SCHOLIUM. If the angle HAG be obtuse, viz, if the point H, and the straight line BC fall on contrary sides of AB; the angle KDI will be equally obtuse.

PROPOSITION VII.

(308) *In any two spherical triangles, if the three sides of the one be equal to the three sides of the other, each to each, the angles which are opposite to the equal sides will be equal.*



Let ABC be any triangle on the surface of the sphere.— From A as a pole, with the distance AC, describe the small circle CmD; from B as a pole, with the distance BC, describe the small circle CGD, crossing CmD in the point D. From the points A and B draw the great circles AD and BD, then AC=AD and BC=BD, by construction, also AB is common to the two triangles ABC and ABD, therefore the three sides of the one are equal to the three sides of the other, each to each.

The angles which are opposite to the equal sides in each triangle are equal. For,

Conceive O to be the centre of the sphere, and let a solid angle be formed at the point O by the three planes AOB, AOD, BOD; also let another solid angle be formed at the same point O by the three planes AOB, AOC, BOC.

The sides of the triangles ABC and ABD are equal, each to each, by construction; and as these sides are the respective measures of the plane angles which constitute the solid angles at the centre of the sphere, it follows, that the plane angles which form one of these solid angles, are equal to the plane angles which form the other solid angle, each to each. But the planes in which such angles are situated have the same inclination to each other (307); therefore the angles of the spherical triangle ABC are equal to those of the spherical triangle ABD, viz. ∠ DAB = ∠ CAB, ∠ DBA = ∠ ABC, and ∠ ADB = ∠ ACB.

SCHOLIUM. The equal angles ACB and ADB being situated on contrary sides of the centre of the sphere, the triangle ABC will not coincide with the triangle ADB, unless we conceive the convexity of the one to be applied to the concavity of the other.

If the equal angles ACB and ADB were situated on the same side of the centre of the sphere, the triangle ABC when applied to the triangle ADB would coincide with it.

Triangles which are equilateral, or isosceles, will coincide when applied to each other.

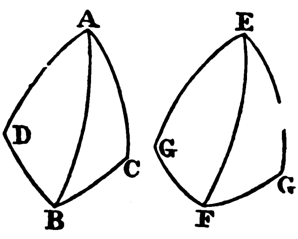
PROPOSITION VIII.

(309) *In two triangles, if the three angles of the one be equal to the three angles of the other, each to each, these triangles are equal in all respects.*

The supplements of the three angles in one triangle are equal to the supplements of the three angles in the other: and each of these three supplements form a new spherical triangle, whose sides and angles are respectively equal (Prop. V. and VII.); and the supplements of the equal angles in the new spherical triangles are equal to the sides of the original triangles whose angles were given; but the supplements of the angles of the new triangles being equal, the sides of the original triangles are equal; and if the sides be equal, the triangles are equal in all respects. (Prop. VII.)

PROPOSITION IX.

(310) *If there be two sides and the included angle in one spherical triangle, equal to two sides, and the included angle in another, each to each, these two triangles are equal in all respects.*



Let ABC and EFG be two triangles on the surface of the sphere, and let the side AB = EF, AC = EG, and the angle BAC = ∠ FEG. The triangle ABC is equal to the triangle EFG.

First, let the triangles be on the same side of the centre of the sphere. Then if the triangle EFG be applied to the triangle ABC, EF will coincide with AB and EG with AC (like plane triangles when they have an equal angle included between equal sides.)

The point F coinciding with B, and G with C, the arc FG must coincide with BC, since only *one* * great circle can be drawn through two given points on the surface of the sphere.

Hence, the three sides of the one triangle being equal to the three sides of the other, each to each, the triangles are equal. (Prop. VII.)

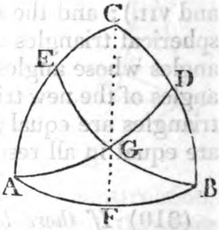
Secondly, let the triangles be situated on contrary sides of the centre of the sphere; then a triangle ADB may be constituted contiguous to ABC, and equal to EGF (as in Prop. VII.), and consequently EGF will coincide with ADB. Therefore the side FG = BC, the angle EFG = ∠ ABC, and ∠ EGF = ∠ ACB.

(311) Corollary. *If a side and the two adjacent angles in one spherical triangle be equal to a side and the two adjacent angles in another, each to each, these triangles are equal in all respects.*

For by taking the supplements of each of the two angles, and of the sides contained between them, two new triangles will be formed which will be equal, by the proposition, and therefore their supplements, viz. the original triangles will be equal.

PROPOSITION X.

(312) *The angles at the base of an isosceles spherical triangle are equal to each other.*



Let ABC be an isosceles triangle having the side AC equal to the side BC.

Make CE equal to CD, and draw EB and DA; now AC being equal to BC by hypothesis, and EC equal to CD by construction, and the angle ACB common to both the triangles ACD and BCE, these two triangles are equal in all respects (310); therefore AD is equal to EB.

Again, since AC is equal to BC, and EC equal to CD, the remaining parts AE and DB are equal; hence the triangles BAE and ABD are equal in all respects: for the two sides AE and EB are equal to the two sides BD and DA, each to each, and the base AB is common to both, therefore (308) the angle BAE is equal to the angle ABD.

(313) Corollary I. *In any spherical triangle if two of the angles be equal, the sides opposite to them will be equal.*

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* For the centre of every great circle is the centre of the sphere, and three given points determine the position of a plane. (*Keith's Geom.*, Prop. 2 Bk. IX. Corol.)

Make BD equal to AE, and draw EB and AD, then the triangle BAE is equal to the triangle ABD (310); therefore AD is equal to BE, the angle AEB equal to the angle ADB, and the angle DAB equal to the angle EBA; now the angle BEC is equal to the angle ADC, being each of them supplements of the equal angles AEB and BDA, therefore the triangle ADC is equal to the triangle BEC (311); consequently AE+EC is equal to BD+DC, or AC is equal to BC.

(314) COROLLARY II. *Hence every equilateral triangle is likewise equiangular, and the contrary.*

(315) COROLLARY III. *A line drawn from the vertex of an isosceles triangle to the middle of the base, is perpendicular to the base.*

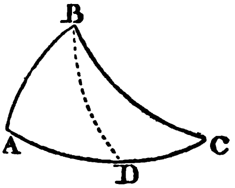
For the two sides FB and BC are equal to the two sides FA and AC, and the angle FBC is equal to the angle FAC, therefore the angle CFB is equal to the angle CFA (310); but CFA and CFB are equal to two right angles (294), therefore CF is perpendicular to AB.

(316) COROLLARY IV. *A perpendicular drawn from the vertex of an isosceles triangle upon the base, always bisects the base and the vertical angle, except when the two sides are quadrants, in which case there may be an indefinite number of perpendiculars.*

For the triangles FBC and FAC are equal in all respects. But if AC and BC are quadrants, that is, if C is the pole of the base, then an infinite number of lines may be drawn from the point C upon the base to cut it at right angles.

PROPOSITION XI.

(317) *In a spherical triangle* (ABC), *if the angle* (B) *be greater than the angle* (A), *the side* (AC) *opposite to* (B), *will be greater than the side* (BC) *opposite to* (A); *and conversely, if the side* (AC) *is greater than* (BC), *the angle* (B) *will be greater than the angle* (A).

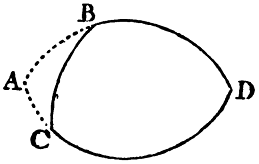


FIRST. Suppose the angle B greater than A, and make the angle DBA equal to the angle BAD, then AD is equal to BD (313): but BD + DC is greater than BC (300); hence, putting AD in place of BD, we shall have AD + DC or AC greater than BC.

SECONDLY. If we suppose AC greater than BC, the angle ABC will be greater than BAC. For if the angle ABC were equal to BAC, we should have AC=BC; if the angle ABC were less than BAC, we should then, as has just been shown, have AC less than BC. Both these conclusions are contrary to the supposition: hence the angle ABC is greater than BAC.

PROPOSITION XII.

(318) *If one side* (DC) *of a spherical triangle* (DCB) *be produced, and if the sum of the other sides* (CB + BD) *be greater, equal to, or less than a semicircle, the internal angle* (D) *at the base* (DC) *is accordingly greater, equal to, or less than the outward and opposite angle* (BCA) *adjacent to the side produced.*



Produce the sides DB and DC till they meet in A. Then the arcs ABD and ACD are each of them a semicircle (285), and the angle at D is equal to the angle at A (298).

1. If CB + BD be greater than 180°, BC will be greater than AB, and the angle CAB, or which is the same thing, the angle D will be greater than the angle BCA (317).

2. If CB + BD be equal to 180°, then BA and BC will be equal, and the angle CAB will be equal to the angle BCA, or the angle D will be equal to the angle BCA (312).

3. If CB + BD be less than 180°, BC will be less than AB, and the angle CAB, or the angle D will be less than the angle ACB (317).

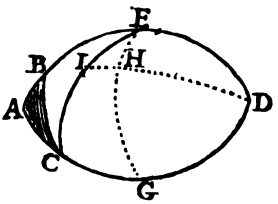
(319) COROLLARY. *In any spherical triangle, if the sum of any two sides be greater, equal to, or less than a semicircle, the sum of the opposite angles will be greater, equal to, or less than two right angles.*

For, according as CB + BD is greater, equal to, or less than a semicircle, the angle D opposite to the side BC is greater, equal to, or less than the angle BCA; but the angle BCA and the angle BCD, opposite to the side DB, are always equal to two right angles (294); therefore whenever D is greater than BCA, the sum of the angles BCD and BDC is greater than two right angles; *et cetera.*

PROPOSITION XIII.

(320) A right-angled spherical triangle may have either,

1. *One right angle and two acute angles.*
2. *One right angle and two obtuse angles.*
3. *One obtuse angle and two right angles.*
4. *One acute angle and two right angles.* OR,
5. *Three right angles.*



Let the angles A and D be right angles; and the arcs AE, ED, AG, and GD quadrants; then E will be the pole of the circle ACGD, and EC and EG will be quadrants (290).

1. Because AE and EC are quadrants, the angles EAC and ACE are right angles (291); therefore the angle ACB is less than a right angle. Suppose AC+BC less than a semicircle, then the sum of the angles ABC and BAC are less than two right angles (319); but BAC is a right angle, therefore ABC is less than a right angle; so that the right-angled triangle CAB has two acute angles and one right angle.

2. The triangle BDC right-angled at D, has two obtuse angles and one right angle. For the angles DBC and DCB are supplements of the two acute angles ABC and ACB, and therefore must be obtuse.

3. The triangle CED has one obtuse angle, and two right angles. For, since DG and DE are quadrants, the angle DEG is a right angle, therefore DEC is an obtuse angle. But EC is a quadrant, and E is the pole of CGD, therefore ECD is a right angle (291).

4. The triangle AEC has already been shown to have two right angles; and AEG is a right angle, therefore AEC is less than a right angle.

5. The right-angled spherical triangle GAE has three right angles, for A is a right angle by the proposition, and A is the pole of the circle EG, therefore AEG and AGE are right angles.

(321) COROLLARY I. In any right-angled triangle, *if the two legs be of the same species, viz. both acute, or both obtuse, the hypothenuse will be less than a quadrant, and the contrary.*

For BC the hypothenuse, or side opposite to the right angles A and D, is common to the triangle BAC having two acute angles, and to the triangle BDC having two obtuse angles.

(322) COROLLARY II. In any right-angled spherical triangle, *if the legs be of different species, the hypothenuse will be greater than a quadrant, and the contrary: but one of the legs, at least, will be a quadrant, if the hypothenuse be a quadrant.*

For, in the triangle ICD, right-angled at C, CI is less than a quadrant and CD greater, from what has already been demonstrated; but since D is the pole of the circle EHG, DH is a quadrant, therefore DI the hypothenuse is greater than a quadrant. And when CI becomes equal to CE, DI will become equal to DE; but DE is a quadrant, therefore if the hypothenuse be a quadrant, one of the legs, at least, will be a quadrant. But both the legs will be quadrants when all the angles are right angles, as in the triangle AEG.

(323) COROLLARY III. *In any right-angled spherical triangle, each of the oblique angles is of the same species as the opposite side; and the sides containing the right angle are of the same species as their opposite angles.*

In the triangle BAC, right-angled at C, let AB be less than a quadrant; then, since E is the pole of AC, the angle ACE is a right angle, and consequently ACB is less than a right angle. If AB be taken greater than a quadrant, it is obvious that ACB will become greater than a right angle. And if AE be a quadrant, then ACE is a right angle.

On the contrary, if ACE be a right angle, then E is the pole of AC, and AE a quadrant. Now it is obvious that if ACB be less or greater than a right angle, the opposite side AB will be less or greater than a quadrant.

(324) COROLLARY IV. *If the hypothenuse be less than* 90°, *the legs are of the same species as their adjacent angles. If the hypothenuse be greater than* 90°, *the legs and their adjacent angles are of different species.*

This is evident from the triangles BAC, CDB, and DCI.

(325) COROLLARY V. *According as the hypothenuse and one side (or its opposite angle) are of the same, or different species; the other side (and its opposite angle) will be less or greater than a quadrant.*

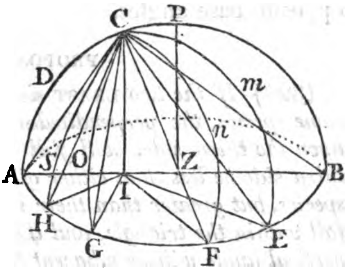
This is also evident from the triangle DCI.

PROPOSITION XIV.

(326) *If any number of arcs of circles* CPB, CmE, CnF, CDA, CSH, COG, *be drawn from one and the same point* C, *on the surface of the sphere, to the circumference of a great circle* AHGFEG; *the greatest arc will be that which passes through the pole* P *of the great circle* AHGFEB; *and the least arc will be the supplement* ADC *of the greatest* CPB. *And that arc which is the nearest to the greatest will be greater than any one more remote; likewise, that arc which is nearest to the least arc, will be less than any one more remote from it.*

Let Z be the centre of the sphere, or of the great circle AHGFEB, and AB a diameter; from C, the point where all the circles intersect each other, draw CI perpendicular to AB, and join IE, IF, IG and IH.

Then, of all the straight lines which can be drawn from the point I to the circumference of the circle AHGFEB, that line IB is the greatest which passes through the centre Z, and its opposite line IA is the least; that line IE which is nearest to IB, is greater than the line IF which is more remote; and the line IH which is nearest to IA, is shorter than the line IG which is more remote. (7 Euclid III.)

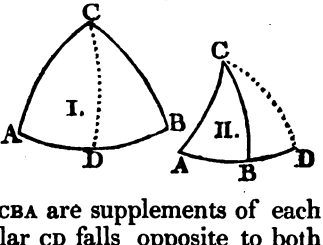


Let AC, HC, GC, FC, &c. be joined by straight lines; then will CIB, CIE, CIF, CIG, CIH and CIA be all right-angled plane triangles, having the perpendicular CI common to each; therefore that triangle CIB which has the greatest base IB, will have the greatest hypothenuse BC; and that triangle CIA which has the least base IA, will have the least hypothenuse AC, &c.; of the others, EC will be greater than FC, and HC less than GC.

But these several hypothenuses are all chords of equal circles, and the greater chord cuts off the greater arc, or circumference, therefore CPB is the greatest arc, and ADC the least; C*m*E is greater than C*n*F, and CSH is less than COG.

PROPOSITION XV.

(327) *If the angles at the base of an oblique-angled spherical triangle be of the same species, viz. both acute or both obtuse, a perpendicular from the vertical angle upon the base will fall within the triangle; if they be of different species, it will fall without.*



In the right-angled triangle CDB, CD is of the same species as CBD (323), and in the right-angled triangle CDA, CD is of the same species as CAD, therefore the perpendicular in the first figure falls between the angles A and B.

In the second figure, CBD and CBA are supplements of each other (294), and the perpendicular CD falls opposite to both CBD and CAD being of the same species, therefore it cannot fall opposite to CBA at the same time, hence it must necessarily fall without the triangle.

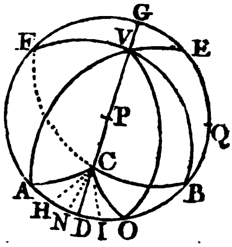
(328) COROLLARY. *In an isosceles triangle, the equal angles are of the same species as their adjacent, or opposite sides.*

For then AD = DB (316) and these segments are always acute, since their sum can never be equal to a semi-circle.

But according as DB and DC, or AD and DC, are of the same, or different species, BC or its equal AC is acute or obtuse; or since the segments are always acute, BC and AC will be of the same species as DC, but DC is of the same species as A and B, therefore the sides are of the same species as their adjacent or opposite base angles.

PROPOSITION XVI.

(329) *If the two* LEAST *sides of a spherical triangle be of the same species, the perpendicular drawn from their included angle upon the third side, will fall within the triangle.— Or, if the third side be less than either of the others, they being of the same species, but greater than their supplements, the perpendicular will fall within the triangle, and the less segment of the base and less vertical angle will be adjacent to the greater side, and the greater segment, and greater vertical angle will join the less side.*



First. Let ABC be the triangle, CD a perpendicular upon its base AB; make BN equal to the hypothenuse BC, and AI equal to the hypothenuse AC, then BCN and ACI are isosceles triangles; the angle BNC is of the same species as BC, and the angle AIC is of the same species as AC. (328) But AC and BC are of the same species, by the hypothesis, therefore the angles BNC and AIC are of the same species; consequently CD falls between them (327), and of course within the triangle ABC.

Secondly. In the triangle AVB, where AV and BV are each greater than AB, the two least sides of the supplemental triangle FVE, which by the hypothesis are each less than AB or FE, are of the same species, and consequently VG falls between them, therefore VD falls between AV and BV. But AV is greater than BV, and AD is less than DB; consequently AVD is less than BVD.

(330) Corollary I. *The perpendicular let fall on the base of an oblique spherical triangle is either less or greater than each side.*

For since CD is perpendicular to AB, it must pass through P the pole of AB (291); and if C be below the pole P, that is, if CD be less than a quadrant, it will be the least of all the lines that can be drawn from the point C to the circumference of the circle ABEF, and CG will be the greatest (326).

If C the vertex of the triangle ABC fall above the pole P, suppose at V, then VD will be the perpendicular, and of all the arcs that can be drawn from V upon the arc AB, VD is the greatest (326), therefore the perpendicular is either less or greater than each side. When it becomes equal to one of the sides, the triangle will be right-angled, as BDC.

If the perpendicular fall without the triangle, as in BOC, or BOV, the demonstration will hold equally true.

(331) COROLLARY II. *When the perpendicular falls without the triangle, it may be either less or greater than a quadrant.*

In the triangle BOC, either CD or CG may be esteemed the perpendicular upon the base OB continued; the former CD is less than a quadrant, and less than either CO or CB, and the latter CG is greater.

(332) COROLLARY III. *When the perpendicular falls without the triangle, the less perpendicular is next the less side, and the greater perpendicular is next the greater side; and either of them may be considered as the proper perpendicular on the base produced.*

In the triangle BOC, CD the less perpendicular is next the less side CO, and CG the greater perpendicular is next the greater side BC, and either of them may be used for determining the several parts of the triangle BOC, but one is sometimes more convenient than the other.

(333) COROLLARY IV. *In any triangle* ABC *or* BOC, *the perpendicular* CD *falling within the triangle* ABC *upon the base* AB, *or* NEAREST *perpendicular* CD *falling without the triangle* BOC, *is always of the same species as half the sum of the two sides of the triangle; and if half the sum of the two sides of the triangle be acute, this perpendicular falls nearest the less side; if obtuse, it falls nearest the greater side.*

In the triangle ABC the perpendicular DC is less than a quadrant, and each side AC and BC is less than a quadrant, therefore half their sum is less than a quadrant, and the perpendicular CD falls nearest to AC.

In the triangle BOC, CD is less than a quadrant, and each of the sides CO and CB is less than a quadrant, therefore half their sum is less than a quadrant, and the perpendicular falls nearest to CO.

And in the triangles ABV and BOV, where the perpendicular VD is greater than a quadrant, and each of the sides AV, BV, OV greater than a quadrant; consequently half their sum must be greater than a quadrant, and the perpendicular VD falls nearest to AV in the former triangle, and to OV in the latter.

(334) COROLLARY V. *If an oblique spherical triangle have two acute sides and one obtuse, a perpendicular drawn on the longest side will always fall within the triangle* (329).—*But if it has two obtuse sides and one acute, a perpendicular drawn to fall*

on the longest side may sometimes fall within, and sometimes without the triangle.

Take the right-angled triangle BDC, and produce BD, BC to meet in F; now because DB and DC are each less than a quadrant, the hypothenuse BC will be less than a quadrant (322), consequently its supplement CF will be greater than a quadrant. The angles B and C being acute or each less than a right angle, BC is greater than BD (317), therefore DF is greater than CF.

Make FA equal to FC, then a great circle as HC may be drawn between A and D; now in the triangle FHC, FH is greater than FC; FC, HC are of different species, and the perpendicular CD falls without the triangle FHC.

In the triangle FOC, OF is greater than FC; FC and OC are of different species, and the perpendicular CD falls within the triangle FOC.

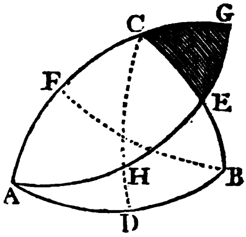
Therefore there can be no *general* rule for drawing a perpendicular when the three sides of a triangle are given, especially if two of those sides be obtuse.

(335) COROLLARY VI. *If the two obtuse sides of an oblique-angled triangle be* EQUAL, *a perpendicular drawn from their included angle will fall within the triangle.*

For then the triangle is isosceles, and the angles at the base are equal to each other (312), and are therefore of the same species.

PROPOSITION XVII.

(336) *If the three angles of a spherical triangle be each acute, each side will be less than a quadrant; and if the three angles be each right angles, each side will be a quadrant.*



Let all the angles A, B, and C of the triangle ABC be acute, then a perpendicular as CD drawn from any angle will fall within the triangle.

In the right-angled triangle ADC, the angles A and ACD are acute, and therefore by (321) and (323) AC is less then a quadrant. By the same manner of reasoning from the triangle BDC, BC is less than a quadrant; and because the angles A and ABF are acute, AB is less than a quadrant.

When the three angles are each right angles, it is obvious that each side is a quadrant.

(337) COROLLARY I. *In any spherical triangle having two*

obtuse angles and one acute, the sides are of the same species as their opposite angles.

Imagine the triangle AEC to have an acute angle at A, and two obtuse ones at E and C; then the triangle CEG has three acute angles, viz. the angle G is equal to the angle A (298), and the acute angles at E and C are supplements of the obtuse angles in the triangle AEC. But when the three angles are acute, each side of the triangle CEG is less than a quadrant, therefore AE and AC, opposite to the obtuse angles C and E, are greater than quadrants, and CE opposite to the acute angle A, is less than a quadrant.

(338) COROLLARY II. *If a spherical triangle have two sides each less, and one greater than a quadrant, the angles will be of the same species as their opposite sides.*

In the triangle AHC, let AH and HC be each of them lesth an a quadrant, and AC greater; the supplemental triangle to it (304) will have two obtuse angles and one acute, and consequently the sides thereof, by the preceding corollary, are of the same species as their opposite angles: but the supplements of these sides are the *angles* of the original triangle AHC; therefore this triangle has two acute angles HAC, ACH, and one obtuse, viz. AHC.

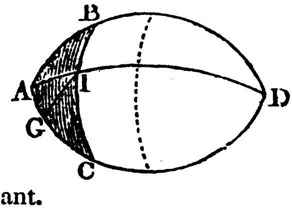
(339) COROLLARY III. *If the three sides are each quadrants, the angles will be right angles; and if the three sides are each greater than quadrants, the three angles will be each obtuse.*

These follow from considering the supplemental triangles (304).

PROPOSITION XVIII.

(340) *If two sides of a spherical triangle be of the same species, and the angle included acute, the third side is less than a quadrant; but if the two sides be of different species, and the included angle obtuse, the third side is greater than a quadrant.* OR,

When two angles are of the same species, and the included side greater than a quadrant, the third angle is obtuse; but if the two angles be of different species and the included side less than a quadrant, the third angle is acute.



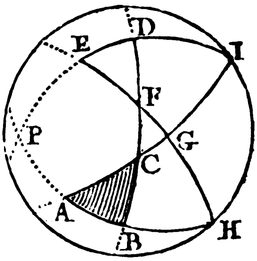
Let the triangles BAC and BDC be right-angled at A and D, then BC will be less than a quadrant (322); but as the arc ABD approaches nearer to ACD, it will diminish the angles A and D, and CI will consequently be less than CB, which has been proved to be less than a quadrant.

In the triangle CID right-angled, suppose at I, when ID is greater and IC less than a quadrant, the other side, or hypothenuse, CD is greater than a quadrant (322): and as the angle DIC increases, viz. when it becomes equal to DIG, the side DC increases till it becomes equal to DG; hence the first part of the proposition is evident.

In the second part, the supplemental triangle (304), will have exactly the same properties as the first part of this proposition.

PROPOSITION XIX.

(341) *In any right-angled spherical triangle* ABC, *if the sides be produced, viz.* AC *to* I, AB *to* H, *and* BC *to* D, *so that* CI, CD, *and* AH *be quadrants, or each* 90°; *and if from the point* A, *as a pole, the great circle* HGFE *be described, and from the point* C, *as a pole,* IDE *be drawn to intersect* HGFE *in* E; *then shall the triangles* CGF *and* EDF *have their respective sides and angles either equal to those of the triangle* ABC, *or they will be complements of each other; and eight right-angled spherical triangles will be formed, having (every two of them) equal angles at their bases.*



Since A is the pole of the circle HGFE, AG and AH are quadrants and perpendiculars to FGH; and the arcs FGH, FCB, being perpendicular to ABH, are quadrants, and F is the pole of ABH. Therefore the triangle CGF is right-angled at G; CG is the complement of AC, CF of BC; BH, the complement of AB, is the measure of the angle F; and GH, the complement of FG, is the measure of the angle A.

The arcs EFG, EDI, being each perpendicular to ACGI, are quadrants, and E is the pole of ACGI; and the arcs CFD and CGI being quadrants, for C is the pole of IDE, are perpendicular to IDE. Therefore the triangle EDF is right-angled at D; ED is the complement of ID, and ID measures the angle ACB equal to DCI (295); DF is equal to BC; the hypothenuse EF is equal to the angle A, for it is equal to GH the measure of A, and for the same reason FED is equal to AC, and EFD is equal to the complement of AB.

The right-angled triangles ABC, AHG, have the angle A common; FGC, FHB have the angle F common; CGF, CID have the angle C common; and EDF, EIG have the angle E common.

CHAP. II.

THE STEREOGRAPHIC PROJECTION OF THE SPHERE.

(342) *The stereographic projection of the sphere*, is such a representation of the various parts of its surface, on the plane of one of its great circles, as would be formed by lines drawn from the pole of that great circle, to every point of the circle, to be projected, viz.

Conceive a point E (Plate IV. Fig. 8.) situated any where on the surface of a globe, and a plane CD to stand at right angles to an imaginary line EO, connecting the centre of the globe and the point E. Then, if an indefinite number of lines be supposed to be drawn from the point E, to every point of the circumference of any circle described upon the spherical surface, they will trace out upon the cutting plane CD a stereographic projection of the sphere.

(343) The *plane* on which the points, lines, and circles of the sphere are represented, is called the *plane of projection;* and the point from which all the lines are drawn through the several parts of the circles of the sphere, to this plane, is called the *projecting point.*

(344) *The primitive circle* is situated in the plane of projection; and the projecting point, on the sphere, is one of the poles of this circle; but on the plane of projection, the projected poles of the primitive are in its centre.

(345) A circle, the plane of which is parallel to the plane of the primitive, is called *a parallel circle,* and is represented on the plane of projection by a circle parallel to, and comprehended within, the primitive.

(346) A circle, whose plane is perpendicular to the plane of the primitive, is called a *right circle;* because, passing through the projecting point, its circumference becomes a straight line on the plane of projection.

(347) A circle whose plane is oblique to the plane of the primitive, is called an *oblique circle.*

(348) Lines drawn from the projecting point to every part of the circumference of a circle to be projected, will form the convex surface of a cone, whose vertex is the projecting point. Thus if AB be a diameter of a circle to be projected, and E the projecting point, then AEB will be a cone, of which E is the vertex.

(349) All writers on conic sections have demonstrated, that if a cone be cut by a plane *ab* parallel to the base AB, the

section will be a circle; hence it follows, that all small circles parallel to the primitive will be projected into circles. And the radii of all projected small circles, parallel to the primitive, will be equal to the semi-tangent of their distances from the remoter pole; thus $ao = ob$ is the tangent of the angle aEO, or the semi-tangent of the arc Ae where e is the remoter pole.

(350) It is also shown by writers on conic sections, that if a scalene, or oblique cone, be cut by a plane not parallel to the base, but in such a manner that the cone cut from the base towards the vertex be equiangular with the original cone, the section shall be a circle.

Thus (Plate IV. Fig. 9.) if the cone ABE be so cut by a plane CD, that the angle CDE be equal to the angle BAE, and DCE equal to ABE, then the section will be a circle, having CD for its diameter. And this is what is termed cutting a cone in a *subcontrary position.*

Emerson's Conic Sections, Book I. Prop. 89th.

PROPOSITION I. (*Plate IV. Fig.* 10.)

(351) *Every circle of the sphere which does not pass through the poles of the primitive, is projected into a circle.*

Let DC be the diameter of a circle to be projected on the plane of the primitive FB, from the point E. Lines from the point E to the circumference of that circle form a cone, whose triangular section is CED.

Now the extremity C, of the diameter CD, will be projected into the point A, and the other extremity D into the point B in the plane of the primitive. Through D draw Dd parallel to FB, that is, perpendicular to Ee, then will the arc Ed be equal to the arc ED; *and since an angle at the circumference of a circle is measured by half the arc on which it stands* (20 Euclid III), the angle ECD will be equal to the angle dDE, or, which is the same thing, equal to the angle ABE, because AB is parallel to dD.

Hence the cone ECD is equiangular with the cone EBA, and it is cut by the plane of projection in a *subcontrary position,* therefore (350) the section is a circle.

(352) COROLLARY. *The centre, and consequently the poles of every circle oblique to the primitive, have their projections in the line of common section of the primitive, and a great circle perpendicular both to the primitive and to the oblique circle.*

For the centre and poles of the circle whose diameter is CD, have their projections in the line FB, the common section of the primitive, and the great circle FCDE which is perpendicular to the primitive and to the circle whose diameter is CD.

PROPOSITION II. (*Plate IV. Fig.* 11.)

(353) *The distance of the centre of any projected oblique great circle from the centre of the primitive, is equal to the tangent, and the radius of the projected circle is equal to the secant of the inclination of the oblique circle and primitive.*

Let E be the projecting point, FG the diameter of the primitive EF*e*G, and CD the diameter of a circle to be projected. The point C will appear at A, and D at B; therefore AB is the projected diameter. Bisect AB in *p*, and *p* will be the centre of the projected circle *e*AEB.

Now CED = CEB, being an angle in a semicircle, is a right angle; and EPA and EPB are likewise right angles; hence the triangles AEB, APE, and EPB are equiangular and similar. The angle *e*PC is double of the angle *e*EC, the one being at the centre and the other at the circumference of the circle, and for the same reason A*p*E is double of ABE; but the angle *e*EC has been shown to be equal to ABE, therefore *e*PC = P*p*E, and CPF = PE*p*: and it is plain that P*p* is the tangent of PE*p* to the radius of the sphere, therefore it is equal to the tangent of the angle CPF, the angle of inclination of the oblique circle and primitive.

It is also obvious that A*p* = E*p*, the radius of the projected circle, is the secant of the angle PE*p* or CPF.

(354) COROLLARY. *The distance* (AP) *of the extremity of the diameter of any projected circle from the centre of the primitive, is the semi-tangent of the complement of* (CF) *the circle's inclination to the primitive.*

For AP is the tangent of the angle AEP, which is the half of CP*e*, the complement of FPC.

PROPOSITION III. (*Plate IV. Fig.* 11.)

(355) *The distances of the projected poles of any oblique great circle from the centre of the primitive, are equal to the tangent and cotangent of half the inclination of the oblique circle and primitive.*

Let E be the projecting point, FG the diameter of the primitive, and CD the diameter of the circle to be projected.

Make *e*M = CF, then CM will be a quadrant; draw MPN, which will be the axis of CD; and M, N, its poles. If EM be joined, the pole M will appear at *m* in the primitive; but P*m* is the tangent of the angle PE*m*, which is the half of *e*PM, or of its equal CPF, the angle of inclination.

Likewise a line drawn from E through N, the other pole, will cut PF, produced beyond F, in the exterior projected pole. Now PF, produced to meet EN, is evidently the cotangent of the angle PE*m*, or half of (CPF) the angle of inclination.

PROPOSITION IV. (*Plate IV. Fig.* 12.)

(356) *The distance of the centre of a projected small circle (perpendicular to the primitive) from the centre of the primitive, is equal to the secant of its distance from its own pole; and the radius thereof is equal to the tangent of that distance.*

Let CD be the diameter of a small circle cutting the primitive at right angles, or, which amounts to the same thing, whose poles G and F lie in the circumference of the plane of projection. Now AB is the diameter of its representative on the plane of projection FG; bisect AB in *p*, and draw *p*C, CP, and CB; then the radius *p*C of the projected small circle is the tangent of the arc GC, the distance of the small circle from its pole G, and P*p* is equal to the secant of the same arc.

For since the triangles EC*e*, BP*e*, are right-angled at C and P, and have the angle *e* common, the remaining angle PEA will be equal to the remaining angle CB*p* = *p*CB, for *p*C = *p*B; but the angle PCA is equal to PEA, because PC is equal to PE, hence PCA = *p*CB, therefore the angle ACB is equal to the angle PC*p*, but ACB is a right angle, therefore PC*p* is a right angle; and consequently *p*C is the tangent of the arc GC, and P*p* its secant.

PROPOSITION V. (*Plate IV. Fig.* 13.)

(357) *The angle comprehended between two great circles, passing through the same point, on the surface of the sphere, is equal to the angle comprehended between their representatives on the plane of projection.*

Let BGK be any spherical angle. Through the angular point G and the centre P, draw the plane of a great circle GDE, perpendicular to the plane PHL of projection. Let a plane GHL touch the sphere in G; then since the plane of the circle GDE is perpendicular to this plane and to the plane of projection, it is perpendicular to their intersection HL. The angles made by great circles at any point on the surface of a sphere are equal to the angles made by their tangents (299); therefore in the plane GHL draw the tangents GH, GF, GL to the arcs GB, GD, GK, and their projections are the lines *g*H, *g*F, *g*L: the angle HGL is equal to H*g*L. For the angle PGF = a right angle = P*g*E + PE*g*; therefore taking away the equal angles PGE and PEG, we have the angle *g*GF = P*g*E = G*g*F, and consequently *g*F = GF. Hence in the right-angled triangles

GFL and *g*FL, there are two sides equal and the included right angle, and therefore the hypothenuse GL = *g*L. And, for the same reason, in the triangles GFH and *g*FH, we have GH = *g*H. Lastly, in the triangles GHL and *g*HL, all the sides are respectively equal, and therefore the angle HGL = H*g*L; but the angle HGL is equal to the spherical angle BGK (299): consequently the angle BGK = H*g*L.

PROPOSITION VI. (*Plate IV. Fig.* 11.)

(358) *The projected extremities of the diameter of any circle on the sphere inclined to the primitive, are distant from the centre of the primitive the semi-tangents of the circle's nearest and greatest distance from the pole opposite to the projecting point; and the diameter is equal to the sum of these semi-tangents.*

For the diameter CD of a circle of the sphere inclined to the plane of projection in an angle CPF, is projected into the line AB from the projecting point at E. Now AP is the semi-tangent of *e*C the circle's nearest distance from the pole *e*, and PB is the semi-tangent of *e*D the circle's greatest distance from the pole *e*. Also the diameter AB is the sum of these semi-tangents.

(359) COROLLARY I. *The projected diameter of a small circle inclined to the plane of projection and encompassing the pole* (*e*) *of the plane of projection, is equal to the sum of the semi-tangents of that circle's greatest and least distance from the pole of projection.*

For such a small circle may be conceived to be parallel to the great circle, and therefore its projected diameter will fall on both sides of the centre P of the primitive.

(360) COROLLARY II. *The projected diameter of a small circle inclined to the plane of projection, and situated wholly on one side of the pole* (*e*), *is equal to the difference of the semi-tangents of that circle's greatest and least distance from the pole* (*e*) *of projection.*

For a small circle on the sphere whose diameter is MO, will be projected into *mp*; P*p* is the semi-tangent of *e*O, the circle's greatest distance from the pole; P*m* the semi-tangent of *e*M, the circle's least distance from the pole; and the difference between these semi-tangents is *mp*.

(361) COROLLARY III. *The points where the projected diameter of a great circle terminates, are distant from the centre of the primitive, the tangent and cotangent of half the complement of the circle's inclination to the primitive.*

For C*e* is the complement of FC the circle's inclination to the primitive, and AP is the tangent of the half of C*e*, or the angle

AEP. And, because AEB is a right angle, PEB is the complement thereof; but PB is the tangent of PEB, that is, the tangent of the complement of AEP, or the cotangent of $\frac{1}{2}$ *ce*.

PROPOSITION VII. (*Plate V. Fig.* 5.)

(362) *In any projected great circle, inclined to the primitive, the radius of the projected circle, is to the radius of the primitive, as the distance of the projected pole from the centre of the projected circle, is to the distance of the projected pole from the centre of the primitive.*

Let FA*e*BC be the projection of a great circle cutting the primitive ENF*e*G in E and *e*, and the line FG produced, in A and C; also let P be the centre of the primitive, O the centre of the projected circle, and *p* and *r* its projected poles. Then

$$EO : EP :: pO : pP; \text{ or } EO : EP :: rO : rP.$$

Join EO, E*p*, and E*r*; then the angle PEO is the measure of the circle's inclination to the primitive: for PO is the tangent of PEO to the radius of the sphere, and it is likewise the tangent of the circle's inclination to the primitive (353).

The angle PE*p* is the measure of half the circle's inclination to the primitive (355); therefore E*p* bisects the angle PEO.

And, because E*p* is perpendicular to E*r* (for *p* and *r* are the two poles), the angle *r*E*p* is equal to the angle *p*EK, being each of them a right angle; also PE*p* and *p*EO have been shown to be equal, therefore the angle *r*EP is equal to the angle OEK.

Consequently E*p* makes equal angles internally, and *r*EK equal angles externally with the sides EP and EO of the triangle PEO. Therefore (by Props. 3 and A Euclid VI.)

$$EO : EP :: pO : pP; \text{ and } EO : EP :: rO : rP.$$

(363) COROLLARY I. *The line* rO, *comprehended between the centre of the projected circle and its external pole, is harmonically* divided by the internal pole of the projected circle, and the centre of the primitive.*

$$\begin{aligned} &\text{For } EO : EP :: pO : pP \\ &\text{and } EO : EP :: rO : rP; \\ &\therefore\ rO : rP :: pO : pP. \end{aligned}$$

(364) COROLLARY II. *Every projected great circle, making an angle with the primitive, is greater than the primitive.*

For the radius EO of the projected great circle EA*e*BC, is the secant of that circle's inclination to the primitive (353);

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* When a straight line is divided into three such parts, that *the whole is to the first part, as the third part is to the second,* it is said to be divided in harmonical ratio.

and therefore is always greater than EP, the radius of the primitive.

PROPOSITION VIII. (*Plate V. Fig.* 6.)

(365) *If through the internal projected pole of a great circle, and the extremities of an arc of that projected circle, straight lines be drawn to cut the primitive, the intercepted arcs of the projected circle and the primitive will be similar.*

Let EFeG represent the primitive circle, and ELA*ehf* the projection of a great circle inclined to the primitive; P the centre of the primitive, and *p* the internal projected pole of the great circle; draw A*p*G and L*p*K, then will the arc GK be similar to the arc AL.

For, it is obvious that if a straight line be drawn from a point within a circle to meet the circumference, the angle formed at the circumference between this line and the radius, will be less than a right angle: therefore in the triangles PK*p*, *p*LO, the angles PK*p* and *p*LO are each less than a right angle. The angle P*p*K is equal to the angle L*p*O (15 Euclid I), and the sides about the remaining angles at P and O are proportional; for OL : PK : : O*p* : P*p* (362), or PK : P*p* : : OL : O*p*. Therefore the triangles *p*PK and *p*OL are equiangular and similar (7 Euclid VI), and consequently the angle KP*p* is equal to the angle *p*OL.

But equal angles at the centres of circles are subtended by similar circumferences, therefore the arc GK is similar to the arc AL; and because OL : PK : : arc AL : arc GK, whatever part of the circumference of the projected circle AL is, the same part of the circumference of the primitive circle will GK be.

(366) COROLLARY I. *If straight lines be drawn from the projected pole of a great circle (towards the pole of the primitive) to cut the projected great circle and the primitive, the arc of the primitive intercepted between these lines, will be the measure of an arc on the sphere, represented by that part of the projected great circle intercepted between these lines.*

Let *p*F, *p*H, and *p*E be drawn; FH will be the measure of a part of a great circle on the sphere represented by AL; and FE will be the measure of a part of a great circle on the sphere represented by AE; for, join OE and P*m*, then, art. (365) OL : PK : : arc AL : arc GK, and OE : P*m* : : arc AE : arc G*m*. But OE = OL and P*m* = PK, therefore arc AL : arc GK : : arc AE : arc G*m*.

Hence, whatever part of the great circle EA*ehf* is represented by AE, the same part of the primitive G*m* must represent; but AE is the projection of a quadrant of a circle on the sphere, having the same radius as the primitive, therefore it must be measured by FE, and not by G*m*; for the same reason AL must be measured by FH and not by GK.

(367) Corollary II. *Hence it follows that all great circles inclined to the primitive, have equal arcs on the sphere, represented by unequal arcs on the plane of projection.*

PROPOSITION IX. *Problem.* (*Plate IV. Fig.* 14.)

(368) *To find the pole of any great circle.*

I. The pole of the primitive *a*EBD is the centre P.

II. Let *a*B be a right circle. Through P the centre of the primitive draw DE perpendicular to *a*B, then D and E are the poles of *a*B.

III. Let CA*e* be an oblique circle. Through the centre of the primitive draw *m*P*n* at right angles to CP*e*, cutting the oblique circle in O; draw *ev* through O, and make *vw* an arc of 90 degrees; join *ew*, then *p* is the pole.

Or, the semi-tangent * of the complement of PO set from P to *p*, will give the pole.

The pole of a small circle is the same with the pole of its parallel great circle.

The truth of this proposition is shown at (355); for O*n*, the measure of the angle C, is the complement of PO, and the semi-tangent of O*n* is applied from P to *p*.

PROPOSITION X. *Problem.* (*Plate IV. Fig.* 14.)

(369) *Through any given point in the circumference of the primitive circle, to describe a great circle making any given angle with the primitive.*

Let E*a*DB be the primitive circle, and C the given point.

Through C draw CP*e*, and *m*P*n* at right angles to it; make the angle PC*r* equal to the given angle, and draw C*r*; with *r* as a centre, and distance *r*C, describe the circle CA*e*, then ACB will be equal to the given angle.

Or, make P*r* equal to the tangent of the given angle to the radius CP; or make C*r* the secant thereof.

Or thus: Set off the semi-tangent of the complement of the given angle from P to O, and through the three points CO*e* describe a circle.

These constructions are obvious from arts. (353) and (354).

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* The common plane scale generally contains a line of semi-tangents, marked S. T. See Book I. Chap. IV. page 16.

PROPOSITION XI. *Problem.* (*Plate IV. Fig.* 15.)

(370) *Through a given point* A *in a right circle a*B, *to describe a great circle* C*A*e, *making an angle at* A, *equal to a given number of degrees.*

Draw DE at right angles to *a*B; find the centre S of a circle which will pass through the three points DAE. Through S draw *dsm* parallel to DE; with the centre A, and a radius equal to PB, describe an arc *ww*, set off *bo* from a scale of chords equal to the complement of the given angle A, and draw A*od*; then *d* is the centre of the circle CA*e* required.

The truth of this construction may be shown thus: S is the centre of the circle DAE, and since all great circles cut each other at the distance of a semicircle (285), all circles passing through A must necessarily pass through the point G, and therefore their centres must be somewhere in the line *dm*. Now *d*A*e* is a right angle, therefore SA*e*=CAB (295) taken from *d*A*e*, leaves *d*AS the complement of the angle CAB, agreeably to the construction.

PROPOSITION XII. *Problem.* (*Plate IV. Fig.* 16.)

(371) *Through a given point* C, *within the primitive, to draw a great circle* AC*e making a given angle* BAC *with the primitive.*

From the centre P of the primitive, with the tangent of the given angle as a radius, describe an arc; and from the point C, with the secant of the same angle as a radius, cross it in O. The point O is the centre, and OC is the radius of the circle AC*e* required.

The truth of this construction is shown in art. (353).

PROPOSITION XIII. *Problem.* (*Plate IV. Fig.* 17.)

(372) *Through two given points within the primitive, to draw a great circle.*

Let O and *m* be the two points. Through one of the points, as O, and the centre of the primitive, draw AB of an unlimited length towards B, and draw CD at right angles to AB. Join CO, and produce it till it cuts the primitive in *v*, draw the diameter *vw* meeting the circumference in *w*, and draw C*w*E meeting AB in E. Then a circle described through the three points, O, *m*, E, will be the great circle required.

The angle *vcw* being in a semicircle is a right angle (31 Euclid III), therefore the distances PO and PE, measured on the line of semi-tangents are together equal to 180 degrees; consequently the point E is diametrically opposite to O, and O*m*E is a great circle.

PROPOSITION XIV. *Problem.* (*Plate IV. Fig.* 18.)

(373) *Through a given point in any projected great circle, to draw another great circle perpendicular to the given one.*

General Rule. Find the pole of the given circle (368), and through that pole and the given point draw a great circle (372), which will be the circle required. For if one great circle pass through the pole of another, it cuts it at right angles (291).

Or, I. The pole of the primitive is in the centre, therefore a diameter of the primitive always cuts its circumference at right angles.

II. *To draw an oblique circle perpendicular to an oblique circle.*

Let BAD be the given oblique circle and A the given point, find *p* the pole of the oblique circle BAD (368), and through the points *p* and A, draw the great circle ECA*p* (372).

III. *To draw an oblique circle perpendicular to a right circle.*

Let HG be the right circle and A the given point, draw DB at right angles to HG, then B and D are the poles of HG, through the three points B, A, D, draw the circle BAD, and it will cut HG at right angles.

PROPOSITION XV. *Problem.* (*Plate IV. Fig.* 19, 20, *and* 21.)

(374) *About any given point, as the pole of a great circle, to describe a small circle at a given distance from that pole. Or, at a proposed distance from a given great circle to describe a parallel circle.*

I. If the small circle be parallel to the primitive. With the radius PC equal to the semi-tangent of the circle's distance from the pole of projection opposite the projecting point (Plate IV. Fig. 19.), describe the circle CDE.

II. If the small circle be parallel to a right circle. Let the given right circle be AB, whose poles are C and D (Plate IV. Fig. 20). Set off the chord of the small circle's distance from its pole, from C to *n* and *m*; or the chord of its distance from the right circle AB, from B to *n*, and from A to *m*; and draw A*m*F and A*on* intersecting PF in *o* and F, which are the extremities of the diameter of the circle *mon* to be projected, the middle point E being the centre.

Or, Having found the points *m* and *n*, join P*n* and draw *n*E at right angles to it; it will cut the axis PC produced in E, the centre of the parallel circle *mon* (356).

Or, Find the three points, *m*, *o*, and *n*, as above, and draw a circle through them.

III. If the small circle be parallel to an oblique circle. Let the oblique circle be DAB (Plate IV. Fig. 21). Find *p* the pole of the oblique circle DAB, and through *p* draw D*p*E; make E*v* and E*w* each equal to the proposed distance from the pole, and draw D*w* and D*v*, cutting GF produced in *o* and F, the two extremities of the diameter of the circle *mon* to be projected, the middle point C being the centre.

Or, Find *p* the pole of the oblique circle DAB (368), and measure its distance P*p* from the centre of the primitive by a scale of semi-tangents. Then add and subtract this distance to and from the complement of the parallel circle's distance from the oblique circle; set off the sum by a scale of semi-tangents, from P to F, and the difference from P to *o*; the middle point between *o* and F, (*viz.* C) is the centre (359).

PROPOSITION XVI. *Problem.* (*Plate V. Fig.* 1.)

(375) *About any given point, as the pole of a great circle, to describe a great circle in a given primitive circle.*

I. If the given point be (P) in the centre of the primitive, the primitive is the great circle required.

II. If the given point be (C) in the circumference of the primitive, through C, draw a diameter CP*e*, and another *b*PB at right angles to it. Then *b*PB is the great circle required.

III. If the given point be (*p*), neither in the centre nor in the circumference of the primitive. Through *p* and the centre of the primitive, draw a straight line *r*PB, and cross it at right angles with the diameter *e*C; through *p* draw *epw*, make *wm* equal to *w*C, and through *m* draw *emr*, then with *r* as a centre and radius *re*, describe the required great circle CA*e*.

This problem is easily deduced from arts. (353) and (355).

PROPOSITION XVII. *Problem.* (*Plate V. Fig.* 2.)

(376) *To measure any arc of a great circle.*

General Rule. Find the pole of the given circle (368), from which draw straight lines through the ends of the arc to be measured, cutting the primitive in two points, the distance between these points applied to a scale of chords, will give the measure of the arc.

Or, I. The pole of the primitive is the centre P, therefore any arc of the primitive is measured by taking the extent of the arc (as *be*) and applying it to a scale of chords.

II. To measure any part of a right circle as P*a*, P*p*, *pv*, &c. From C the pole, draw C*af*, CP*e*, C*pg*, C*vw*; then *ef* applied to a scale of chords is the measure of P*a*, *eg* of P*p*, and *gw* of *pv*.

Or P*a*, P*p*, P*v* applied to the line of semi-tangents, will give their respective measures. And it is evident that *vp* is the difference between the measures of P*v* and P*p*, and that *ap* is the sum of the measures of P*a* and P*p*.

III. To measure any part of an oblique circle as C*r*, *ro*, *oa*, &c. Find *p* the pole thus: draw C*af*, make *fg* an arc of 90 degrees, and join *g*C, then *p* is the pole. From *p* through *r*, *o*, *a*, *m*, draw *p*R, *p*O, *p*A, *p*M. Then RO, applied to a scale of chords, is the measure of an arc on the sphere represented by *ro*; OA is the measure of an arc on the sphere represented by *oa*; and AM is the measure of an arc on the sphere represented by *am*, &c. (366).

PROPOSITION XVIII. *Problem.* (*Plate V. Fig.* 2.)

(377) *To cut any number of degrees from the arc of a great circle; or, from a given point, in a projected great circle, to cut off an arc equal to a given arc.*

This proposition is the reverse of Prop. XVII.

I. If the projected great circle be the primitive whose pole is P, and *e* the given point. Take the number of degrees in the given arc from a scale of chords and apply the extent from *e* to *b*, then *eb* is the arc required.

II. If the projected great circle be a right circle (as *x*A) whose pole is C, and *a* the given point. From the pole C, through *a*, draw C*af*, make *fg* equal to the given arc, by a scale of chords, and join C*g*, then will *pa* contain the required number of degrees.

Or, any number of degrees may be cut from *x*A by the scale of semi-tangents, in the same manner as the arcs were measured in the XVIIth Problem.

III. If the projected circle be an oblique circle (as C*ae*) whose pole is *p*, and *o* the given point. From *p* through *o* draw *p*O*o*, make OR equal to the given number of degrees, and join *p*R, then *or* will contain the number of degrees required.

PROPOSITION XIX. *Problem.* (*Plate V. Fig.* 3.)

(378) *Any great circle* CA*e in the plane of projection being given: to describe another great circle* BAD, *which shall cut the given circle* CA*e*, *and also the primitive in any assigned angles.*

About the pole P of the primitive describe the parallel circle *no*, at a distance equal to the angle which BAD is required to make with the primitive (374). About *p*, the pole of CA*e*, at a distance equal to the measure of the angle which BAD is required to make with CA*e*, describe the parallel circle *ab* cutting *no* in *d*.

About *d* as a pole describe the great circle BAD (375), cutting the primitive in B, and CA*e* in A: then BAD is the great circle required.

For P*d*, *pd*, the distances of the poles on the sphere are the measures of the inclinations of the plane of the great circle BAD with the planes of the primitive and the great circle CA*e*, or of the projected angles which that circle makes with the primitive and with CA*e*.

When the small circles thus described do not meet each other, the problem is impossible.

PROPOSITION XX. *Problem.* (*Plate V. Fig.* 4.)

(379) *To measure any spherical angle.*

General Rule. Find the poles of the two great circles which form the angle (368). Straight lines drawn from the angular point through these poles, will cut the primitive in two points; the distance between which, applied to a scale of chords, will give the measure of the required angle.

The truth of this appears from (298).

Or, I. If the angular point be at P, and *r*P*v* the angle to be measured, then P*r*, P*v* are quadrants; and as a spherical angle is always measured on the arc of a great circle at a quadrant's distance from the angular point (297), *rv* applied to a scale of chords is the measure of the angle *r*P*v*.

II. If the angle be formed by the primitive, and an oblique circle, as *o*CE.

Find *m* the pole of CA*e*, and from C draw C*mw*; through P, the pole of the primitive, draw CP*e*, then *we* applied to a scale of chords gives the measure of *o*CE. To measure *o*CP, the angle formed by the right circle CP*e*, and the oblique circle CA*e*; through *m* the pole of CA*e*, and *r* the pole of CP*e*, draw C*w* and C*r*; then *wr* applied to a scale of chords is the measure of *o*CP.

Or, The angle *o*CE may be measured by the line of semi-tangents; thus, let *o*E be applied to the scale of semi-tangents from 90 towards the left hand, the number of degrees contained between the points of the compasses, will be the measure of *o*CE; and P*o*, applied to the scale of semi-tangents, from the beginning of the scale towards 90, will give the measure of *o*CP.

III. If the angle be formed by two oblique circles, as CA*e* and BAD.

Find *m* the pole of CA*e* and *n* the pole of BAD (368). From the angular point A, through *m* and *n*, draw A*ms* and A*nv*, cutting the primitive in *s* and *v*; then *sv* applied to a scale of chords, will give the measure of the angle BAC, or of its equal *e*AD.

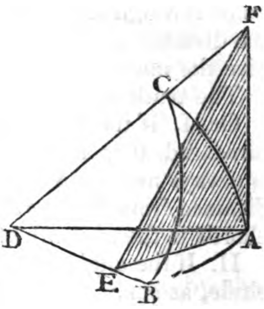
CHAP. III.

INVESTIGATION OF GENERAL RULES FOR CALCULATING THE SIDES AND ANGLES OF RIGHT-ANGLED SPHERICAL TRIANGLES.

PROPOSITION XXI.

(380) In any right-angled spherical triangle,

Radius is to the sine of either side, as the tangent of the adjacent angle is to the tangent of the opposite side.



Let ABC be a spherical triangle, right-angled at A,

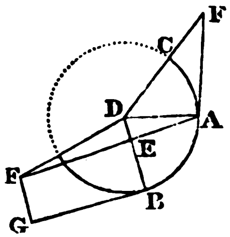
rad : sin AB : : tan B : tan AC.

For, let D be the centre of the sphere, join DC, DA, and DB, and draw AE perpendicular to DB; also, in the plane DBC, draw EF perpendicular to DB, meeting DC produced in F, and join AF.

Then since DE is perpendicular to both EA and EF, it will be perpendicular to the plane AEF; also because the plane ADB passes through DE, it will also be perpendicular to the plane AEF; or, which is the same thing, the plane AEF will be perpendicular to the plane ADB. But the angle BAC being a right angle (by hypothesis), the plane ADC or ADF will also be perpendicular to the same plane ADB. Hence the planes AEF, ADF being each perpendicular to the same plane ADB, their common section AF will also be perpendicular to ADB. And since AE, EF which lie in the planes ADB, BDC, are each perpendicular to BD, the angle AEF will be the measure of their inclination, or of the spherical angle ABC. Also FAD, AED being right angles, AF will be the tangent of the arc AC, and AE the sine of the arc AB, to the radius AD of the sphere. Hence FAE being a right-angled triangle, we have

rad : AE : : tan FEA : AF
or rad : sin AB : : tan ABC : tan AC.

SCHOLIUM.



The foregoing proposition is of considerable importance in spherical trigonometry, and therefore ought to be clearly understood.—It will perhaps be of advantage to some students, to illustrate the nature of it in a mechanical way, in order to render the preceding demonstration more perspicuous.

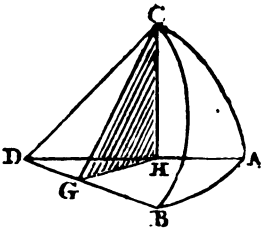
Upon a piece of pasteboard, with any radius describe a circle, and make AC of any length less than a quadrant; draw AF perpendicular to the radius AD, and draw DCF through C; let AB be of any length less than a semicircle, and draw DB. From A draw AEF cutting BD at right angles, make DF equal to DCF, and draw BG a tangent to BD.

Then cut out the figure FABGFDCF, next cut the back of the line AD half through, do the same with the line DB, and raise AF till it is perpendicular to the plane ADB; then raise up the plane DEBGF, so that the points F may coincide, and you will have a plane triangle FAE right-angled at A, which will show the nature of the problem as clearly as possible.

PROPOSITION XXII.

(381) In any right-angled spherical triangle,

Radius is to the sine of the hypothenuse, as the sine of any angle is to the sine of its opposite side.



Let ABC be a spherical triangle, right-angled at A.

rad : sin BC : : sin B : sin AC.

For, let D be the centre of the sphere, and join DC, DB, DA.

In the plane DBC draw CG perpendicular to DB, and it will be the sine of the hypothenuse BC.

From the point G in the plane DBA, draw GH perpendicular to DB; and join CH; then, it may be proved as in art. (380), that GHC will be a right angle, and the angle CGH will be equal to the spherical angle ABC.

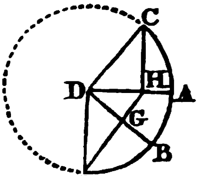
The plane triangle GHC is right-angled at H, the hypothenuse GC is the sine of the spherical hypothenuse BC, the perpendicular CH is the sine of the spherical perpendicular AC, and the angle CGH is equal to the spherical angle ABC. Hence,

rad : GC : : sin CGH : CH; that is,

rad : sin BC : : sin CBA : sin AC.

SCHOLIUM.

This proposition may perhaps be rendered more familiar to the understanding by a mechanical illustration.

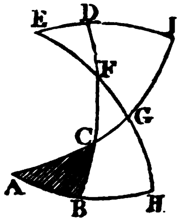


Upon a piece of pasteboard, with any radius describe a circle, make BC, the hypothenuse of any length less than a quadrant; and draw the radii BD, CD; from C draw CG at right angles to BD; set off AB of such a length that when CG is produced it may cut AD somewhere, as in H; at H erect the perpendicular HC, and let DC be drawn.

Cut out the figure CDCABC, then cut the lines AD and BD half through, as in the XXIst proposition, raise the plane ADC perpendicular to the plane ADB, and the plane BDC, till the points C coincide; you will then have a right-angled plane triangle CHG, from which the whole proposition will appear exceedingly simple and easy.

PROPOSITION XXIII.

(382) In any two right-angled spherical triangles, having the same acute angle at the base,



The sines of their bases have the same ratio to each other as the tangents of their perpendiculars; and the sines of their hypothenuses have the same ratio to each other as the sines of their perpendiculars.

Let the right-angled spherical triangles ABC and AHG have the acute ∠ A common; then

rad : sin AB : : *tan* A : tan BC (380).
rad : sin AH : : *tan* A : tan HG (380).
Therefore, sin AB : sin AH : : tan BC : tan HG.
rad : sin AC : : *sin* A : sin BC (381).
rad : sin AG : : *sin* A : sin HG (381).
Hence, sin AC : sin AG : : sin BC : sin HG.

SCHOLIUM.

(383) The different cases or varieties that may happen in the solution of right-angled spherical triangles, wherein two things, together with the right angle, are always given to find a third, are in all sixteen, and from this proposition alone, by producing the sides of the triangle to quadrants (341) the whole may be solved. The five following corollaries, or cases, in-

clude the whole practice of right-angled spherical triangles, and are the foundation of Baron Napier's rules: —

CASE I. sin AH : sin AB : : tan GH : tan BC (382)
Viz. rad : sin AB : : tan A : tan BC (341)
But rad : tan : : cot : rad (209)
∴ cot A : rad : : sin AB : tan BC
Also rad : sin AC : : sin C : sin AB (381)

CASE II. rad : sin BC : : tan C : tan AB (380)
But rad : tan : : cot : rad (209)
∴ cot C : rad : : sin BC : tan AB
Also sin AG : sin AC : : sin HG : sin BC (382)
Viz. rad : sin AC : : sin A : sin BC (341)

CASE III. sin EI : sin ED : : tan IG : tan DF (382)
Viz. rad : cos C : : tan AC : tan BC (341)
But rad : tan : : cot : rad (209)
∴ cot AC : rad : : cos C : tan BC
Also sin DC : sin ID : : sin CF : sin FG (382)
Viz. rad : sin C : : cos BC : cos A (341)
OR rad : sin A : : cos AB : cos C

CASE IV. sin FH : sin FG : : tan BH : tan CG (382)
Viz. rad : cos A : : cot AB : cot AC (341)
But rad : cot : : tan : rad (209)
∴ tan AB : rad : : cos A : cot AC
Also sin CF : sin CD : : sin FG : sin ID (382)
Viz. cos BC : rad : : cos A : sin C (341)

CASE V. sin CG : sin CI : : tan FG : tan DI (382)
Viz. cos AC : rad : : cot A : tan C (341)
But rad : tan : : cot : rad (209)
∴ cos AC : cot A : : cot C : rad
Also sin BH : sin CG : : sin FB : sin CF (382)
Viz. cos AB : cos AC : : rad : cos BC (341)

(384) If the extremes and means of each case be multiplied together, we shall obtain the same equations as are produced by Baron Napier's rules.

I. rad × sin AB = tan BC × cot A
rad × sin AB = sin AC × sin C
II. rad × sin BC = tan AB × cot C
rad × sin BC = sin AC × sin A
III. rad × cos C = cot AC × tan BC
rad × cos C = sin A × cos AB
IV. rad × cos A = cot AC × tan AB
rad × cos A = cos BC × sin C
V. rad × cos AC = cot A × cot C
rad × cos AC = cos AB × cos BC

(385) If we use the notation of Legendre[*], and call the hypothenuse b, the base c, and the perpendicular a, and their opposite angles B, C, and A, the equations may be thus expressed.

I. $\text{rad}\times\sin c=\tan a\times\cot A=\sin b\times\sin C$
II. $\text{rad}\times\sin a=\tan c\times\cot C=\sin b\times\sin A$
III. $\text{rad}\times\cos C=\cot b\times\tan a=\cos c\times\sin A$
IV. $\text{rad}\times\cos A=\cot b\times\tan c=\cos a\times\sin C$
V.[†] $\text{rad}\times\cos b=\cot A\times\cot C=\cos c\times\cos a$

Any two of these quantities, together with the right angle, being given, the rest may be found.

BARON NAPIER'S UNIVERSAL RULES FOR SOLVING RIGHT-ANGLED SPHERICAL TRIANGLES.

PROPOSITION XXIV.

(386) I. *Radius* × *sine of the middle part*=*rectangle of the tangents of the extremes when conjunct.*

And,

II. *Radius* × *sine of the middle part*=*rectangle of the cosines of the extremes when disjunct.*

Observing to use the complements of the hypothenuse and angles.

EXPLANATION OF THE RULES.

In every right-angled spherical triangle there are FIVE CIRCULAR PARTS, exclusive of the right angle, which is not taken into consideration: and these five parts are the hypothenuse, the two legs or sides, and their opposite angles; they are called circular parts, because the measure of each of them is the arc of a circle.

Now in every case proposed for solution, there are three of these five parts concerned, *viz.* two given and a third required.

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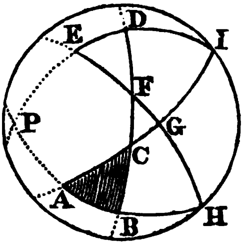
[*] Eléments de Géométrie, page 380.

[†] A late writer on trigonometry, after telling us in his preface that the well-known rules of Napier, called the FIVE CIRCULAR PARTS, are "too artificial and restricted to be generally employed in the present advanced state of the science;" gives the rules above, and actually solves all his cases of right-angled spherics by them, though they are exactly the same as those of Napier, but less commodiously expressed. And whatever improvements may have been made in trigonometry, either by the English or Foreign mathematicians, it is a certain truth that "Mathematical science cannot boast of a neater compendium of results, or a more valuable aid to the memory of the student," than Napier's rules. *Vide* Monthly Review, Vol. LIII. page 280.

It is therefore obvious that when the three parts follow each other in a successive order, the middle one is the *middle part*, and the two others the *extremes conjunct*, that is, joined to the middle part.



Suppose in the triangle ABC, that the angles A and C, and the hypothenuse AC are the parts concerned, it is evident AC is the middle part, and that A and C are joined, or adjacent to AC, and therefore are *conjunct*.

Again, if the three parts do not follow each other in a successive order, that which is not connected with either of the other two is *invariably* called the *middle part;* and the other two, which are not connected with it, are called the *extremes disjunct;* that is, not joined to the middle part.

Suppose the hypöthenuse AC, the base AB, and the perpendicular BC, the parts concerned; AC will be the middle part, for it is not connected either with AB or BC, the angle A intervening in the former case, and the angle C in the latter; the sides AB and BC are *extremes disjunct*, that is, not joined to AC, because of the intervention of the angles A and C, but the sides AB and BC are considered as joined, though the right angle B is between them, for, as we have before observed, it is excluded, or not taken into consideration.

It must be remembered that, in speaking of the hypothenuse or either of the angles as a middle part, or extremes conjunct or disjunct, their complements are to be used; but when we speak of the sides or legs, *their* complements are not to be used, but the *real* sides or legs.

Hence the middle part must *universally* be either the base AB, the perpendicular BC, the complement of the angle C, the complement of the angle A, or the complement of the hypothenuse AC, that is, it must be some one of the five circular parts.

(387) CASE I. FIRST, *let* AB, BC, *and the angle* A, *be the parts under consideration, in the triangle* ABC.

Here the three parts are joined together, because the right angle B is not regarded. Therefore AB is the *middle part*, BC and the complement of A, are the *extremes conjunct*.

$$\text{Hence, rad} \times \sin AB = \tan BC \times \cot A.$$

SECONDLY. *Let* AB, AC, *and the angle* C, *be the parts under consideration, in the triangle* ABC.

The base AB is still the *middle part*, for it is not connected either with AC or the angle C, therefore these parts are the *extreme disjunct;* the former being separated from AB by the angle A, and the latter by the side BC. Here we apply the second rule, recollecting that the cosine of the complement of an angle, or the hypothenuse, is the sine itself.

Hence, rad × sin AB = sin AC × sin C.

(388) CASE II. FIRST, *Let* AB, BC, *and the angle* C, *be the parts under consideration, in the triangle* ABC.

This is exactly like the first part of the preceding case; the three parts are connected, BC is the *middle part*, AB and the complement of C are the *extremes conjunct.*

Hence, rad × sin BC = tan AB × cot C.

SECONDLY. *Let* BC, *the hypothenuse* AC, *and the angle* A, *be the parts concerned in the triangle* ABC.

This is similar to the second part of the preceding case; the perpendicular BC is still the *middle part*, for the angle C separates it from AC, and the side AB from the angle A: therefore AC and the angle A are *extremes disjunct;* that is, not joined to BC the middle part. Here we must apply the second rule, taking care to remember that the cosine of the complement of the hypothenuse, or an angle, is the sine itself.

Hence, rad × sin BC = sin AC × sin A.

(389) CASE III. FIRST, *Let* BC, AC, *and the angle* C, *be the parts under consideration, in the triangle* ABC.

The three parts follow each other, without the intervention of any other quantity; therefore the complement of C is the *middle part*, BC and the complement of AC are the *extremes conjunct*, that is, joined to the angle C.

Hence, rad × cos C = cot AC × tan BC.

SECONDLY. *Let* AB, *the angle* A, *and the angle* C, *be the parts under consideration, in the triangle* ABC.

The complement of the angle C is here the *middle part*, being separated from the angle A by AC; and from the side AB by the perpendicular BC; therefore AB and the complement of the angle A, are the *extremes disjunct*, or not joined to C.

Hence, rad × cos C = sin A × cos AB.

(390) CASE IV. FIRST, *Let* AB, AC, *and the angle* A, *be the parts under consideration, in the triangle* ABC.

This is exactly similar to the first part of Case III.

The complement of the angle A is the *middle part*, AB and the complement of AC, are the *extremes conjunct*, that is, they are joined to the angle A.

Hence, rad × cos A = cot AC × tan AB.

SECONDLY. *Let* BC, *the angle* A, *and the angle* C, *be the parts under consideration, in the triangle* ABC.

This is exactly of the same nature with the second part of Case III. The complement of the angle A is the *middle part;* BC and the complement of C, are the *extremes disjunct*, the former being separated from the middle part A, by the base AB, and the latter by the hypothenuse AC.

Hence, rad × cos A = cos BC × sin C.

(391) CASE V. FIRST, *Let the hypothenuse* AC, *the angle* A, *and the angle* C, *be the parts under consideration, in the triangle* ABC.

The three parts follow each other; therefore the complement of AC is the *middle part*, the complement of A and the complement of C are the *extremes conjunct*, that is, they are joined to the middle part AC.

Hence, rad × cos AC = cot A × cot C.

SECONDLY. *Let the hypothenuse* AC, *the base* AB, *and the perpendicular* BC, *be the parts under consideration, in the triangle* ABC.

The complement of AC is here the *middle part*, being separated from AB by the angle A, and from BC by the angle C; therefore AB and BC are the *extremes disjunct*.

Hence, rad × cos AC = cos AB × cos BC.

SCHOLIUM.

The preceding cases include all the varieties that can possibly happen in the practice of right-angled spherical triangles.

Any of the equations may be turned into a proportion by putting the required term last, that with which it is connected first, and the other two in the middle of any order. These equations are exactly the same as those already given (384), and therefore Napier's rules are universally true.

[illegible] is not connected [illegible] these parts are the [illegible] separated from AB by the [illegible] Here we apply the [illegible] complement of [illegible]

[illegible]

[illegible] angle C, be the [illegible] ABC.

[illegible] preceding case; the [illegible] middle part, AB and the [illegible] extremes conjunct.

Hence, [illegible] C.

[illegible] and the angle A, be [illegible] ABC.

[illegible] the preceding case; [illegible] middle part, for the angle C [illegible] the angle A: there[illegible] extremes disjunct; that is, not [illegible] Here we must apply the second [illegible] remember that the cosine of the complement of the [illegible], or an angle, is the sine itself.

Hence, rad × sin BC = sin AC × sin A.

(389) Case III. First, *Let* BC, AC, *and the angle* C, *be the parts under consideration, in the triangle* ABC.

The three parts follow each other, without the intervention of any other quantity; therefore the complement of C is the *middle part*, BC and the complement of AC are the *extremes conjunct*, that is, joined to the angle C.

Hence, rad × cos C = cot AC × tan BC.

Secondly. *Let* AB, *the angle* A, *and the angle* C, *be the parts under consideration, in the triangle* ABC.

The complement of the angle C is here the *middle part*, being separated from the angle A by AC; and from the side AB by the perpendicular BC; therefore AB and the complement of the angle A, are the *extremes disjunct*, or not joined to C.

Hence, rad × cos C = sin A × cos AB.

(390) Case IV. First, *Let* AB, AC, *and the angle* [illegible] *parts under considerati*[illegible] *triangle* ABC.

This is ex[illegible] first part of C[illegible]

The com[illegible]

the complement [illegible] ; thus, if are joined to [illegible] tions be [illegible] that with [illegible] middle in

[illegible]

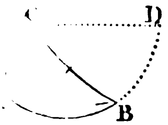
SECONDLY. [illegible] *under consideration* [illegible] equations

This is exac[illegible] lower equa- Case III. The [illegible] as the other *part;* BC and [illegible] marked or re- the former bei[illegible] base AB, and the [illegible]

[illegible] *an oblique-* [illegible] *angles were*

(391) CASE [illegible] *and the angle* [illegible]

ABC.



The three p[illegible] ment of AC is [illegible] complement o[illegible] joined to the [illegible] are the *ex-*

[illegible]

SECONDLY. *Let the* [illegible] BCD. *perpendicular* BC, *be* [illegible] DC × sin ACD. *angle* ABC. [illegible] × sin BCD.

The complement [illegible] sin ACD. rated from AB by [illegible] required, we must have therefore AB and [illegible] only the ∠ A must have

Hence, [illegible] proportion would have [illegible] cos ∠ A.

[illegible] BCD, gives ACB according [illegible] thout the triangle.

The prec[illegible] *of the angles at the base,* sibly happen[illegible] *of the angles at the* angles. [illegible] *from the vertical angle*

Any of [illegible] putting the [illegible] first, and [illegible] *of the cosines of the angles* equation[illegible] and then [illegible]

[illegible] are given; if the perpendicular fall [illegible] venient to draw it so that *two known* parts [illegible] and the *given* angle to which it is opposite; [illegible] in this case.

[illegible] out of the equations are printed in *italics.*

CHAP. IV.

INVESTIGATION OF GENERAL RULES FOR SOLVING THE DIFFERENT CASES OF OBLIQUE SPHERICAL TRIANGLES, BY DRAWING A PERPENDICULAR FROM THE VERTICAL ANGLE UPON THE BASE.

PROPOSITION XXV.

Showing the manner of applying Baron Napier's *rules to oblique spherical triangles, from which several useful corollaries are deduced.*

(392) When the three *given* parts do not follow each other in a regular order, viz. when an unknown part intervenes, a perpendicular should always be drawn from the end of a *given* side, and opposite to an *adjacent given angle.*

But when the three *given* quantities follow each other without the intervention of an unknown quantity, the perpendicular should be drawn in such a manner, as to fall not only from the end of a *given* side and opposite to *an adjacent given angle*, but likewise from the end of a *required* side, or opposite to a *required* angle, according as a side or an angle is the subject of inquiry.†

(393) Having drawn a perpendicular, agreeably to the foregoing directions; then, if the vertical angle of the oblique triangle be given or sought, find the vertical angle of that right-angled triangle wherein two things are given; but if the base of the oblique triangle be given or sought, find the base of that right-angled triangle wherein two quantities are given.

Compare the *perpendicular*, the part given, and the part sought, in that triangle wherein only *one* quantity is given; find the middle part, and make an equation agreeably to Napier's Rules, marking the term sought with an asterisk (*).

Compare the *perpendicular* and the similar parts in the triangle where *two* quantities are given, and make an equation, which place exactly under the former, and strike out such terms as are common to both the equations.

---

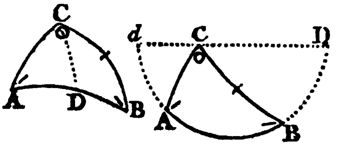
† The reason of these rules for drawing a perpendicular is founded on practice and observation. For in every right-angled triangle there must be two *given* quantities, exclusive of the right angle, and it is obvious that the perpendicular must be so drawn as to form *one* right-angled triangle wherein two quantities are given.

Turn the remaining quantities into a proportion; thus, if all the terms which are to be struck out of the equations be on the same side of both, put the required term last, that with which it is connected first, and the other two in the middle in any order.

But if the terms which are not struck out of the equations stand on both sides; one unmarked term, in the lower equation, is to the unmarked term exactly above it, as the other unmarked term in the lower equation, is to the marked or required term which stands above it.

(394) CASE I. *Suppose two angles at the base of an oblique-angled triangle, and a side opposite to one of these angles were given, to find the vertical angle.*

First, find the vertical angle BCD† in the triangle BDC, where two entire quantities are given.



Here BC is the *middle part*, and the other parts are the *extremes conjunct*. Hence,

rad × cos BC = cot ∠ B × cot $\overset{*}{\text{BCD}}$.

Therefore cot ∠ B : rad : : cos BC : cot BCD.

In the triangle ADC, *rad*‡ × cos ∠ A = *cos* DC × sin $\overset{*}{\text{ACD}}$.

In the triangle BDC, *rad* × cos ∠ B = *cos* DC × sin BCD.

Therefore cos ∠ B : cos ∠ A : : sin BCD : sin ACD.

Had the ∠ C been given, and the ∠ A required, we must have proceeded exactly in the same manner, only the ∠ A must have been marked instead of ACD, and the proportion would have been thus; sin BCD : sin ACD : : cos ∠ B : cos ∠ A.

The sum, or difference, of ACD and BCD, gives ACB according as the perpendicular falls within or without the triangle.

(395) COROLLARY I. *The cosines of the angles at the base, are in proportion to each other as the sines of the angles at the vertex, made by a perpendicular drawn from the vertical angle upon the base.*

(396) COROLLARY II. *The sum of the cosines of the angles at the base.*
Is to their difference,

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† When both the angles at the base are given; if the perpendicular fall without the triangle, it is the most convenient to draw it so that *two known* parts may come between the perpendicular and the *given* angle to which it is opposite; hence CD is the proper perpendicular in this case.

‡ The terms which are to be struck out of the equations are printed in *italics*.

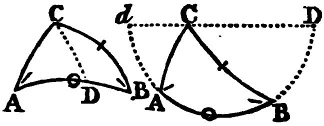
As the sum of the sines of the angles at the vertex,
Is to their difference. For,

$$\cos B : \cos A :: \sin BCD : \sin ACD;$$

And by composition and division,

$$\cos A + \cos B : \cos A \sim \cos B :: \sin ACD + \sin BCD : \sin ACD \sim \sin BCD.$$

(397) CASE II. *Suppose the two angles at the base of an oblique spherical triangle, and a side opposite to one of these angles were given to find the base.*



First find the base DB in the triangle BDC, where two entire quantities are given. Here the complement of CBD is the *middle part*, and the other parts are the *extremes conjunct.* Hence,

$$\text{rad} \times \cos\angle B = \tan \overset{*}{DB} \times \cot BC;$$

therefore $\cot BC : \text{rad} :: \cos\angle B : \tan DB.$

In the triangle ADC, $\textit{rad} \times \sin \overset{*}{AD} = \cot A \times \textit{tan}\ DC.$

In the triangle BDC, $\textit{rad} \times \sin DB = \cot B \times \textit{tan}\ DC.$

Therefore $\cot\angle B : \cot\angle A :: \sin DB : \sin AD.$

Had the base AB been given, and the angle A required, the same method of solution must have been observed, only the angle A must have been marked instead of the segment AD, and the last proportion would have been,

$$\sin DB : \sin AD :: \cot\angle B : \cot\angle A.$$

The sum, or difference, of AD and DB, gives the base AB, according as the perpendicular falls within or without the triangle.

(398) COROLLARY I. *The cotangents of the angles at the base are in proportion to each other, as the sines of the segments of the base.*

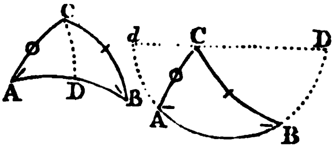
(399) COROLLARY II. *The sum of the cotangents of the angles at the base,*
Is to their difference,
As the sum of the sines of the segments of the base,
Is to their difference. For,

$$\cot\angle B : \cot\angle A :: \sin DB : \sin AD,$$

And by composition and division,

$$\cot\angle B + \cot\angle A : \cot\angle B \sim \cot\angle A :: \sin DB + \sin AD : \sin DB \sim \sin AD.$$

(400) CASE III. *Suppose the two angles at the base, and a side opposite to one of them were given, to find the side opposite to the other.*



Neither the base nor vertical angle being here concerned, it would be superfluous to find any thing in the triangle BDC.

In the triangle ADC, *rad* × *sin* DC = sin $\overset{*}{\text{AC}}$ × sin ∠ A.
In the triangle BDC, *rad* × *sin* DC = sin BC × sin ∠ B.
Therefore sin ∠ A : sin ∠ B : : sin BC : sin AC.

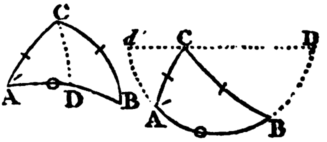
Had the side AC been given, and the angle A required, the angle A must have been marked, and the proportion would have been, sin AC : sin BC : : sin B : sin A.

(401) COROLLARY I. *The sines of the sides are in proportion to each other as the sines of their opposite angles, et contra.*

(402) COROLLARY II. *The sum of the sines of the sides,*
Is to their difference,
As the sum of the sines of the angles at the base,
Is to their difference. For,

sin AC : sin BC : : sin ∠ B : sin ∠ A;
and by composition and division,
sin AC + sin BC : sin AC ∼ sin BC : : sin ∠ B + sin ∠ A : sin ∠ B ∼ sin ∠ A.

(403) CASE IV. *Suppose two sides and an angle opposite to one of them were given, to find the base.*



First find the segment AD† in the triangle ADC where two entire quantities are given.

Here the complement of A is the *middle part*, and the other two parts are the *extremes conjunct.*

Hence, rad × cos ∠ A = cot AC × tan $\overset{*}{\text{AD}}$;
therefore, cot AC : rad : : cos ∠ A : tan AD.

In the triangle BDC, *rad* × cos $\overset{*}{\text{BC}}$ = *cos* DC × cos DB.
In the triangle ADC, *rad* × cos AC = *cos* DC × cos AD;
therefore cos AC : cos BC : : cos AD : cos DB.

Had the side AB been given, and BC required, the operation would have been the same, only BC must have been marked instead of DB; then we should have had this proportion,

---

† When *only one* angle at the base is given; if the perpendicular fall without the triangle, let it be drawn so as to fall opposite to that given angle, whether it be acute or obtuse, hence CD is the proper perpendicular in this case.

cos AD : cos DB : : cos AC : cos BC.

The difference between the base AB and the segment AD gives the segment DB; if AB exceeds AD the perpendicular falls within the triangle, if not, it falls without.

(404) Corollary I. *The cosines of the sides are in proportion to each other, as the cosines of the segments of the base.*

(405) Corollary II. *The sum of the cosines of the sides,*
Is to their difference,
As the sum of the cosines of the segments of the base,
Is to their difference. For,

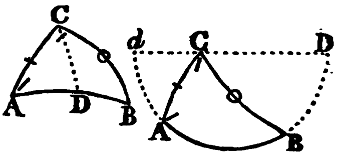
cos AC : cos BC : : cos AD : cos DB;
and by composition and division,

cos AC + cos BC : cos AC ∼ cos BC : : cos AD + cos DB : cos AD ∼ cos DB.

(406) Case V. *Suppose a side, and its two adjacent angles were given, to find a side opposite to one of these angles.*

Find the vertical angle ACD, in the triangle ADC.

Here the complement of AC is the *middle part*, and the complements of A and C are the *extremes conjunct.*



Hence rad × cos AC = cot ∠ A × cot ACD;
therefore cot ∠ A : rad : : cos AC : cot ACD.

Then the difference between ACB and ACD gives BCD. If ACD be less than ACB the perpendicular falls within the triangle, if greater, it falls without.

In the triangle BDC, *rad* × cos BCD = *tan* DC × cot BC.
In the triangle ADC, *rad* × cos ACD = *tan* DC × cot AC.
Therefore cos ACD : cos BCD : : cot AC : cot BC.

Had the side BC been given, and the angle ACB required, the same mode of operation must have been observed, only marking BCD instead of BC, and the proportion would have been thus,

cot AC : cot BC : : cos ACD : cos BCD.

(407) Corollary I. *The cosines of the vertical angles are in proportion to each other as the cotangents of the sides. Or the tangents of the sides are reciprocally as the cosines of the verical angles.* For,

cot AC . cot BC : : cos ACD : cos BCD·

but the tangents are inversely as the cotangents:
hence tan BC : tan AC :: cos ACD : cos BCD.

(408) COROLLARY II. *The sum of the cotangents of the sides is to their difference, as the sum of the cosines of the vertical angles is to their difference.*

And, *The sum of the tangents of the sides,*
Is to their difference,
As the sum of the cosines of the vertical angles,
Is to their difference. For,
cot AC : cot BC :: cos ACD : cos BCD,
and tan BC : tan AC :: cos ACD : cos BCD.
By composition and division,

cot AC + cot BC : cot AC ∼ cot BC :: cos ACD + cos BCD : cos ACD ∼ cos BCD, and

tan BC + tan AC : tan BC ∼ tan AC :: cos ACD + cos BCD : cos ACD ∼ cos BCD.

SCHOLIUM.

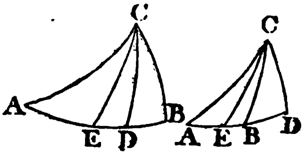
From the preceding cases and their corollaries, we deduce the following general rules for solving ten cases of oblique spherical triangles, viz.

I. *The sines of the sides are directly proportional to the sines of their opposite angles, et contra.*

II. *The cosines of the vertical angles (made by a perpendicular) are as the cotangents of their adjacent sides; and the sines thereof as the cosines of the angles at the base.*

III. *The cosines of the segments of the base (made by a perpendicular) are as the cosines of their adjacent sides; and the sines thereof as the cotangents of their adjacent angles at the base.*

PROPOSITION XXVI.



(409) In any spherical triangle.

I. If the perpendicular fall within the triangle,
The cotangent of half the sum of the two sides,
Is to tangent of half their difference,
As the cotangent of half the base,
Is to the tangent of the distance of a perpendicular from the middle of the base.

II. Or, if the perpendicular fall without the triangle,
Tangent of half the difference of the sides is to cotangent of half their sum, as tangent of half the base is to cotangent of the distance of a perpendicular from the middle of the base.

By art. (405) $\cos AC+\cos BC : \cos AC\sim\cos BC :: \cos AD+\cos DB : \cos AD\sim\cos DB$.

By art. (233) $\cos AC+\cos BC : \cos AC\sim\cos BC :: \cot\frac{1}{2}(AC+BC) : \tan\frac{1}{2}(AC\sim BC)$.

and $\cos AD+\cos BD : \cos AD\sim\cos BD :: \cot\frac{1}{2}(AD+BD) : \tan\frac{1}{2}(AD\sim DB)$.

$\therefore \cot\frac{1}{2}(AC+BC) : \tan\frac{1}{2}(AC\sim BC) :: \cot\frac{1}{2}(AD+DB) : \tan\frac{1}{2}(AD\sim DB)$.

When the perpendicular falls within the triangle, we have, since AB is bisected in E, $\frac{1}{2}(AD+DB)=AE$, and $\frac{1}{2}(AD\sim DB)=DE$;

$\therefore \cot\frac{1}{2}(AC+BC) : \tan\frac{1}{2}(AC\sim BC) :: \cot AE : \tan DE$.

When the perpendicular falls without the triangle,

$\frac{1}{2}(AD\sim DB)=AE$ and $\frac{1}{2}(AD+DB)=DE$.

$\therefore \cot\frac{1}{2}(AC+BC) : \tan\frac{1}{2}(AC\sim BC) :: \cot DE : \tan AE$; and by inversion, $\tan\frac{1}{2}(AC\sim BC) : \cot\frac{1}{2}(AC+BC) :: \tan AE : \cot DE$.

(410) Corollary I. The *distance* of a perpendicular from the middle of the base, or as some writers call it the *altern*, or *alternate* base, is always equal to half the difference of the segments of the base, when the perpendicular falls within the triangle; or equal to half the sum of the segments, when a perpendicular falls without the triangle. Either of the above rules will bring the same conclusion, whether the perpendicular falls within or without the triangle; only in the first case, the fourth number will be less than half the base, and in the second case, it will be greater.

That the rules are both the same may be shown thus,

When the perpendicular falls within the triangle:

$\cot\frac{1}{2}(AC+BC) : \tan\frac{1}{2}(AC\sim BC) :: \cot AE : \tan DE$; and inversely, $\tan\frac{1}{2}(AC\sim BC) : \cot\frac{1}{2}(AC+BC) :: \tan DE : \cot AE$.

But tangents are reciprocally as their cotangents.

$\therefore \tan\frac{1}{2}(AC\sim BC) \cdot \cot\frac{1}{2}(AC+BC) :: \tan AE : \cot DE$; the same conclusion as when the perpendicular falls without the triangle.

(411) Corollary II. *The tangent of half the base,*

Is to the tangent of half the sum of the sides,

As the tangent of half the difference of the sides,

Is to the tangent of the distance of a perpendicular from the middle of the base. Or,

The tangent of half the sum of the sides, is to the cotangent of half their difference, as the tangent of half the base, is to the cotangent of the distance of a perpendicular from the middle of the base. And,

According as this distance is less or greater than half the base, the perpendicular falls within or without the triangle. For we have already shown, that

cot $\frac{1}{2}$ (AC + BC) : tan $\frac{1}{2}$ (AC ∼ BC) : : cot AE : tan DE;
and tan $\frac{1}{2}$ (AC ∼ BC) : cot $\frac{1}{2}$ (AC + BC) : : tan AE : cot DE.

But since tangents are reciprocally as their cotangents,

∴ tan AE : tan $\frac{1}{2}$ (AC + BC) : : tan $\frac{1}{2}$ (AC ∼ BC) : tan DE;
and tan $\frac{1}{2}$ (AC + BC) : cot $\frac{1}{2}$ (AC ∼ BC) : : tan AE : cot DE.

If the triangle be isosceles or equilateral, the perpendicular will fall on the middle of the base, except the sides be quadrants, and then it may fall on any part of the base.

(412) COROLLARY III. In a right-angled triangle, *the rectangle of the tangents of half the sum and half the difference of the hypothenuse and one leg, is equal to the square of the tangent of half the other leg.* For in this case B and D will coincide, and DE will be equal to $\frac{1}{2}$ AB, or equal AE.

PROPOSITION XXVII. (*See the Fig. to* PROP. XXVI.)

(413) In any spherical triangle,

I. If the perpendicular fall within the triangle:
The cotangent of half the sum of the angles at the base,
Is to the tangent of half their difference,
As the tangent of half the vertical angle,
Is to the tangent of the excess of the greater of the two vertical angles, made by a perpendicular, above half the aforesaid vertical angle.

II. Or, if the perpendicular fall without the triangle:
The tangent of half the difference between the base angles,
Is to cotangent of half their sum,
As the cotangent of half the vertical angle,
Is to the cotangent of the excess of the greater of the two vertical angles, formed by a perpendicular upon the base, above half the aforesaid vertical angle.

By art. (396) cos A + cos B : cos A ∼ cos B : : sin ACD + sin BCD : sin ACD ∼ sin BCD.

By art. (233) *cos* A + *cos* B : *cos* A ∼ *cos* B : : cot $\frac{1}{2}$ (A + B) : tan $\frac{1}{2}$ (A ∼ B).

By art. (231) *sin* ACD + *sin* BCD : *sin* ACD ∼ *sin* BCD : : tan $\frac{1}{2}$ (ACD + BCD) : tan $\frac{1}{2}$ (ACD ∼ BCD).

∴ cot $\frac{1}{2}$ (A + B) : tan $\frac{1}{2}$ (A ∼ B) : : tan $\frac{1}{2}$ (ACD + BCD) : tan $\frac{1}{2}$ (ACD ∼ BCD).

When the perpendicular falls within the triangle, we have, since CE bisects the angle ACB,

$\frac{1}{2}$ (ACD + BCD) = $\frac{1}{2}$ ACB, and $\frac{1}{2}$ (ACD ∼ BCD) = ECD,
∴ cot $\frac{1}{2}$ (A + B) : tan $\frac{1}{2}$ (A ∼ B) : : tan $\frac{1}{2}$ ACB : tan ECD.

But when the perpendicular falls without the triangle, $\frac{1}{2}(\text{ACD}-\text{BCD})=\frac{1}{2}\,\text{ACB}$, and $\text{ECD}=\frac{1}{2}(\text{ACD}+\text{BCD})$.

$\therefore \cot\frac{1}{2}(\text{A}+\text{B}) : \tan\frac{1}{2}(\text{A}\sim\text{B}) : \tan\text{ECD} : \tan\frac{1}{2}\,\text{ACB}$.

But tangents and cotangents are reciprocally proportional.

$\therefore \tan\frac{1}{2}(\text{A}\sim\text{B}) : \cot\frac{1}{2}(\text{A}+\text{B}) :: \cot\frac{1}{2}\,\text{ACB} : \cot\text{ECD}$.

(414) Corollary I. The excess of the greater of the two vertical angles (formed by a perpendicular) above half the vertical angle, is equal to half the difference of those vertical angles, when the perpendicular falls within the triangle; or half their sum, when it falls without. Either of the above rules will bring the same conclusion, whether the perpendicular falls within or without the triangle; only in the former case, the fourth number will be less than half the vertical angle, and in the latter it will be greater. That the rules are the same may be shown thus:

When the perpendicular falls without the triangle.

$\tan\frac{1}{2}(\text{A}\sim\text{B}) : \cot\frac{1}{2}(\text{A}+\text{B}) :: \cot\frac{1}{2}\,\text{ACB} : \cot\text{ECD}$; by inversion, $\cot\frac{1}{2}(\text{A}+\text{B}) : \tan\frac{1}{2}(\text{A}\sim\text{B}) :: \cot\text{ECD} : \cot\frac{1}{2}\,\text{ACB}$.

But tangents are reciprocally as their cotangents.

$\therefore \cot\frac{1}{2}(\text{A}+\text{B}) : \tan\frac{1}{2}(\text{A}\sim\text{B}) :: \tan\frac{1}{2}\,\text{ACB} : \tan\text{ECD}$, the same expression as when the perpendicular falls within the riangle.

(415) Corollary II. *The cotangent of half the sum of the ngles at the base, is to the tangent of half their difference, as he tangent of half the vertical angle, is to tangent of half the difference between the two vertical angles, formed by a perpendicular, or to tangent of half their sum, according as their perpendicular falls within or without the triangle.*

CHAP. V.

INVESTIGATION OF GENERAL RULES FOR CALCULATING THE SIDES AND ANGLES OF OBLIQUE-ANGLED SPHERICAL TRIANGLES WITHOUT MAKING USE OF A PERPENDICULAR.

PROPOSITION XXVIII.*

(416) *If the cosine of any side of a spherical triangle be multiplied by the radius, and the rectangle of the cosines of the other two sides be deducted from the product; the remainder divided by the rectangle of the sines of these two sides, will be equal to the cosine of the included angle divided by the radius.*

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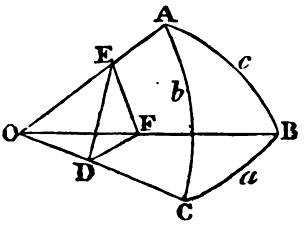
* Legendre's Geometry, 6th Edition, page 386. et seq.

viz. in any spherical triangle ABC,

$$\frac{\cos AB\,.\,\text{rad}-\cos AC\,.\,\cos BC}{\sin AC\,.\,\sin BC}=\frac{\cos C}{\text{rad}}.$$



Let ABC be the proposed triangle, and O the centre of the sphere; join AO, BO, and CO. Take any point D in OC, and in the planes AOC, BOC, draw DE and DF each at right angles to OC, and join EF.

Then because the $\angle$ EDF is the measure of the inclination of the planes AOC, BOC, it is also the measure of the spherical $\angle$ ACB.

In the plane triangle EDF, $\dfrac{\cos EDF}{\text{rad}}=\dfrac{ED^2+DF^2-EF^2}{2DE\,.\,DF}$ (119);

also in the plane triangle EOF, $\dfrac{\cos EOF}{\text{rad}}=\dfrac{EO^2+OF^2-EF^2}{2EO\,.\,OF}$.

From the second of these equations,

$$EF^2=EO^2+OF^2-\frac{2EO\,.\,OF\,.\,\cos EOF}{\text{rad}},$$

which substituted in the first equation gives

$$\cos EDF=\frac{\text{rad}\,(ED^2+DF^2-EO^2-OF^2)+2EO\,.\,OF\,.\,\cos EOF}{2DE\,.\,DF}.$$

But $OE^2-ED^2=OD^2$ (47 Euclid I), and $OF^2-DF^2=OD^2$, consequently $\cos EDF=\dfrac{EO\,.\,OF\,.\,\cos EOF-OD^2\,.\,\text{rad}}{DE\,.\,DF}$.

Now, $\angle EDF=\angle C$; $\angle EOF=AB$; $\dfrac{OE}{DE}=\dfrac{\text{rad}}{\sin DOE}=\dfrac{\text{rad}}{\sin AC}$; $\dfrac{OF}{DF}=\dfrac{\text{rad}}{\sin DOF}=\dfrac{\text{rad}}{\sin BC}$; $\dfrac{OD}{DE}=\dfrac{\cos DOE}{\sin DOE}=\dfrac{\cos AC}{\sin AC}$; $\dfrac{OD}{DF}=\dfrac{\cos DOF}{\sin DOF}=\dfrac{\cos BC}{\sin BC}$; and if these values be substituted in the last expression for cos EDF, we have

$$\cos EDF=\frac{\text{rad}^2\,.\,\cos AB-\text{rad}\,.\,\cos AC\,.\,\cos BC}{\sin AC\,.\,\sin BC};$$

$$\therefore\ \frac{\cos C}{\text{rad}}=\frac{\cos AB\,.\,\text{rad}-\cos AC\,.\,\cos BC}{\sin AC\,.\,\sin BC}.$$

(417) Since the preceding conclusion does not depend on any peculiar relation which the $\angle$ C has to the other angles, a similar equation will be equally true for the angles A and B. Hence

$$\cos \mathrm{A}=\frac{\text{rad}^2 . \cos a-\text{rad} . \cos b . \cos c}{\sin b . \sin c},$$

$$\cos \mathrm{B}=\frac{\text{rad}^2 . \cos b-\text{rad} . \cos a . \cos c}{\sin a . \sin c},$$

$$\cos \mathrm{C}=\frac{\text{rad}^2 . \cos c-\text{rad} . \cos b . \cos a}{\sin b . \sin a}.$$

These are the formulæ from which Lagrange and Legendre* begin their investigations, and of the four quantities involved, any three being given the fourth may be found. They are applicable to every species of spherical triangles, whether right-angled, quadrantal, or oblique-angled, but the formulæ for right-angled and quadrantal triangles have already been given in arts. (384) and (385).

(418) If A, B, C represent the three angles of a spherical triangle, the opposite sides may be represented by 180°—A, 180°—B, 180°—C; and if *a*, *b*, *c* represent the three sides, their opposite angles may be represented by 180°—*a*, 180°—*b*, 180°—*c* (304). Hence

$$\cos(180^\circ-a)=\frac{\text{rad}^2 . \cos(180^\circ-\mathrm{A})-\text{rad} . \cos(180^\circ-\mathrm{B}) . \cos(180^\circ-\mathrm{C})}{\sin(180^\circ-\mathrm{B}) . \sin(180^\circ-\mathrm{C})}.$$

But $\cos(180^\circ-a)=-\cos a$; $\cos(180^\circ-\mathrm{A})=-\cos \mathrm{A}$, &c. (203); consequently

$$\cos a=\frac{\text{rad}^2 . \cos \mathrm{A}+\text{rad} . \cos \mathrm{B} . \cos \mathrm{C}}{\sin \mathrm{B} . \sin \mathrm{C}}.$$

In the same manner the cosines of the other sides may be determined.

(419) It has been shown (401) that the sines of the sides of any spherical triangle have the same ratio to each other as the sines of their opposite angles; hence by using the notation of Legendre we shall have

$$\text{I. } \sin \mathrm{A}=\frac{\sin a . \sin \mathrm{B}}{\sin b}=\frac{\sin a . \sin \mathrm{C}}{\sin c}.$$

$$\sin \mathrm{B}=\frac{\sin b . \sin \mathrm{A}}{\sin a}=\frac{\sin b . \sin \mathrm{C}}{\sin c}.$$

$$\sin \mathrm{C}=\frac{\sin c . \sin \mathrm{A}}{\sin a}=\frac{\sin c . \sin \mathrm{B}}{\sin b}.$$

$$\text{II. } \sin a=\frac{\sin b . \sin \mathrm{A}}{\sin \mathrm{B}}=\frac{\sin c . \sin \mathrm{A}}{\sin \mathrm{C}}.$$

$$\sin b=\frac{\sin a . \sin \mathrm{B}}{\sin \mathrm{A}}=\frac{\sin c . \sin \mathrm{B}}{\sin \mathrm{C}}.$$

$$\sin c=\frac{\sin a . \sin \mathrm{C}}{\sin \mathrm{A}}=\frac{\sin b . \sin \mathrm{C}}{\sin \mathrm{B}}.$$

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* The former in the *Journal de L'Ecole Polytechnique*, and the latter in his *Eléments de Géométrie.*

(420) The general expressions for the cosines, which have been obtained by this proposition, may be arranged thus:

$$\text{III. } \cos A = \frac{\text{rad}^2 . \cos a - \text{rad} . \cos b . \cos c}{\sin b . \sin c}.$$

$$\cos B = \frac{\text{rad}^2 . \cos b - \text{rad} . \cos a . \cos c}{\sin a . \sin c}.$$

$$\cos C = \frac{\text{rad}^2 . \cos c - \text{rad} . \cos b . \cos a}{\sin b . \sin a}.$$

And by reducing these last equations

$$\text{IV. } \cos a = \frac{\cos A . \sin b . \sin c + \text{rad} . \cos b . \cos c}{\text{rad}^2}.$$

$$\cos b = \frac{\cos B . \sin a . \sin c + \text{rad} . \cos a . \cos c}{\text{rad}^2}.$$

$$\cos c = \frac{\cos C . \sin b . \sin a + \text{rad} . \cos b . \cos a}{\text{rad}^2}.$$

$$\text{V. } \cos a = \frac{\text{rad}^2 . \cos A + \text{rad} . \cos B . \cos C}{\sin B . \sin C} \quad (418).$$

$$\cos b = \frac{\text{rad}^2 . \cos B + \text{rad} . \cos A . \cos C}{\sin A . \sin C}.$$

$$\cos c = \frac{\text{rad}^2 . \cos C + \text{rad} . \cos A . \cos B}{\sin A . \sin B}.$$

And by reducing these equations, we shall have

$$\text{VI. } \cos A = \frac{\cos a . \sin B . \sin C - \text{rad} . \cos B . \cos C}{\text{rad}^2}.$$

$$\cos B = \frac{\cos b . \sin A . \sin C - \text{rad} . \cos A . \cos C}{\text{rad}^2}$$

$$\cos C = \frac{\cos c . \sin A . \sin B - \text{rad} . \cos A . \cos B}{\text{rad}^2}.$$

(421) The six sets of equations in the two preceding articles afford solutions to all the different cases of oblique-angled spherical triangles.

The Ist. Finds the angles, when two sides and an angle opposite to one of them are given.

The IId. Finds a side, when two angles and a side opposite to one of them are given.

The IIId. Finds the angles, from the three sides being given.

The IVth. Finds the third side, when two sides and their contained angle are given.

The Vth. Finds the sides, from the three angles being given.

The VIth. Finds the third angle, when two angles and the side adjacent to both of them are given.

(422) But as none of the foregoing formulæ are conveniently adapted to logarithmic calculation, let the value of the cosine of C (420) be substituted in the formula $2\sin^2\frac{1}{2}\mathrm{C} = \mathrm{rad}^2 - \mathrm{rad}\,.\cos\mathrm{C}$ (249), we shall have

$$\frac{2\sin^2\frac{1}{2}\mathrm{C}}{\mathrm{rad}^2} = 1 - \frac{\cos\mathrm{C}}{\mathrm{rad}} = 1 - \frac{\mathrm{rad}^2\,.\cos c - \mathrm{rad}\,.\,\cos a\,.\,\cos b}{\sin a\,.\,\sin b\,.\,\mathrm{rad}}$$

$$= \frac{\sin a\,.\,\sin b + \cos a\,.\,\cos b - \mathrm{rad}\,.\,\cos c}{\sin a\,.\,\sin b};$$

but $\sin a\,.\,\sin b + \cos a\,.\,\cos b = \mathrm{rad}\,.\,\cos(a-b)$ (244),

hence $\dfrac{2\sin^2\frac{1}{2}\mathrm{C}}{\mathrm{rad}^2} = \dfrac{\mathrm{rad}\,.\,[\cos(a-b) - \cos c]}{\sin a\,.\,\sin b}$.

From art. (246),

$\mathrm{rad}\,.\,(\cos\mathrm{Q} - \cos\mathrm{P}) = 2\sin\frac{1}{2}(\mathrm{P}+\mathrm{Q})\,.\,\sin\frac{1}{2}(\mathrm{P}-\mathrm{Q})$,

which, by putting $\mathrm{Q}=(a-b)$, and $\mathrm{P}=c$, becomes

$\mathrm{rad}\,.\,[\cos(a-b) - \cos c] = 2\sin\frac{1}{2}(c+a-b)\,.\,\sin\frac{1}{2}(c-a+b)$,

hence we obtain

$$\frac{2\sin^2\frac{1}{2}\mathrm{C}}{\mathrm{rad}^2} = \frac{2\sin\frac{1}{2}(c+a-b)\,.\,\sin(c-a+b)}{\sin a\,.\,\sin b},$$

$$\therefore \sin\tfrac{1}{2}\mathrm{C} = \mathrm{rad}\sqrt{\frac{\sin\frac{1}{2}(c+a-b)\,.\,\sin\frac{1}{2}(c-a+b)}{\sin a\,.\,\sin b}}.$$

Let $a+b+c=2s$; then $\frac{1}{2}(c+a-b)=s-b$, and $\frac{1}{2}(c-a+b) = s-a$;

$$\text{hence } \sin\tfrac{1}{2}\mathrm{C} = \mathrm{rad}\sqrt{\frac{\sin(s-a)\,.\,\sin(s-b)}{\sin a\,.\,\sin b}},$$

and it is evident that the same formula will be obtained, with only a change of letters for the angles A and B.

When $\frac{1}{2}\angle\mathrm{C}$ is near 90° this will not be a convenient rule for producing an accurate result, because the difference of the logarithmic sines for 1″ is then very small (see the note, page 54.); if $\frac{1}{2}\angle\mathrm{C}$ be less than 45°, it will be proper to use this rule.

(423) Again, if the value of cos C (420) be substituted in $2\cos^2\frac{1}{2}\mathrm{C} = \mathrm{rad}^2 + \mathrm{rad}\,.\,\cos\mathrm{C}$ (249), we shall have

$$\frac{2\cos^2\frac{1}{2}\mathrm{C}}{\mathrm{rad}^2} = 1 + \frac{\cos\mathrm{C}}{\mathrm{rad}} = 1 + \frac{\mathrm{rad}\,.\,\cos c - \cos b\,.\,\cos a}{\sin a\,.\,\sin b} =$$

$$\frac{\sin a\,.\,\sin b + \mathrm{rad}\,.\,\cos c - \cos b\,.\,\cos a}{\sin a\,.\,\sin b};$$

but $\cos a\,.\,\cos b - \sin a\,.\,\sin b = \mathrm{rad}\,.\,\cos(a+b)$ (244),

or $\sin a\,.\,\sin b - \cos a\,.\,\cos b = -\mathrm{rad}\,.\,\cos(a+b)$,

$$\therefore \frac{2\cos^2\frac{1}{2}\mathrm{C}}{\mathrm{rad}^2} = \frac{\mathrm{rad}\,.\,[\cos c - \cos(a+b)]}{\sin a\,.\,\sin b}$$

From art. (246),

$$\text{rad}\,.\,(\cos Q-\cos P)=2\sin\tfrac{1}{2}(P+Q)\,.\,\sin\tfrac{1}{2}(P-Q),$$

which, by putting $Q=c$, and $P=(a+b)$, becomes

$$\text{rad}\,.\,[\cos c-\cos(a+b)]=2\sin\tfrac{1}{2}(a+b+c)\,.\,\sin\tfrac{1}{2}(a+b-c),$$

hence, as $a+b+c=2s$, and $\therefore \tfrac{1}{2}(a+b-c)=s-c$, we obtain

$$\frac{2\cos^2\frac{1}{2}C}{\text{rad}^2}=\frac{2\sin s\,.\,\sin(s-c)}{\sin a\,.\,\sin b},$$

$$\therefore \cos\tfrac{1}{2}C=\text{rad}\sqrt{\frac{\sin s\,.\,\sin(s-c)}{\sin a\,.\,\sin b}},$$

and it is obvious that the same formula will be obtained, with only a change of letters, for the angles A and B.

When $\frac{1}{2}\angle C$ is very small, this rule should not be used where a very accurate result is wanted, because the logarithmic cosines of very small arcs, in a table carried to seven places of figures, differ but little from each other (see the note, page 54); if $\frac{1}{2}\angle C$ be between 45° and 90°, this rule may be used with advantage.

(424) Also, because $\tan\frac{1}{2}C=\dfrac{\text{rad}\,.\,\sin\frac{1}{2}C}{\cos\frac{1}{2}C}$, and $\cot\frac{1}{2}C=\dfrac{\text{rad}\,.\,\cos\frac{1}{2}C}{\sin\frac{1}{2}C}$, we shall obtain by division,

$$\tan\tfrac{1}{2}C=\text{rad}\sqrt{\frac{\sin(s-a)\,.\,\sin(s-b)}{\sin s\,.\,\sin(s-c)}},$$

$$\cot\tfrac{1}{2}C=\text{rad}\sqrt{\frac{\sin s\,.\,\sin(s-c)}{\sin(s-a)\,.\,\sin(s-b)}}.$$

(425) Any one of the formulæ in the three preceding articles will determine an angle when the three sides are given, and by continuing Legendre's* mode of investigation, formulæ for determining a side in terms of the three angles may be obtained; thus, let the value of $\cos a$ (418) be substituted in the formula $2\sin^2\frac{1}{2}a=\text{rad}^2-\text{rad}\,.\,\cos a$ (252), we have

$$\frac{2\sin^2\frac{1}{2}a}{\text{rad}^2}=1-\frac{\cos a}{\text{rad}}=1-\frac{\text{rad}^2\,.\,\cos A+\text{rad}\,.\cos B\,.\,\cos C}{\sin B\,.\,\sin C\,.\,\text{rad}}=$$

$$\frac{\sin B\,.\,\sin C-\text{rad}\,.\,\cos A-\cos B\,.\,\cos C}{\sin B\,.\,\sin C};$$

but $\text{rad}\,.\,\cos(B+C)=\cos B\,.\,\cos C-\sin B\,.\,\sin C$ (244),
or $\sin B\,.\,\sin C-\cos B\,.\,\cos C=-\text{rad}\,.\,\cos(B+C)$;

$$\therefore \frac{2\sin^2\frac{1}{2}a}{\text{rad}^2}=\frac{-\text{rad}\,.\,[\cos(B+C)+\cos A]}{\sin B\,.\,\sin C}.$$

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* Elémens de Géométrie, 6th Edition, page 391. et seq.

By art. (246) $\text{rad} \,.\, (\cos \mathrm{P} + \cos \mathrm{Q}) = 2 \cos \frac{1}{2} (\mathrm{P}+\mathrm{Q}) \,.\, \cos \frac{1}{2} (\mathrm{P}-\mathrm{Q})$, which, by putting $\mathrm{P}=\mathrm{B}+\mathrm{C}$, and $\mathrm{Q}=\mathrm{A}$, becomes $\text{rad} \,.\, [\cos (\mathrm{B}+\mathrm{C}) + \cos \mathrm{A}] = 2 \cos \frac{1}{2} (\mathrm{A}+\mathrm{B}+\mathrm{C}) \,.\, \cos \frac{1}{2} (\mathrm{B}+\mathrm{C}-\mathrm{A})$; hence we obtain

$$\frac{2 \sin^2 \frac{1}{2} a}{\text{rad}^2} = \frac{-2 \cos \frac{1}{2} (\mathrm{A}+\mathrm{B}+\mathrm{C}) \,.\, \cos \frac{1}{2} (\mathrm{B}+\mathrm{C}-\mathrm{A})}{\sin \mathrm{B} \,.\, \sin \mathrm{C}},$$

$$\text{or } \sin \tfrac{1}{2} a = \text{rad} \sqrt{\frac{-\cos \frac{1}{2} (\mathrm{A}+\mathrm{B}+\mathrm{C}) \,.\, \cos \frac{1}{2} (\mathrm{B}+\mathrm{C}-\mathrm{A})}{\sin \mathrm{B} \,.\, \sin \mathrm{C}}}.$$

Let $\mathrm{A}+\mathrm{B}+\mathrm{C}=2\,\mathrm{S}$; then $\frac{1}{2} (\mathrm{B}+\mathrm{C}-\mathrm{A}) = \mathrm{S}-\mathrm{A}$, and

$$\sin \tfrac{1}{2} a = \text{rad} \sqrt{\frac{-\cos \mathrm{S} \,.\, \cos (\mathrm{S}-\mathrm{A})}{\sin \mathrm{B} \,.\, \sin \mathrm{C}}}.$$ *

It is evident that the same formula will be obtained for the sides b and c, by a change of letters only.

When $\frac{1}{2} a$ is near 90°, this rule should not be used where extraordinary accuracy is required; but if $\frac{1}{2} a$ be less than 45°, it may then be used with advantage. See the conclusion of art. (422).

(426) Again, if the value of $\cos a$ (418) be substituted in $2 \cos^2 \frac{1}{2} a = \text{rad}^2 + \text{rad} \,.\, \cos a$ (249), we have

$$\frac{2 \cos^2 \frac{1}{2} a}{\text{rad}^2} = 1 + \frac{\cos a}{\text{rad}} = 1 + \frac{\text{rad} \,.\, \cos \mathrm{A} + \cos \mathrm{B} \,.\, \cos \mathrm{C}}{\sin \mathrm{B} \,.\, \sin \mathrm{C}} = \frac{\sin \mathrm{B} \,.\, \sin \mathrm{C} + \cos \mathrm{B} \,.\, \cos \mathrm{C} + \text{rad} \,.\, \cos \mathrm{A}}{\sin \mathrm{B} \,.\, \sin \mathrm{C}};$$

but $\text{rad} \,.\, \cos (\mathrm{B}-\mathrm{C}) = \sin \mathrm{B} \,.\, \sin \mathrm{C} - \cos \mathrm{B} \,.\, \cos \mathrm{C}$ (244),

$$\therefore \frac{2 \cos^2 \frac{1}{2} a}{\text{rad}^2} = \frac{\text{rad} \,.\, [\cos (\mathrm{B}-\mathrm{C}) + \cos \mathrm{A}]}{\sin \mathrm{B} \,.\, \sin \mathrm{C}}.$$

By art. (246) $\text{rad} \,.\, (\cos \mathrm{P} + \cos \mathrm{Q}) = 2 \cos \frac{1}{2} (\mathrm{P}+\mathrm{Q}) \,.\, \cos \frac{1}{2} (\mathrm{P}-\mathrm{Q})$, which, by putting $\mathrm{Q}=\mathrm{B}-\mathrm{C}$, and $\mathrm{P}=\mathrm{A}$, becomes $\text{rad} \,.\, [\cos \mathrm{A} + \text{rad} \,.\, \cos (\mathrm{B}-\mathrm{C})] = 2 \cos \frac{1}{2} (\mathrm{A}+\mathrm{B}-\mathrm{C}) \,.\, \cos \frac{1}{2} (\mathrm{A}+\mathrm{C}-\mathrm{B})$;

$$\therefore \frac{2 \cos^2 \frac{1}{2} a}{\text{rad}^2} = \frac{2 \cos \frac{1}{2} (\mathrm{A}+\mathrm{B}-\mathrm{C}) \,.\, \cos \frac{1}{2} (\mathrm{A}+\mathrm{C}-\mathrm{B})}{\sin \mathrm{B} \,.\, \sin \mathrm{C}},$$

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* Though the quantity under the radical sign appears under a negative form, it is always positive; for, since the three angles of every spherical triangle are together greater than two and less than six right angles (303), $\mathrm{S} = \frac{1}{2} (\mathrm{A}+\mathrm{B}+\mathrm{C})$ must be greater than 90°, and the cosine of an arc greater than 90° and less than 270°, is negative (200); therefore $-\cos \mathrm{S}$ is positive. The other part under the radical sign is always positive; for if the supplements of the three angles A, B, C be taken, they will give the three sides of a new triangle $= 180° - \mathrm{A}$, $180° - \mathrm{B}$, $180° - \mathrm{C}$ (304), any two of which taken together are greater than the third (300); hence $180° - \mathrm{A}$ *is less than* $180° - \mathrm{B} + 180° - \mathrm{C}$, that is, $-\mathrm{A}+\mathrm{B}+\mathrm{C}$ *is less than* 180°, consequently $\frac{1}{2} (\mathrm{B}+\mathrm{C}-\mathrm{A})$ or its equal $\mathrm{S}-\mathrm{A}$ *is less than* 90°, and the cosine of an arc less than 90° is always affirmative (200). Legendre page 392.

or $\cos \frac{1}{2} a = \text{rad} \sqrt{\dfrac{\cos (\mathrm{A}+\mathrm{B}-\mathrm{C}) \,.\, \cos \frac{1}{2} (\mathrm{A}+\mathrm{C}-\mathrm{B})}{\sin \mathrm{B} \,.\, \sin \mathrm{C}}}$,

but $\frac{1}{2} (\mathrm{A}+\mathrm{B}-\mathrm{C}) = \mathrm{S}-\mathrm{C}$, and $\frac{1}{2} (\mathrm{A}+\mathrm{C}-\mathrm{B}) = \mathrm{S}-\mathrm{B}$,

$$\therefore \cos \tfrac{1}{2} a = \text{rad} \sqrt{\frac{\cos (\mathrm{S}-\mathrm{C}) \,.\, \cos (\mathrm{S}-\mathrm{B})}{\sin \mathrm{B} \,.\, \sin \mathrm{C}}}.$$

And it is plain that the same formula, with only a change of letters, will be found for the sides b and c.

When $\frac{1}{2} a$ is very small it will not be proper to use this rule, but if $\frac{1}{2} a$ be between 45° and 90°, it may be used with advantage. (See the conclusion of art. 423).

(427) Also by art. (218) $\tan \frac{1}{2} a = \text{rad} \,.\, \dfrac{\sin \frac{1}{2} a}{\cos \frac{1}{2} a}$, and $\cot \frac{1}{2} a = \dfrac{\text{rad} \,.\, \cos \frac{1}{2} a}{\sin \frac{1}{2} a}$, we shall obtain by division

$$\tan \tfrac{1}{2} a = \text{rad} \sqrt{\frac{-\cos \mathrm{S} \,.\, \cos (\mathrm{S}-\mathrm{A})}{\cos (\mathrm{S}-\mathrm{B}) \,.\, \cos (\mathrm{S}-\mathrm{C})}},$$

$$\cot \tfrac{1}{2} a = \text{rad} \sqrt{\frac{\cos (\mathrm{S}-\mathrm{B}) \,.\, \cos (\mathrm{S}-\mathrm{C})}{-\cos \mathrm{S} \,.\, \cos (\mathrm{S}-\mathrm{A})}}.$$

And by the same process a similar formula will be obtained for the other sides. Also for the reasons already given (Note on page 188) $-\cos \mathrm{S}$, which appears with a negative sign, is positive.

Any one of the formulæ in the three preceding articles will determine a side when the three angles are given.

(428) From the third set of equations (420) we have

$$\cos \mathrm{A} \,.\, \sin b \,.\, \sin c = \text{rad}^2 \,.\, \cos a - \text{rad} \,.\, \cos b \,.\, \cos c,$$
$$\cos \mathrm{C} \,.\, \sin b \,.\, \sin a = \text{rad}^2 \,.\, \cos c - \text{rad} \,.\, \cos b \,.\, \cos a.$$

By eliminating $\cos c$ and reducing, we get

$$\text{rad} \,.\, \cos \mathrm{A} \,.\, \sin c = \text{rad} \,.\, \cos a \,.\, \sin b - \cos \mathrm{C} \,.\, \sin a \,.\, \cos b.$$

If in this last equation we interchange the letters A and B, and also a and b, we shall have

$$\text{rad} \,.\, \cos \mathrm{B} \,.\, \sin c = \text{rad} \,.\, \cos b \,.\, \sin a - \cos \mathrm{C} \,.\, \sin b \,.\, \cos a.^{*}$$

By adding the last two equations together, and reducing them, we obtain $\sin c \,.\, (\cos \mathrm{A} + \cos \mathrm{B}) = (\text{rad} - \cos \mathrm{C}) \,.\, \sin (a+b)$.

But since $\dfrac{\sin c}{\sin \mathrm{C}} = \dfrac{\sin a}{\sin \mathrm{A}} = \dfrac{\sin b}{\sin \mathrm{B}}$ (419) we have

$$\sin c \,.\, (\sin \mathrm{A} + \sin \mathrm{B}) = \sin \mathrm{C} \, (\sin a + \sin b),$$
and $\sin c \,.\, (\sin \mathrm{A} - \sin \mathrm{B}) = \sin \mathrm{C} \, (\sin a - \sin b)$.

Dividing successively these two equations by the preceding equation, we obtain

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* Legendre's Geometry, page 394.

$$\frac{\sin A+\sin B}{\cos A+\cos B}=\frac{\sin C}{\text{rad}-\cos C}\cdot\frac{\sin a+\sin b}{\sin\,(a+b)},$$

$$\frac{\sin A-\sin B}{\cos A+\cos B}=\frac{\sin C}{\text{rad}-\cos C}\cdot\frac{\sin a-\sin b}{\sin\,(a+b)}.$$

Reducing these equations by the formulæ in arts. (247) and (249),

$$\tan\tfrac{1}{2}\,(A+B)=\cot\tfrac{1}{2}\,C\cdot\frac{\cos\frac{1}{2}\,(a-b)}{\cos\frac{1}{2}\,(a+b)},$$

$$\tan\tfrac{1}{2}\,(A-B)=\cot\tfrac{1}{2}\,C\cdot\frac{\sin\frac{1}{2}\,(a-b)}{\sin\frac{1}{2}\,(a+b)}.$$

These equations give the analogies of Baron Napier for finding the angles A and B, when the two sides a and b, and the angle C comprehended between them are given, viz.

$\cos\frac{1}{2}\,(a+b) : \cos\frac{1}{2}\,(a-b) :: \cot\frac{1}{2}\,C : \tan\frac{1}{2}\,(A+B)$,
$\sin\frac{1}{2}\,(a+b) : \sin\frac{1}{2}\,(a-b) :: \cot\frac{1}{2}\,C : \tan\frac{1}{2}\,(A-B)$.

(429) Again, if A, B, C represent the three angles of a spherical triangle, the opposite sides may be represented by 180° − A, 180° − B, 180° − C; and if a, b, c represent the three sides, their opposite angles may be represented by $180°-a$, $180°-b$, $180°-c$ (304). By substituting 180° − A and 180° − B for a and b; $180°-c$ for C; and $180°-a$ and $180°-b$ for A and B respectively, in the preceding analogies, we shall obtain the following:—

$\cos\frac{1}{2}\,(A+B) : \cos\frac{1}{2}\,(A-B) :: \tan\frac{1}{2}\,c : \tan\frac{1}{2}\,(a+b)$,
$\sin\frac{1}{2}\,(A+B) : \sin\frac{1}{2}\,(A-B) :: \tan\frac{1}{2}\,c : \tan\frac{1}{2}\,(a-b)$.

These are Baron Napier's analogies for finding the sides a and b when two angles A and B, and the side c adjacent to both of them are given.

(430) From the preceding formulæ, which are more than sufficient to solve all the cases of oblique-angled spherical triangles, various others may easily be deduced. By art. (428) $\text{rad}\,.\,\cos A\,.\,\sin c=\text{rad}\,.\,\cos a\,.\,\sin b-\cos C\,.\,\sin a\,.\,\cos b$.

By substituting in this equation $\sin c=\dfrac{\sin a\,.\,\sin C}{\sin A}$ (419), $\dfrac{\text{rad}\,.\,\cos A}{\sin A}=\cot A$, and $\dfrac{\text{rad}\,.\,\cos a}{\sin a}=\cot a$ (218), we shall obtain $\cot A=\dfrac{\cot a\,.\,\sin b-\cos C\,.\,\cos b}{\sin C}$, and by substituting B for A, and b for a,

$$\cot B=\frac{\cot b\,.\,\sin a-\cos C\,.\,\cos a}{\sin C}.$$

These rules may be applied, when two sides and the angle contained between them are given, to find the other two angles;

for it is evident that similar formulæ may be obtained for the angles A and C, or B and C, by merely changing the letters.

(431) By substituting $180°-a$ for A, $180°-B$ for b, $180°-A$ for a, and $180°-c$ for C, and reducing the equation,

$$\cot a=\frac{\cot A\,.\,\sin B+\cos c\,.\,\cos B}{\sin c},$$ and by substituting B for A, &c. as in the preceding article, we get

$$\cot b=\frac{\cot B\,.\,\sin A+\cos c\,.\,\cos A}{\sin c}.$$

These rules may be applied, when two angles and the side adjacent to both of them are given, to find the third angle.

(432) The formulæ in the two preceding articles are not conveniently adapted to logarithmic calculation, but they may be rendered so by the assistance of an auxiliary angle φ; thus,

$\cot A\,.\,\sin C+\cos C\,.\,\cos b=\cot a\,.\,\sin b$ (430).

Let $\tan\varphi=\frac{\cos b\,.\,\tan A}{\text{rad}}$, then because $\tan A=\frac{\text{rad}^2}{\cot A}$, and $\tan\varphi=\frac{\text{rad}\,.\,\sin\varphi}{\cos\varphi}$ (218), we obtain $\cot A=\frac{\cos b\,.\,\cos\varphi}{\sin\varphi}$; hence, by substitution and reduction,

$\frac{\cos b}{\sin\varphi}\,.\,(\cos\varphi\,.\,\sin C+\sin\varphi\,.\,\cos C)=\cot a\,.\,\sin b.$

But $\cos\varphi\,.\,\sin C+\sin\varphi\,.\,\cos C=\text{rad}\,.\,\sin\,(\varphi+C)$ (244), hence $\sin\,(\varphi+C)=\frac{\cot a\,.\,\sin b\,.\,\sin\varphi}{\text{rad}\,.\,\cos b}$, and because $\cot a=\frac{\text{rad}^2}{\tan a}$, and $\frac{\text{rad}\,.\,\sin b}{\cos b}=\tan b$, we derive

$$\sin\,(\varphi+C)=\frac{\tan b\,.\,\sin\varphi}{\tan a}.$$

Now the value of φ is already known, therefore the value of C is easily obtained; these formulæ may be applied when two sides a and b, and an angle A, opposite to one of them, are given, to find the included angle C.

(433) From art. (420) we have

$\cos A\,.\,\sin b\,.\,\sin c+\text{rad}\,.\,\cos b\,.\,\cos c=\text{rad}^2\,.\,\cos a.$

Let $\cos A\,.\,\sin b=\frac{\text{rad}\,.\,\cos b\,.\,\sin\varphi}{\cos\varphi}=\tan\varphi\,.\,\cos b,$

hence $\tan\varphi=\frac{\cos A\,.\,\sin b}{\cos b}=\frac{\cos A\,.\,\tan b}{\text{rad}}.$

By subst. $\frac{\cos b}{\cos\varphi}\,.\,(\cos c\,.\,\cos\varphi+\sin c\,.\,\sin\varphi)=\text{rad}\,.\,\cos a$

But $\cos c \,.\, \cos\varphi + \sin c \,.\, \sin\varphi = \text{rad} \,.\, \cos(c-\varphi)$ (244), hence $\cos(c-\varphi) = \dfrac{\cos a \,.\, \cos\varphi}{\cos b}$. Now the value of φ is already known, therefore the value of c is easily determined. These formulæ may be applied when two sides a and b, and an angle A opposite to one of them, are given to find the third side.

(434) Again, by art. (420)

$$\text{rad}^2 \,.\, \cos c = \sin a \,.\, \sin b \,.\, \cos C + \text{rad} \,.\, \cos a \,.\, \cos b.$$

Let $\tan\varphi = \dfrac{\cos C \,.\, \tan b}{\text{rad}}$;

then $\cos b.\ \tan\varphi = \cos C \,.\, \dfrac{\tan b \,.\, \cos b}{\text{rad}} = \cos C \,.\, \sin b$

hence, by substitution,

$$\text{rad}^2 \,.\, \cos c = \sin a \,.\, \cos b \,.\, \tan\varphi + \text{rad} \,.\, \cos a \,.\, \cos b,$$

or $\dfrac{\text{rad}^2 \,.\, \cos c}{\cos b} = \sin a \,.\, \tan\varphi + \text{rad} \,.\, \cos a$,

but $\tan\varphi = \dfrac{\text{rad} \,.\, \sin\varphi}{\cos\varphi}$,

$$\therefore \frac{\text{rad} \,.\, \cos c}{\cos b} = \frac{\sin a \,.\, \sin\varphi + \cos a \,.\, \cos\varphi}{\cos\varphi} = \frac{\text{rad} \,.\, \cos(a-\varphi)}{\cos\varphi}.$$

Consequently $\cos c = \dfrac{\cos b}{\cos\varphi} \,.\, \cos(a-\varphi)$.

These formulæ may be applied when two sides a and b and the angle C, contained between them, are given to find the third side c.

(435) From art. (420) we have

$$\text{rad}^2 \,.\, \cos C = \cos c \,.\, \sin A \,.\, \sin B - \text{rad} \,.\, \cos A \,.\, \cos B.$$

Let $\cot\varphi = \dfrac{\cos c \,.\, \tan B}{\text{rad}}$; then

$$\cos B \,.\, \cot\varphi = \cos c \,.\, \frac{\tan B \,.\, \cos B}{\text{rad}} = \cos c \,.\, \sin B;$$

hence, by substitution,

$$\text{rad}^2 \,.\, \cos C = \cos B \,.\, \cot\varphi \,.\, \sin A - \text{rad} \,.\, \cos A \,.\, \cos B,$$

but $\cot\varphi = \dfrac{\cos\varphi \,.\, \text{rad}}{\sin\varphi}$; hence, by substitution and reduction,

$$\frac{\text{rad} \,.\, \cos C}{\cos B} = \frac{\sin A \,.\, \cos\varphi - \cos A \,.\, \sin\varphi}{\sin\varphi} = \frac{\text{rad} \,.\, \sin(A-\varphi)}{\sin\varphi};$$

therefore $\cos C = \cos B \,.\, \dfrac{\sin(A-\varphi)}{\sin\varphi}$.

These formulæ may be applied when two angles A and B, and the side c, adjacent to both of them, are given, to find the third angle C.

(436) $\cot a \,.\, \sin c - \cos c \,.\, \cos B = \cot A \,.\, \sin B$ (431).

Let $\tan \phi = \frac{\cos B \,.\, \tan a}{\text{rad}} = \frac{\cos B \,.\, \text{rad}}{\cot a}$, then

$\frac{\cot a}{\text{rad}} = \frac{\cos B}{\tan \phi} = \frac{\cos B \,.\, \cos \phi}{\text{rad} \,.\, \sin \phi}$, and $\therefore \cot a = \cos B \,.\, \frac{\cos \phi}{\sin \phi}$;

hence, by subst. $\frac{\cos B}{\sin \phi}(\sin c \,.\, \cos \phi - \cos c \,.\, \sin \phi) = \cot A \,.\, \sin B$;

but $\sin c \,.\, \cos \phi - \cos c \,.\, \sin \phi = \sin (c - \phi) \,.\, \text{rad}$,

$\therefore \cos B \,.\, \sin (c - \phi) \,.\, \text{rad} = \cot A \,.\, \sin B \,.\, \sin \phi$,

$\therefore \sin (c - \phi) = \frac{\cot A}{\text{rad}} \,.\, \tan B \,.\, \sin \phi = \frac{\tan B \,.\, \sin \phi}{\tan A}$.

These formulæ may be applied when two angles A and B, and a side a, opposite to one of them, are given to find the side c adjacent to both the given angles.

(437) From art. (420) we have

$\cos a \,.\, \sin B \,.\, \sin C - \text{rad} \,.\, \cos B \,.\, \cos C = \text{rad}^2 \,.\, \cos A$.

Let $\cot \phi = \frac{\cos a \,.\, \tan B}{\text{rad}} = \cos a \,.\, \frac{\sin B}{\cos B}$, then

$\cos a \,.\, \sin B = \cos B \,.\, \cot \phi = \frac{\text{rad} \,.\, \cos B \,.\, \cos \phi}{\sin \phi}$,

and $\therefore \frac{\cos B}{\sin \phi}(\sin C \,.\, \cos \phi - \cos C \,.\, \sin \phi) = \text{rad} \,.\, \cos A$.

But $\sin C \,.\, \cos \phi - \cos C \,.\, \sin \phi = \sin (C - \phi) \,.\, \text{rad}$;

$$\therefore \sin (C - \phi) = \frac{\cos A \,.\, \sin \phi}{\cos B}.$$

These formulæ may be applied when two angles A and B, and a side a, opposite to one of them, are given to find the third angle.

(438) We readily infer from what is done in art. (422), that

$$\frac{2}{\text{rad}^3} \,.\, \sin^2 \tfrac{1}{2} C \,.\, \sin a \,.\, \sin b = \cos (a - b) - \cos c,$$

$$\therefore \cos c = \cos (a - b) - \sin a \,.\, \sin b \,.\, \sin^2 \tfrac{1}{2} C \,.\, \frac{2}{\text{rad}^3}.$$

This formula, by the assistance of a table of logarithms and a table of natural sines, furnishes us with a very convenient rule for finding the third side of a spherical triangle, when two sides and the included angle are given.

(439) We infer from art. (426), that

$\frac{2}{\text{rad}^3} \,.\, \cos^2 \tfrac{1}{2} a \,.\, \sin B \,.\, \sin C = \cos (B - C) + \cos A$,

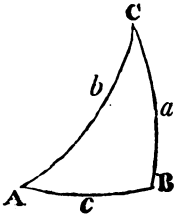
$\therefore \cos A = \left(\sin B \,.\, \sin C \,.\, \cos^2 \tfrac{1}{2} a \,.\, \frac{2}{\text{rad}^3}\right) - \cos (B - C)$.

This formula, by the assistance of a table of logarithms and a table of natural sines, gives a useful rule for finding the third angle of a spherical triangle, when two angles and their adjacent side are given.

SCHOLIUM.

(440) Spherical triangles whose sides are very small arcs, may be considered as plane, and, therefore, the sides may become nearly equal to their sines or tangents; hence, all the foregoing proportions and formulæ, wherein *cosines* or *cotangents* of the sides are not concerned, are equally applicable to plane trigonometry, using the word *side* instead of *sine of a side* or *tangent of a side.*

SOLUTIONS OF THE DIFFERENT CASES OF RIGHT-ANGLED SPHERICAL TRIANGLES.



Rules for solving the different cases both of right-angled and oblique-angled spherical triangles, have already been given; the subject is here resumed for the express purpose of concentrating some of the principal formulæ.

The three angles of the triangle are represented by A, B, C, and their opposite sides by a, b, c, as in the figure annexed; r = radius = sine of 90°.

CASE I. Given the hypothenuse and an angle, to find the side adjacent to the given angle.

$$\tan c = \frac{r \cos A}{\cot b} = \frac{\cos A \,.\, \tan b}{r} = \frac{r \tan b}{\sec A}.$$

$$\cot c = \frac{r \cot b}{\cos A} = \frac{r \sec A}{\tan b} = \frac{\cot b \,.\, \sec A}{r}.$$

$$\tan a = \frac{r \cos C}{\cot b} = \frac{\cos C \,.\, \tan b}{r} = \frac{r \tan b}{\sec C}.$$

$$\cot a = \frac{r \cot b}{\cos C} = \frac{r \sec C}{\tan b} = \frac{\cot b \,.\, \sec C}{r}.$$

CASE II. Given the hypothenuse and an angle, to find the side opposite to that given angle.

$$\sin a = \frac{\sin A \,.\, \sin b}{r} = \frac{r \sin A}{\operatorname{cosec} b} = \frac{r \sin b}{\operatorname{cosec} A}.$$

$$\operatorname{cosec} a = \frac{r \operatorname{cosec} b}{\sin \mathrm{A}} = \frac{\operatorname{cosec} b \,.\, \operatorname{cosec} \mathrm{A}}{r} = \frac{r \operatorname{cosec} \mathrm{A}}{\sin b}.$$

$$\sin c = \frac{\sin \mathrm{C} \,.\, \sin b}{r} = \frac{r \sin \mathrm{C}}{\operatorname{cosec} b} = \frac{r \sin b}{\operatorname{cosec} \mathrm{C}}.$$

$$\operatorname{cosec} c = \frac{r \operatorname{cosec} b}{\sin \mathrm{C}} = \frac{\operatorname{cosec} b \,.\, \operatorname{cosec} \mathrm{C}}{r} = \frac{r \operatorname{cosec} \mathrm{C}}{\sin b}.$$

CASE III. Given the hypothenuse and one angle, to find the other angle.

$$\tan \mathrm{C} = \frac{r \sec b}{\tan \mathrm{A}} = \frac{\sec b \,.\, \cot \mathrm{A}}{r} = \frac{r \cot \mathrm{A}}{\cos b}.$$

$$\cot \mathrm{C} = \frac{r \cos b}{\cot \mathrm{A}} = \frac{\cos b \,.\, \tan \mathrm{A}}{r} = \frac{r \tan \mathrm{A}}{\sec b}.$$

$$\tan \mathrm{A} = \frac{r \sec b}{\tan \mathrm{C}} = \frac{\sec b \,.\, \cot \mathrm{C}}{r} = \frac{r \cot \mathrm{C}}{\cos b}.$$

$$\cot \mathrm{A} = \frac{r \cos b}{\cot \mathrm{C}} = \frac{\cos b \,.\, \tan \mathrm{C}}{r} = \frac{r \tan \mathrm{C}}{\sec b}.$$

CASE IV. Given the hypothenuse and a leg, to find the angle adjacent to that given leg.

$$\cos \mathrm{A} = \frac{\tan c \,.\, \cot b}{r} = \frac{r \tan c}{\tan b} = \frac{r \cot b}{\cot c}.$$

$$\sec \mathrm{A} = \frac{r \cot c}{\cot b} = \frac{\cot c \,.\, \tan b}{r} = \frac{r \tan b}{\tan c}.$$

$$\cos \mathrm{C} = \frac{\tan a \,.\, \cot b}{r} = \frac{r \tan a}{\tan b} = \frac{r \cot b}{\cot a}.$$

$$\sec \mathrm{C} = \frac{r \cot a}{\cot b} = \frac{\cot a \,.\, \tan b}{r} = \frac{r \tan b}{\tan a}.$$

$$\tan \tfrac{1}{2}\mathrm{A} = \sqrt{\frac{\sin (b-c)}{\sin (b+c)}}, \text{ and } \tan \tfrac{1}{2}\mathrm{C} = r\sqrt{\frac{\sin (b-a)}{\sin (b+a)}}.$$

CASE V. Given the hypothenuse and a leg, to find the angle opposite to that given leg.

$$\sin \mathrm{C} = \frac{r \sin c}{\sin b} = \frac{\sin c \,.\, \operatorname{cosec} b}{r} = \frac{r \operatorname{cosec} b}{\operatorname{cosec} c}.$$

$$\operatorname{cosec} \mathrm{C} = \frac{r \operatorname{cosec} c}{\operatorname{cosec} b} = \frac{r \sin b}{\sin c} = \frac{\sin b \,.\, \operatorname{cosec} c}{r}.$$

$$\sin \mathrm{A} = \frac{r \sin a}{\sin b} = \frac{\sin a \,.\, \operatorname{cosec} b}{r} = \frac{r \operatorname{cosec} b}{\operatorname{cosec} a}.$$

$$\operatorname{cosec} \mathrm{A}=\frac{r \operatorname{cosec} a}{\operatorname{cosec} b}=\frac{r \sin b}{\sin a}=\frac{\sin b \,.\, \operatorname{cosec} a}{r}.$$

$$\tan\left(45^\circ+\tfrac{1}{2}\mathrm{C}\right)=\sqrt{\tan\tfrac{1}{2}(b+c) \,.\, \cot\tfrac{1}{2}(b-c)}.$$

$$\tan\left(45^\circ\pm\tfrac{1}{2}\mathrm{A}\right)=\sqrt{\tan\tfrac{1}{2}(b+a) \,.\, \cot\tfrac{1}{2}(b-a)}.$$

Case VI. Given the hypothenuse and either leg, to find the other leg.

$$\cos a=\frac{r \cos b}{\cos c}=\frac{\cos b \,.\, \sec c}{r}=\frac{r \sec c}{\sec b}.$$

$$\sec a=\frac{r \cos c}{\cos b}=\frac{\cos c \,.\, \sec b}{r}=\frac{r \sec b}{\sec c}.$$

$$\cos c=\frac{r \cos b}{\cos a}=\frac{\cos b \,.\, \sec a}{r}=\frac{r \sec a}{\sec b}.$$

$$\sec c=\frac{r \cos a}{\cos b}=\frac{\cos a \,.\, \sec b}{r}=\frac{r \sec b}{\sec a}.$$

$$\tan\tfrac{1}{2}a=\sqrt{\tan\tfrac{1}{2}(b+c) \,.\, \tan\tfrac{1}{2}(b-c)}.$$

$$\tan\tfrac{1}{2}c=\sqrt{\tan\tfrac{1}{2}(b+a) \,.\, \tan\tfrac{1}{2}(b-a)}.$$

Case VII. Given a side of a right-angled spherical triangle, and the angle adjacent or joining to it, to find the side opposite to the given angle.

$$\tan a=\frac{r \sin c}{\cot \mathrm{A}}=\frac{\tan \mathrm{A} \,.\, \sin c}{r}=\frac{r \tan \mathrm{A}}{\operatorname{cosec} c}.$$

$$\cot a=\frac{r \cot \mathrm{A}}{\sin c}=\frac{r \operatorname{cosec} c}{\tan \mathrm{A}}=\frac{\operatorname{cosec} c \,.\, \cot \mathrm{A}}{r}.$$

$$\tan c=\frac{r \sin a}{\cot \mathrm{C}}=\frac{\tan \mathrm{C} \,.\, \sin a}{r}=\frac{r \tan \mathrm{C}}{\operatorname{cosec} a}.$$

$$\cot c=\frac{r \cot \mathrm{C}}{\sin a}=\frac{r \operatorname{cosec} a}{\tan \mathrm{C}}=\frac{\operatorname{cosec} a \,.\, \cot \mathrm{C}}{r}$$

Case VIII. Given a side of a right-angled spherical triangle, and its adjacent angle, to find the angle opposite to the given side.

$$\cos \mathrm{C}=\frac{\cos c \,.\, \sin \mathrm{A}}{r}=\frac{r \cos c}{\operatorname{cosec} \mathrm{A}}=\frac{r \sin \mathrm{A}}{\sec c}.$$

$$\sec \mathrm{C}=\frac{r \operatorname{cosec} \mathrm{A}}{\cos c}=\frac{\operatorname{cosec} \mathrm{A} \,.\, \sec c}{r}=\frac{r \sec c}{\sin \mathrm{A}}.$$

$$\cos \mathrm{A} = \frac{\cos a \,.\, \sin \mathrm{C}}{r} = \frac{r \cos a}{\operatorname{cosec} \mathrm{C}} = \frac{r \sin \mathrm{C}}{\sec a}.$$

$$\sec \mathrm{A} = \frac{r \operatorname{cosec} \mathrm{C}}{\cos a} = \frac{\operatorname{cosec} \mathrm{C} \,.\, \sec a}{r} = \frac{r \sec a}{\sin \mathrm{C}}.$$

CASE IX. Given a side and its adjacent angle, to find the hypothenuse.

$$\cot b = \frac{r \cos \mathrm{A}}{\tan c} = \frac{\cot c \,.\, \cos \mathrm{A}}{r} = \frac{r \cot c}{\sec \mathrm{A}}.$$

$$\tan b = \frac{r \tan c}{\cos \mathrm{A}} = \frac{r \sec \mathrm{A}}{\cot c} = \frac{\sec \mathrm{A} \,.\, \tan c}{r}.$$

$$\cot b = \frac{r \cos \mathrm{C}}{\tan a} = \frac{\cot a \,.\, \cos \mathrm{C}}{r} = \frac{r \cot a}{\sec \mathrm{C}}.$$

$$\tan b = \frac{r \tan a}{\cos \mathrm{C}} = \frac{r \sec \mathrm{C}}{\cot a} = \frac{\sec \mathrm{C} \,.\, \tan a}{r}.$$

CASE X. Given a side of a right-angled spherical triangle, and its opposite angle, to find the side adjacent to that angle.

$$\sin c = \frac{\tan a \,.\, \cot \mathrm{A}}{r} = \frac{r \tan a}{\tan \mathrm{A}} = \frac{r \cot \mathrm{A}}{\cot a}.$$

$$\operatorname{cosec} c = \frac{r \tan \mathrm{A}}{\tan a} = \frac{\tan \mathrm{A} \,.\, \cot a}{r} = \frac{r \cot a}{\cot \mathrm{A}}.$$

$$\sin a = \frac{\tan c \,.\, \cot \mathrm{C}}{r} = \frac{r \tan c}{\tan \mathrm{C}} = \frac{r \cot \mathrm{C}}{\cot c}.$$

$$\operatorname{cosec} a = \frac{r \tan \mathrm{C}}{\tan c} = \frac{\tan \mathrm{C} \,.\, \cot c}{r} = \frac{r \cot c}{\cot \mathrm{C}}.$$

$$\tan (45^\circ + \tfrac{1}{2} c) = r \sqrt{\frac{\sin (\mathrm{A} + a)}{\sin (\mathrm{A} - a)}}.$$

$$\tan (45^\circ + \tfrac{1}{2} a) = r \sqrt{\frac{\sin (\mathrm{C} + c)}{\sin (\mathrm{C} - c)}}.$$

CASE XI. Given a side of a right-angled spherical triangle, and its opposite angle, to find the adjacent angle.

$$\sin \mathrm{C} = \frac{r \cos \mathrm{A}}{\cos a} = \frac{\cos \mathrm{A} \,.\, \sec a}{r} = \frac{r \sec a}{\sec \mathrm{A}}.$$

$$\operatorname{cosec} \mathrm{C} = \frac{r \sec \mathrm{A}}{\sec a} = \frac{\sec \mathrm{A} \,.\, \cos a}{r} = \frac{r \cos a}{\cos \mathrm{A}}.$$

$$\sin \mathrm{A} = \frac{r \cos \mathrm{C}}{\cos c} = \frac{\cos \mathrm{C} \,.\, \sec c}{r} = \frac{r \sec c}{\sec \mathrm{C}}.$$

$$\text{cosec}\ \mathrm{A}=\frac{r\ \sec\ \mathrm{C}}{\sec\ c}=\frac{\sec\ \mathrm{C}\ .\ \cos\ c}{r}=\frac{r\ \cos\ c}{\cos\ \mathrm{C}}.$$

$$\tan\ (45^\circ+\tfrac{1}{2}\ \mathrm{C})=\sqrt{\cot\ \tfrac{1}{2}\,(\mathrm{A}+a)\ .\ \cot\ \tfrac{1}{2}\ (\mathrm{A}-a)}.$$

$$\tan\ (45^\circ+\tfrac{1}{2}\ \mathrm{A})=\sqrt{\cot\ \tfrac{1}{2}\ (\mathrm{C}+c)\ .\ \cot\ \tfrac{1}{2}\ (\mathrm{C}-c)}.$$

Case XII. Given a side of a right-angled spherical triangle, and its opposite angle, to find the hypothenuse.

$$\sin\ b=\frac{r\ \sin\ a}{\sin\ \mathrm{A}}=\frac{\sin\ a\ .\ \text{cosec}\ \mathrm{A}}{r}=\frac{r\ \text{cosec}\ \mathrm{A}}{\text{cosec}\ a}.$$

$$\text{cosec}\ b=\frac{r\ \sin\ \mathrm{A}}{\sin\ a}=\frac{\sin\ \mathrm{A}\ .\ \text{cosec}\ a}{r}=\frac{r\ \text{cosec}\ a}{\text{cosec}\ \mathrm{A}}.$$

$$\sin\ b=\frac{r\ \sin\ c}{\sin\ \mathrm{C}}=\frac{\sin\ c\ .\ \text{cosec}\ \mathrm{C}}{r}=\frac{r\ \text{cosec}\ \mathrm{C}}{\text{cosec}\ c}.$$

$$\text{cosec}\ b=\frac{r\ \sin\ \mathrm{C}}{\sin\ c}=\frac{\sin\ \mathrm{C}\ .\ \text{cosec}\ c}{r}=\frac{r\ \text{cosec}\ c}{\text{cosec}\ \mathrm{C}}.$$

$$\tan\ (45^\circ+\tfrac{1}{2}\,b)=\sqrt{\tan\ \tfrac{1}{2}\ (\mathrm{C}+c)\ .\ \cot\ \tfrac{1}{2}\ (\mathrm{C}-c)}.$$

$$\tan\ (45^\circ+\tfrac{1}{2}\,b)=\sqrt{\tan\ \tfrac{1}{2}\ (\mathrm{A}+a)\ .\ \cot\ \tfrac{1}{2}\ (\mathrm{A}-a)}.$$

Case XIII. Given the two sides, or legs, of a right-angled spherical triangle, to find an angle.

$$\cot\ \mathrm{A}=\frac{r\ \sin\ c}{\tan\ a}=\frac{\sin\ c\ .\ \cot\ a}{r}=\frac{r\ \cot\ a}{\text{cosec}\ c}.$$

$$\tan\ \mathrm{A}=\frac{r\ \tan\ a}{\sin\ c}=\frac{\tan\ a\ .\ \text{cosec}\ c}{r}=\frac{r\ \text{cosec}\ c}{\cot\ a}.$$

$$\cot\ \mathrm{C}=\frac{r\ \sin\ a}{\tan\ c}=\frac{\sin\ a\ .\ \cot\ c}{r}=\frac{r\ \cot\ c}{\text{cosec}\ a}.$$

$$\tan\ \mathrm{C}=\frac{r\ \tan\ c}{\sin\ a}=\frac{\tan\ c\ .\ \text{cosec}\ a}{r}=\frac{r\ \text{cosec}\ a}{\cot\ c}.$$

Case XIV. Given the two sides, or legs, of a right-angled spherical triangle, to find the hypothenuse.

$$\cos\ b=\frac{\cos\ a\ .\ \cos\ c}{r}=\frac{r\ \cos\ a}{\sec\ c}=\frac{r\ \cos\ c}{\sec\ a}.$$

$$\sec\ b=\frac{r\ \sec\ a}{\cos\ c}=\frac{\sec\ a\ .\ \sec\ c}{r}=\frac{r\ \sec\ c}{\cos\ a}.$$

CASE XV. Given two angles of a right-angled spherical triangle, to find a side, or leg.

$$\cos a = \frac{r \cos \text{A}}{\sin \text{C}} = \frac{\cos \text{A} \,.\, \text{cosec C}}{r} = \frac{r \,\text{cosec C}}{\sec \text{A}}.$$

$$\sec a = \frac{r \sin \text{C}}{\cos \text{A}} = \frac{\sin \text{C} \,.\, \sec \text{A}}{r} = \frac{r \sec \text{A}}{\text{cosec C}}.$$

$$\cos c = \frac{r \cos \text{C}}{\sin \text{A}} = \frac{\cos \text{C} \,.\, \text{cosec A}}{r} = \frac{r \,\text{cosec A}}{\sec \text{C}}.$$

$$\sec c = \frac{r \sin \text{A}}{\cos \text{C}} = \frac{\sin \text{A} \,.\, \sec \text{C}}{r} = \frac{r \sec \text{C}}{\text{cosec A}}.$$

$$\tan \tfrac{1}{2} c = \sqrt{\tan\left(\frac{\text{A}+\text{C}}{2} - 45^\circ\right) . \cot\left(\frac{\text{A}-\text{C}}{2} + 45^\circ\right)}.$$

$$\tan \tfrac{1}{2} a = \sqrt{\tan\left(\frac{\text{C}+\text{A}}{2} - 45^\circ\right) . \cot\left(\frac{\text{C}-\text{A}}{2} + 45^\circ\right)}.$$

CASE XVI. Given two angles, of a right-angled spherical triangle, to find the hypothenuse.

$$\cos b = \frac{\cot \text{C} \,.\, \cot \text{A}}{r} = \frac{r \cot \text{A}}{\tan \text{C}} = \frac{r \cot \text{C}}{\tan \text{A}}.$$

$$\sec b = \frac{r \tan \text{C}}{\cot \text{A}} = \frac{\tan \text{C} \,.\, \tan \text{A}}{r} = \frac{r \tan \text{A}}{\cot \text{C}}.$$

$$\tan \tfrac{1}{2} b = r \sqrt{\frac{-\cos(\text{A}+\text{C})}{\cos(\text{A} \sim \text{C})}}.$$

GENERAL OBSERVATIONS ON THE SPECIES AND AMBIGUITY OF THE CASES.

(441) The species of the sides and angles may be determined from the equations produced by Baron Napier's Rules, or from the preceding formulæ, by attending to the signs of the quantities which compose the equations or formulæ.

The sides which contain the right angle are each of the same species as their opposite angles, viz. a is of the same species with A, and b is of the same species with B. (323).

It may be proper to observe, that where a quantity is to be determined by the sines only, and a side or angle opposite to the quantity sought does not enter into the equation, the case will be ambiguous: thus, in the XIIth case, where $\sin b = \frac{r \sin a}{\sin \text{A}}$, the hypothenuse b is ambiguous.

Again, in the IId case, where $\sin a = \frac{\sin \text{A} . \sin b}{r}$, the sine of a is evidently determinate, because it is of the same species with A, which is a given quantity.

(442) When an unknown quantity is to be determined by its cosine, tangent, or cotangent, the sign of this value will always determine its species; for, if its proper sign be +, the arc will be less than 90°; if the proper sign be −, the arc will be greater than 90°.

(443) Again, in Case VIth, where $\text{rad} \times \cos b = \cos c \times \cos a$, it is obvious that the three sides are each less than 90°, or that two of them are greater than 90°, and the third less; as no other combination can render the sign of $\cos c \times \cos a$ like that of $\cos b$ as the equation requires.*

QUADRANTAL TRIANGLES.

(444) Any spherical triangle of which A, B, C, are the angles and a, b, c, the opposite sides, may be changed into a spherical triangle of which the angles are supplements of the sides a, b, c, and the sides supplements of the angles A, B, C, (304) viz. if we call A′, B′, C′ the angles of the supplemental triangle, and a', b', c' the sides opposite to these angles, we shall have

$$\text{A}' = 180° - a;\ \text{B}' = 180° - b;\ \text{C}' = 180° - c$$
$$a' = 180° - \text{A};\ b' = 180° - \text{B};\ c' = 180° - \text{C}.$$

Hence it is plain, that if a spherical triangle has a side b equal to a quadrant, the corresponding angle B′ of the supplemental triangle will be a right angle, and since there are always three *given* parts in a triangle, the supplemental triangle will be a right-angled triangle, having *two* parts given to find the rest; consequently, by finding the required parts in the supplemental right-angled triangle, the different parts of the quadrantal triangle will be known.

(445) Formulæ might have been inserted for solving the different cases of quadrantal triangles, but this would be making an increase of formulæ to very little purpose, since all quadrantal triangles are easily turned into right-angled triangles.

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* Legendre's Geometry, 6th edition, page 381.

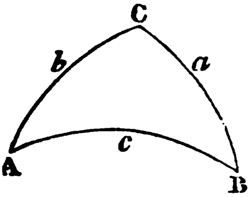
SOLUTIONS OF THE DIFFERENT CASES OF OBLIQUE-ANGLED SPHERICAL TRIANGLES.

CASE I. Given two sides of an oblique-angled spherical triangle, and an angle opposite to one of them, to find the angle opposite to the other.

$$\sin A = \frac{\sin B \,.\, \sin a}{\sin b} = \frac{\sin C \,.\, \sin A}{\sin c}$$

$$\sin B = \frac{\sin A \,.\, \sin b}{\sin a} = \frac{\sin C \,.\, \sin b}{\sin c}$$

$$\sin C = \frac{\sin A \,.\, \sin c}{\sin a} = \frac{\sin B \,.\, \sin c}{\sin b}$$



CASE II. Given two sides of an oblique-angled spherical triangle, and an angle opposite to one of them, to find the angle contained between these sides.

Find the angle opposite to the other given side by Case I, then

$$\cot \tfrac{1}{2} A = \frac{\sin \frac{1}{2}(c+b)}{\sin \frac{1}{2}(c \sim b)} \,.\, \tan \tfrac{1}{2}(C \sim B)$$

$$\cot \tfrac{1}{2} B = \frac{\sin \frac{1}{2}(c+a)}{\sin \frac{1}{2}(c \sim a)} \,.\, \tan \tfrac{1}{2}(C \sim A)$$

$$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a \sim b)} \,.\, \tan \tfrac{1}{2}(A \sim B)$$

OTHERWISE,

Let $\tan \varphi = \frac{\cos c \,.\, \tan B}{\text{rad}}$; then $\sin(\varphi + A) = \frac{\tan c \,.\, \sin \varphi}{\tan b}$

Let $\tan \varphi = \frac{\cos a \,.\, \tan C}{\text{rad}}$; then $\sin(\varphi + B) = \frac{\tan a \,.\, \sin \varphi}{\tan c}$

Let $\tan \varphi = \frac{\cos b \,.\, \tan A}{\text{rad}}$; then $\sin(\varphi + C) = \frac{\tan b \,.\, \sin \varphi}{\tan a}$

CASE III. Given two sides of an oblique-angled spherical triangle, and an angle opposite to one of them, to find the other side.

Find the angle opposite to the other given side by Case I, then

$$\tan \tfrac{1}{2} a = \frac{\sin \frac{1}{2}(C+B)}{\sin \frac{1}{2}(C \sim B)} \,.\, \tan \tfrac{1}{2}(c \sim b)$$

$$\tan \tfrac{1}{2} b = \frac{\sin \frac{1}{2}(C+A)}{\sin \frac{1}{2}(C \sim A)} \,.\, \tan \tfrac{1}{2}(c \sim a)$$

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2} (A+B)}{\sin \frac{1}{2} (A \sim B)} \cdot \tan \tfrac{1}{2} (a \sim b)$$

OTHERWISE,

Let $\tan \varphi = \dfrac{\cos B \,.\, \tan c}{\text{rad}}$. then $\cos (a \sim \varphi) = \dfrac{\cos b \,.\, \cos \varphi}{\cos c}$

Let $\tan \varphi = \dfrac{\cos C \,.\, \tan a}{\text{rad}}$; then $\cos (b \sim \varphi) = \dfrac{\cos c \,.\, \cos \varphi}{\cos a}$

Let $\tan \varphi = \dfrac{\cos A \,.\, \tan b}{\text{rad}}$; then $\cos (c \sim \varphi) = \dfrac{\cos a \,.\, \cos \varphi}{\cos b}$

Case IV. Given two angles of an oblique-angled spherical triangle, and a side opposite to one of them, to find the other opposite side.

$$\sin a = \frac{\sin A \,.\, \sin b}{\sin B} = \frac{\sin A \,.\, \sin c}{\sin C}$$

$$\sin b = \frac{\sin B \,.\, \sin a}{\sin A} = \frac{\sin B \,.\, \sin c}{\sin C}$$

$$\sin c = \frac{\sin C \,.\, \sin a}{\sin A} = \frac{\sin C \,.\, \sin b}{\sin B}$$

Case V. Given two angles of an oblique-angled spherical triangle, and a side opposite to one of them, to find the side adjacent to these angles.

Find the side opposite to the other given angle by Case IV, then

$$\tan \tfrac{1}{2} a = \frac{\sin \frac{1}{2} (C+B)}{\sin \frac{1}{2} (C \sim B)} \cdot \tan \tfrac{1}{2} (c \sim b)$$

$$\tan \tfrac{1}{2} b = \frac{\sin \frac{1}{2} (A+C)}{\sin \frac{1}{2} (A \sim C)} \cdot \tan \tfrac{1}{2} (a \sim c)$$

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2} (A+B)}{\sin \frac{1}{2} (A \sim B)} \cdot \tan \tfrac{1}{2} (a \sim b)$$

OTHERWISE,

Let $\tan \varphi = \dfrac{\cos C \,.\, \tan b}{\text{rad}}$; then $\sin (a \sim \varphi) = \dfrac{\sin \varphi \,.\, \tan C}{\tan B}$

Let $\tan \varphi = \dfrac{\cos A \,.\, \tan c}{\text{rad}}$; then $\sin (b \sim \varphi) = \dfrac{\sin \varphi \,.\, \tan A}{\tan C}$

Let $\tan \varphi = \dfrac{\cos B \,.\, \tan a}{\text{rad}}$; then $\sin (c \sim \varphi) = \dfrac{\sin \varphi \,.\, \tan B}{\tan A}$

Case VI. Given two angles of an oblique-angled spherical triangle, and a side opposite to one of them, to find the third angle.

Find the side opposite to the other given angle by Case IV, then

$$\cot \tfrac{1}{2} A = \frac{\sin \frac{1}{2}(c+b)}{\sin \frac{1}{2}(c \sim b)} \cdot \tan \tfrac{1}{2}(C \sim B)$$

$$\cot \tfrac{1}{2} B = \frac{\sin \frac{1}{2}(a+c)}{\sin \frac{1}{2}(a \sim c)} \cdot \tan \tfrac{1}{2}(A \sim C)$$

$$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a \sim b)} \cdot \tan \tfrac{1}{2}(A \sim B)$$

OTHERWISE,

$$\text{Let } \cot \varphi = \frac{\cos b \cdot \tan C}{\text{rad}}; \text{ then } \sin(A \sim \varphi) = \frac{\cos B \cdot \sin \varphi}{\cos C}$$

$$\text{Let } \cot \varphi = \frac{\cos c \cdot \tan A}{\text{rad}}; \text{ then } \sin(B \sim \varphi) = \frac{\cos C \cdot \sin \varphi}{\cos A}$$

$$\text{Let } \cot \varphi = \frac{\cos a \cdot \tan B}{\text{rad}}; \text{ then } \sin(C \sim \varphi) = \frac{\cos A \cdot \sin \varphi}{\cos B}$$

CASE VII. Given two sides of an oblique-angled spherical triangle, and the angle contained between them, to find the other angles.

$$\left\{\begin{array}{l} \tan \frac{1}{2}(A+B) = \dfrac{\cos \frac{1}{2}(a \sim b)}{\cos \frac{1}{2}(a+b)} \cdot \cot \frac{1}{2} C \\[2ex] \tan \frac{1}{2}(A \sim B) = \dfrac{\sin \frac{1}{2}(a \sim b)}{\sin \frac{1}{2}(a+b)} \cdot \cot \frac{1}{2} C \end{array}\right.$$

$$\left\{\begin{array}{l} \tan \frac{1}{2}(A+C) = \dfrac{\cos \frac{1}{2}(a \sim c)}{\cos \frac{1}{2}(a+c)} \cdot \cot \frac{1}{2} B \\[2ex] \tan \frac{1}{2}(A \sim C) = \dfrac{\sin \frac{1}{2}(a \sim c)}{\sin \frac{1}{2}(a+c)} \cdot \cot \frac{1}{2} B \end{array}\right.$$

$$\left\{\begin{array}{l} \tan \frac{1}{2}(C+B) = \dfrac{\cos \frac{1}{2}(c \sim b)}{\cos \frac{1}{2}(c+b)} \cdot \cot \frac{1}{2} A \\[2ex] \tan \frac{1}{2}(C \sim B) = \dfrac{\sin \frac{1}{2}(c \sim b)}{\sin \frac{1}{2}(c+b)} \cdot \cot \frac{1}{2} A \end{array}\right.$$

OTHERWISE,

$$\left\{\begin{array}{l} \cot A = \dfrac{\cot a \cdot \sin b - \cos C \cdot \cos b}{\sin C} \\[2ex] \cot B = \dfrac{\cot b \cdot \sin a - \cos C \cdot \cos a}{\sin C} \end{array}\right.$$

$$\left\{\begin{array}{l} \cot C = \dfrac{\cot c \cdot \sin a - \cos B \cdot \cos a}{\sin B} \\[2ex] \cot A = \dfrac{\cot a \cdot \sin c - \cos B \cdot \cos c}{\sin B} \end{array}\right.$$

$$\begin{cases} \cot \mathrm{B} = \dfrac{\cot b \,.\, \sin c - \cos \mathrm{A} \,.\, \cos b}{\sin \mathrm{A}} \\ \cot \mathrm{C} = \dfrac{\cot c \,.\, \sin b - \cos \mathrm{A} \,.\, \cos b}{\sin \mathrm{A}} \end{cases}$$

OTHERWISE,

$$\begin{cases} \tan \varphi = \dfrac{\cos \mathrm{A} \,.\, \tan b}{\mathrm{rad}};\ \text{then } \cot \mathrm{B} = \dfrac{\cot \mathrm{A}}{\sin \varphi} \,.\, \sin (c \sim \varphi) \\ \tan \varphi = \dfrac{\cos \mathrm{A} \,.\, \tan c}{\mathrm{rad}};\ \text{then } \cot \mathrm{C} = \dfrac{\cot \mathrm{A}}{\sin \varphi} \,.\, \sin (b \sim \varphi) \end{cases}$$

Here b, c, and the included ∠ A are given, and formulæ will be obtained by a mere change of letters, if a, b, and the ∠ C; or if a, c, and the ∠ B are given.

CASE VIII. Given two sides of an oblique-angled spherical triangle, and the angle contained between them, to find the third side.

Find the other two angles by Case VII., and then find the third side by Case V.

OTHERWISE,

$$\text{Let } \tan \varphi = \frac{\cos \mathrm{A} \,.\, \tan c}{\mathrm{rad}};\ \text{then } \cos a = \frac{\cos c}{\cos \varphi} \,.\, \cos (b \sim \varphi)$$

$$\text{Let } \tan \varphi = \frac{\cos \mathrm{B} \,.\, \tan a}{\mathrm{rad}};\ \text{then } \cos b = \frac{\cos a}{\cos \varphi} \,.\, \cos (c \sim \varphi)$$

$$\text{Let } \tan \varphi = \frac{\cos \mathrm{C} \,.\, \tan b}{\mathrm{rad}};\ \text{then } \cos c = \frac{\cos b}{\cos \varphi} \,.\, \cos (a \sim \varphi)$$

Or, the sides a, b, c, respectively, may be found by the *fourth* set of equations, page 185.

CASE IX. Given two angles of an oblique-angled spherical triangle, and the side adjacent to both of them, to find a side opposite to one of the given angles.

$$\begin{cases} \tan \tfrac{1}{2} (a+b) = \dfrac{\cos \frac{1}{2} (\mathrm{A} \sim \mathrm{B})}{\cos \frac{1}{2} (\mathrm{A}+\mathrm{B})} \,.\, \tan \tfrac{1}{2} c \\ \tan \tfrac{1}{2} (a \sim b) = \dfrac{\sin \frac{1}{2} (\mathrm{A} \sim \mathrm{B})}{\sin \frac{1}{2} (\mathrm{A}+\mathrm{B})} \,.\, \tan \tfrac{1}{2} c \end{cases}$$

$$\begin{cases} \tan \tfrac{1}{2} (a+c) = \dfrac{\cos \frac{1}{2} (\mathrm{A} \sim \mathrm{C})}{\cos \frac{1}{2} (\mathrm{A}+\mathrm{C})} \,.\, \tan \tfrac{1}{2} b \\ \tan \tfrac{1}{2} (a \sim c) = \dfrac{\sin \frac{1}{2} (\mathrm{A} \sim \mathrm{C})}{\sin \frac{1}{2} (\mathrm{A}+\mathrm{C})} \,.\, \tan \tfrac{1}{2} b \end{cases}$$

$$\left\{\begin{array}{l}\tan\frac{1}{2}(c+b)=\dfrac{\cos\frac{1}{2}(C\sim B)}{\cos\frac{1}{2}(C+B)}\,.\,\tan\frac{1}{2}a\\[2ex]\tan\frac{1}{2}(c\sim b)=\dfrac{\sin\frac{1}{2}(C\sim B)}{\sin\frac{1}{2}(C+B)}\,.\,\tan\frac{1}{2}a\end{array}\right.$$

OTHERWISE,

$$\left\{\begin{array}{l}\cot a=\dfrac{\cot A\,.\,\sin B+\cos B\,.\,\cos c}{\sin c}\\[2ex]\cot b=\dfrac{\cot B\,.\,\sin A+\cos A\,.\,\cos c}{\sin c}\end{array}\right.$$

$$\left\{\begin{array}{l}\cot c=\dfrac{\cot C\,.\,\sin A+\cos A\,.\,\cos b}{\sin b}\\[2ex]\cot a=\dfrac{\cot A\,.\,\sin C+\cos C\,.\,\cos b}{\sin b}\end{array}\right.$$

$$\left\{\begin{array}{l}\cot b=\dfrac{\cot B\,.\,\sin C+\cos C\,.\,\cos a}{\sin a}\\[2ex]\cot c=\dfrac{\cot C\,.\,\sin B+\cos B\,.\,\cos a}{\sin a}\end{array}\right.$$

OTHERWISE,

$$\left\{\begin{array}{l}\text{Let }\cot\varphi=\dfrac{\tan A\,.\,\cos b}{\text{rad}};\text{ then }\cot a=\dfrac{\cot b}{\cos\varphi}\,.\,\cos(C\sim\varphi)\\[2ex]\text{Let }\cot\varphi=\dfrac{\tan C\,.\,\cos b}{\text{rad}};\text{ then }\cot c=\dfrac{\cot b}{\cos\varphi}\,.\,\cos(A\sim\varphi)\end{array}\right.$$

Here $\angle A$, $\angle C$, and the included side b are given; and formulæ will be obtained by a change of letters, if $\angle A$ and $\angle B$ and the included side c are given; or if $\angle B$ and $\angle C$ and the included side a are given.

CASE X. Given two angles of an oblique-angled spherical triangle, and the side adjacent to both of them, to find the other angle.

Find the other two sides by Case IX., and then find the other angle by Case VI.

OTHERWISE,

$$\cot\varphi=\frac{\cos a\,.\,\tan C}{\text{rad}};\text{ then }\cos A=\cos C\,.\,\frac{\sin(B\sim\varphi)}{\sin\varphi}$$

$$\cot\varphi=\frac{\cos b\,.\,\tan a}{\text{rad}};\text{ then }\cos B=\cos A\,.\,\frac{\sin(C\sim\varphi)}{\sin\varphi}$$

$$\cot\varphi=\frac{\cos c\,.\,\tan B}{\text{rad}};\text{ then }\cos C=\cos B\,.\,\frac{\sin(A\sim\varphi)}{\sin\varphi}$$

Or, the angles A, B, C, respectively, may be found by the *sixth* set of equations, page 185.

Case XI. Given the three sides of an oblique-angled spherical triangle, to find the angles.

Let $a+b+c=2s$, then

$$\sin\tfrac{1}{2}\mathrm{A}=\text{rad}\sqrt{\frac{\sin(s-b)\,.\,\sin(s-c)}{\sin b\,.\,\sin c}}$$

$$\cos\tfrac{1}{2}\mathrm{A}=\text{rad}\sqrt{\frac{\sin s\,.\,\sin(s-a)}{\sin b\,.\,\sin c}}$$

$$\tan\tfrac{1}{2}\mathrm{A}=\text{rad}\sqrt{\frac{\sin(s-b)\,.\,\sin(s-c)}{\sin s\,.\,\sin(s-a)}}$$

$$\cot\tfrac{1}{2}\mathrm{A}=\text{rad}\sqrt{\frac{\sin s\,.\,\sin(s-a)}{\sin(s-b)\,.\,\sin(s-c)}}$$

$$\sin\tfrac{1}{2}\mathrm{B}=\text{rad}\sqrt{\frac{\sin(s-a)\,.\,\sin(s-c)}{\sin a\,.\,\sin c}}$$

$$\cos\tfrac{1}{2}\mathrm{B}=\text{rad}\sqrt{\frac{\sin s\,.\,\sin(s-b)}{\sin a\,.\,\sin c}}$$

$$\tan\tfrac{1}{2}\mathrm{B}=\text{rad}\sqrt{\frac{\sin(s-a)\,.\,\sin(s-c)}{\sin s\,.\,\sin(s-b)}}$$

$$\cot\tfrac{1}{2}\mathrm{B}=\text{rad}\sqrt{\frac{\sin s\,.\,\sin(s-b)}{\sin(s-a)\,.\,\sin(s-c)}}$$

$$\sin\tfrac{1}{2}\mathrm{C}=\text{rad}\sqrt{\frac{\sin(s-b)\,.\,\sin(s-a)}{\sin a\,.\,\sin b}}$$

$$\cos\tfrac{1}{2}\mathrm{C}=\text{rad}\sqrt{\frac{\sin s\,.\,\sin(s-c)}{\sin a\,.\,\sin b}}$$

$$\tan\tfrac{1}{2}\mathrm{C}=\text{rad}\sqrt{\frac{\sin(s-b)\,.\,\sin(s-a)}{\sin s\,.\,\sin(s-c)}}$$

$$\cot\tfrac{1}{2}\mathrm{C}=\text{rad}\sqrt{\frac{\sin s\,.\,\sin(s-c)}{\sin(s-b)\,.\,\sin(s-c)}}$$

Or, the angles A, B, C, respectively, may be found by the formulæ in art. (417).

Case XII. Given the three angles of an oblique-angled spherical triangle, to find the sides.

Let $\mathrm{A}+\mathrm{B}+\mathrm{C}=2\mathrm{S}$, then

$$\sin\tfrac{1}{2}a=\text{rad}\sqrt{\frac{\cos\mathrm{S}\,.\,\cos(\mathrm{S}-\mathrm{A})}{\sin\mathrm{B}\,.\,\sin\mathrm{C}}}$$

$$\cos\tfrac{1}{2}a=\text{rad}\sqrt{\frac{\cos(\mathrm{S}-\mathrm{B})\,.\,\cos(\mathrm{S}-\mathrm{C})}{\sin\mathrm{B}\,.\,\sin\mathrm{C}}}$$

$$\tan\tfrac{1}{2}a=\text{rad}\sqrt{\frac{\cos S\,.\,\cos(S-A)}{\cos(S-B)\,.\,\cos(S-C)}}$$

$$\cot\tfrac{1}{2}a=\text{rad}\sqrt{\frac{\cos(S-B)\,.\,\cos(S-C)}{\cos S\,.\,\cos(S-A)}}$$

$$\sin\tfrac{1}{2}b=\text{rad}\sqrt{\frac{\cos S\,.\,\cos(S-B)}{\sin A\,.\,\sin C}}$$

$$\cos\tfrac{1}{2}b=\text{rad}\sqrt{\frac{\cos(S-A)\,.\,\cos(S-C)}{\sin A\,.\,\sin C}}$$

$$\tan\tfrac{1}{2}b=\text{rad}\sqrt{\frac{\cos S\,.\,\cos(S-B)}{\cos(S-A)\,.\,\cos(S-C)}}$$

$$\cot\tfrac{1}{2}b=\text{rad}\sqrt{\frac{\cos(S-A)\,.\,\cos(S-C)}{\cos S\,.\,\cos(S-B)}}$$

$$\sin\tfrac{1}{2}c=\text{rad}\sqrt{\frac{\cos S\,.\,\cos(S-C)}{\sin A\,.\,\sin B}}$$

$$\cos\tfrac{1}{2}c=\text{rad}\sqrt{\frac{\cos(S-A)\,.\,\cos(S-B)}{\sin A\,.\,\sin B}}$$

$$\tan\tfrac{1}{2}c=\text{rad}\sqrt{\frac{\cos S\,.\,\cos(S-C)}{\cos(S-A)\,.\,\cos(S-C)}}$$

$$\cot\tfrac{1}{2}c=\text{rad}\sqrt{\frac{\cos(S-A)\,.\,\cos(S-B)}{\cos S\,.\,\cos(S-C)}}$$

Or, the sides a, b, c, respectively, may be found by the formulæ in the *fifth* set of equations, page 185.

OF THE AMBIGUITY OF THE DIFFERENT CASES.

(446) When two sides, and an angle opposite to one of them, are given, to find the rest (see Case Ist, IId, and IIId), the values of these required parts are sometimes ambiguous.

Practical Rules for determining whether the quantities sought are acute, obtuse, or ambiguous, are given in the solutions of the different cases.

The two following tables are the same as those given by Legendre at pages 400. and 401. of the 6th edition of his Geometry, and are deduced from Prop. XVII. page 150 and Prop. XVIII. page 151 of this treatise.

TABLE I. Let A, a and b be the given parts. Then,

$$\begin{array}{ll}
A<90^\circ,\ b<90^\circ & \left\{\begin{array}{l} a>b \text{ one solution.} \\ a<b \text{ two solutions.} \end{array}\right. \\
A<90^\circ,\ b>90^\circ & \left\{\begin{array}{l} a+b>180^\circ \text{ one solution.} \\ a+b<180^\circ \text{ two solutions.} \end{array}\right. \\
A>90^\circ,\ b<90^\circ & \left\{\begin{array}{l} a+b>180^\circ \text{ two solutions.} \\ a+b<180^\circ \text{ one solution.} \end{array}\right. \\
A>90^\circ,\ b>90^\circ & \left\{\begin{array}{l} a>b \text{ two solutions.} \\ a<b \text{ one solution.} \end{array}\right.
\end{array}$$

If $A=90^\circ$, $a=b$, or $a+b=180$, the cases will not be ambiguous; but if $b=90^\circ$, there will be two solutions.

When two angles, and a side opposite to one of them, are given, to find the rest (see Case IVth, Vth, and VIth), the values of the required parts are subject to ambiguity; this triangle being supplemental to that wherein two sides and an angle opposite to one of them are given.

TABLE II. Let A, B, and a be the given parts. Then,

$$\begin{array}{ll}
1.\ a>90^\circ,\ B>90^\circ & \left\{\begin{array}{l} A<B \text{ one solution.} \\ A>B \text{ two solutions.} \end{array}\right. \\
2.\ a>90^\circ,\ B<90^\circ & \left\{\begin{array}{l} A+B<180^\circ \text{ one solution.} \\ A+B>180^\circ \text{ two solutions.} \end{array}\right. \\
3.\ a<90^\circ,\ B>90^\circ & \left\{\begin{array}{l} A+B<180^\circ \text{ two solutions.} \\ A+B>180^\circ \text{ one solution.} \end{array}\right. \\
4.\ a<90^\circ,\ B<90^\circ & \left\{\begin{array}{l} A<B \text{ two solutions.} \\ A>B \text{ one solution.} \end{array}\right.
\end{array}$$

If $a=90^\circ$, $A=B$, or $A+B=180^\circ$, there will be but one solution; but if $B=90^\circ$, there will be two solutions.

(447) We may likewise remark, that since any side or angle of a spherical triangle is less than 180°, the half of any angle or half the difference between any two sides, or half the difference between any two angles, must be acute.

Hence in the equation

$\cot \frac{1}{2} C \cdot \cos \frac{1}{2} (a-b) = \tan \frac{1}{2} (A+B) \cdot \cos \frac{1}{2} (a+b)$ (428), it is plain that $\cot \frac{1}{2} C$ and $\cos \frac{1}{2} (a-b)$ are both positive (200), and therefore $\tan \frac{1}{2} (A+B) \cdot \cos \frac{1}{2} (a+b)$ must be positive; consequently, *half the sum of any two sides of a spherical triangle is of the same species as half the sum of their opposite angles.* This rule is applied in the practical solutions of the different cases, and will frequently remove the ambiguity which would otherwise arise, where a quantity sought is to be determined by means of a sine.

CHAP. VI.

I. PRACTICAL RULES FOR THE SOLUTIONS OF ALL THE DIFFERENT CASES OF RIGHT-ANGLED SPHERICAL TRIANGLES, WITH THEIR APPLICATION BY LOGARITHMS.

(448) Every spherical triangle consists of six parts, three sides, and three angles; any three of which being given, the rest may be found.

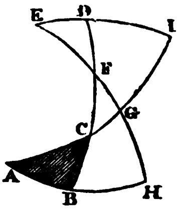
In a right-angled spherical triangle, two given parts, besides the right angle, are sufficient to determine the rest.

The questions arising from a variation of the given and required parts are 16, but if distinguished by the *data*, the number of cases is 6.

THE GIVEN QUANTITIES ARE, EITHER

1. The hypothenuse and an angle.
2. The hypothenuse and one side.
3. A side and its adjacent angle.
4. A side and its opposite angle.
5. The two sides.
6. The two angles.

RULE I.



(449) Draw a rough figure, as in the margin; and let AG, AH; FB, FH; CI, CD; EI, EG, be considered as quadrants, or 90° each; then you have eight right-angled spherical triangles, every two of which will have equal angles at their bases: and the triangles CGF and EDF will have their respective sides and angles either equal to those of ABC, the triangle under consideration, or they will be the complements thereof (341).

In triangles having equal angles at their bases, the sines of their bases have the same ratio to each other, as the tangents of their perpendiculars.

The sines of the hypothenuses have the same ratio to each other, as the sines of the perpendiculars (382).

Illustration. 1*st. In the triangles* ABC *and* AHG.

1. sin AG : sin AC : : sin HG : sin BC } and these are propor-
2. sin AH : sin AB : : tan HG : tan BC } tional by inversion.

2d. In the triangles FGC *and* FHB.

$$\left.\begin{array}{l}\text{3. sin BH : sin CG : : sin FB : sin FC}\\ \text{4. sin FH : sin FG : : tan BH : tan CG}\end{array}\right\}\text{ and inversely, \&c.}$$

3d. In the triangles CGF *and* CID.

$$\left.\begin{array}{l}\text{5. sin DC : sin FC : : sin ID : sin FG}\\ \text{6. sin CG : sin CI : : tan FG : tan ID}\end{array}\right\}\text{ and inversely, \&c.}$$

4th. In the triangles EDF *and* EIG.

7. sin EI : sin ED : : tan IG : tan DF, and inversely, &c.

The student must remember that ABC is the proper triangle in the preceding proportions, and that AG, AH, &c. are each 90°, consequently sin AG, sin AH, &c. are each equal to radius.

$$\left.\begin{array}{l}\text{BH is the complement of the base AB.}\\ \text{CG is the complement of the hypoth. AC.}\\ \text{FC is the complement of the perp. BC.}\\ \text{FG is the complement of the angle A.}\\ \text{ED is the complement of the angle C.}\end{array}\right\}\text{ In any of these cases, for sine, or tangent, write cosine, or cotangent.}$$

HG = EF is the measure of the angle A.
ID is the measure of the angle C.
IG = AC the hypothenuse, and DF = BC the perpendicular.

Since AB and BC are perpendicular to each other, either of them may be considered as the base, and to avoid a number of different figures it will sometimes be necessary to make A and C change places, as is done in some of the succeeding cases; then, in considering these remarks, where AB occurs read BC, and instead of the angle A read C; and the contrary.

RULE II. BARON NAPIER'S RULES.

(450) 1. In every right-angled spherical triangle there are five parts, exclusive of the right angle, which is not taken into consideration; and these five parts are the hypothenuse, the two sides or legs, and their opposite angles. Now in every case proposed for solution, there are three of these five parts concerned, that is, two given (together with the right angle) and a third required.

2. If the three quantities under consideration, viz. the two which are given and that which is required, are joined together, or follow each other in a successive order, without the intervention of a side, or angle, not concerned in the question, the middle one is called the *middle part*, and the other two the *extremes conjunct*, because they are joined to the middle part.

3. If only two of the three things under consideration are connected, or joined together, they are invariably called *ex-*

tremes disjunct, that is, not joined to the middle part, and the other term which is not joined with them, is called the *middle part.*

Then,

4. *Radius × sine of the middle part = rectangle, or product, of the tangents of the extremes when they are conjunct.*

5. *Radius × sine of the middle part = rectangle, or product, of the cosines of the extremes when they are disjunct.*

Observe to write cosine or cotangent, instead of sine or tangent, and sine instead of cosine, &c. when you make use of either of the angles, or the hypothenuse; but not when you use the sides or legs.

6. Having found which is the *middle part*, and written the terms down as expressed in one or other of the above rules, according as the extremes are *conjunct* or *disjunct*, turn the terms into a proportion, *thus:* Put the required term *last*, that with which it is connected *first*, and the remaining two in the middle in any order.

RULE III.

(451) 1. Radius × sine of either side = sine of the opposite angle × sine of the hypothenuse.

2. Radius × sine of either side = tangent of the other side × cot of its opposite ∠.

3. Radius × cosine of either of the angles = tangent of the adjacent side × cot hypothenuse.

4. Radius × cosine of either of the angles = sine of the other angle × cosine of its adjacent side.

5. Radius × cosine of the hypothenuse = cot of one angle × cot of the other ∠.

6. Radius × cosine of the hypothenuse = cosine of one side × cosine of the other side.

Then put the required term last, that with which it is connected first, and the remaining two in the middle in any order, and you will have a correct proportion.

NOTE. This rule is the same as BARON NAPIER'S (384).

OF THE DIFFERENT SPECIES OR AFFECTIONS OF RIGHT-ANGLED SPHERICAL TRIANGLES.

(452) I. *When the hypothenuse and an angle are given.*

1. The side opposite to the *given* angle, is of the same species with the *given* angle.

2. The side adjacent to the *given* angle, is acute or obtuse, according as the hypothenuse is of the same, or of different species with the *given* angle.

3. The other angle is acute or obtuse, according as the hypothenuse and the *given* angle are of the same or of different species.

(453) II. *When the hypothenuse and one side are given.*

1. The angle opposite to the *given* side, is of the same species with the *given* side.

2. The angle adjacent to the *given* side, is acute or obtuse, according as the hypothenuse is of the same or of different species with the *given* side.

3. The other side is acute or obtuse, according as the hypothenuse is of the same, or of different species with the *given* side.

(454) III. *When a side and its adjacent angle are given.*

1. The other angle is of the same species as the *given* side.

2. The other side is of the same species as the *given* angle.

3. The hypothenuse is acute or obtuse, according as the *given* angle is of the same or of different species with the *given* side.

(455) IV. *When a side and its opposite angle are given.*

The required parts are always ambiguous or doubtful, that is, they may be either greater or less than 90°, and therefore admit of two answers.

(456) V. *When the two sides, containing the right angle, are given.*

1. Each angle is of the same species as its opposite side.

2. The hypothenuse is acute or obtuse, according as the sides are of the same or of different species.

(457) VI. *When the two angles, adjacent to the hypothenuse, are given.*

1. Each side is of the same species as its opposite angle.

2. The hypothenuse is acute or obtuse, according as the angles are of the same or of different species.

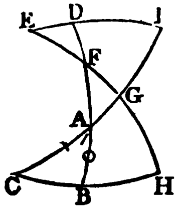
(458) CASE I. *Given the hypothenuse and an angle, to find the side adjacent to the given angle.*

Example. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{ll}\text{The hypothenuse} & AC = 78^\circ\ 20' \\ \text{The angle} & A = 37\ \ 25\end{array}\right\}\begin{array}{l}\text{Required the adja-} \\ \text{cent leg } AB.\end{array}$$

BY NAPIER'S RULE.

The complement of the angle A is the *middle part,* AB and the complement of AC are the *extremes conjunct,* or joined to the middle part.



*

rad × cos A = tan AB × cot AC.
∴ 10 + log cos A − log cot AC = log tan AB.
or, cot AC - - =78° 20′ - 9·31489
: radius, sine of - 90 0 - 10·00000
: : cos ∠ A - - =37 25 - 9·89995
: tan AB - - =75 26 - 10·58506

Here AB *is acute* (452).

BY RULE I.

sin EI : sin ED : : tan IG : tan DF. But EI is radius, DI the measure of the angle A, and ED is complement, IG = AC, and DF = AB. Hence,

rad : cos ∠ A : : tan AC : tan AB.

BY CONSTRUCTION. (*Plate V. Fig.* 8.)

1. With the chord of 60 degrees describe the primitive circle; and through any point A in the primitive, draw a great circle A*ce*, making an angle BAC = 37° 25′ (369).

2. Set off the hypothenuse 78° 20′ from A to *m*, find *p* the pole of the oblique circle A*ce* (368); join *pm*, and through the point C, where it cuts A*ce*, draw BP*n*; then ABC is the triangle required.

To measure the required parts.

3. AB applied to a scale of chords gives the measure of AB = 75° 26′.

4. Take BC in your compasses, set one foot of the compasses on 90° on the scale of semi-tangents, and the other towards the left hand of the scale, the degrees between the points = 36° 31′ will be the measure of BC.

5. Produce *mp* to *b*, and find *a* the pole of *n*PB (368); then *ab* applied to the scale of chords will be the measure of the angle C = 81° 13′.

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC = 61° 4′ 56″ } Required the adjacent leg AB.
{ The angle A = 61 50 29 }

Answer. AB = 40° 30′ 19″ acute (452).

2. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC = 113° 55′ } Required the adjacent leg AB.
{ The angle A = 31 51 }

Answer. AB = 117° 34′ obtuse (452).

(459) Case II. *Given the hypothenuse and one angle, to find the side opposite to that given angle.*

Example. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC = 78° 20′ / The angle A = 37 25 } Required the opposite leg BC.

BY NAPIER'S RULE.

Here BC is *the middle part*, and the complement of A and the complement of AC are the *extremes disjunct;* that is, not joined to the middle part, because of the intervention of AB and the angle C.

rad × sin BC = sin AC × sin A.

∴ log sin AC + log sin A — 10 = log sin BC.

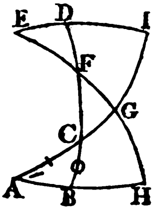
or, radius, sine of - - 90° 0′ - 10·00000

: sin AC - - - = 78 20 - 9·99093

: : sin∠A - - = 37 25 - 9·78362

: sin BC - - = 36 31 - 9·77455

Here BC *is acute* (452).



BY RULE I.

sin AG : sin AC : : sin HG : sin BC. But AG is radius, and HG is the measure of the angle A. Hence,

rad : sin AC : : sin A : sin BC.

BY CONSTRUCTION.

See the construction of Case I. (458).

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC = 61° 4′ 56″ / The angle A = 61 50 29 } Required the opposite leg BC.

Answer. BC = 50° 30′ 30″ acute (452).

2. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC = 113° 55′ / The angle A = 104 8 } Required the opposite leg BC.

Answer. BC = 117° 34′ obtuse (452).

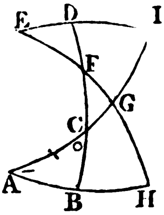
(460) Case III. *Given the hypothenuse and one angle to find the other angle.*

Example. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC = 78° 20′ / The angle A = 37 25 } Required the angle C.

BY NAPIER'S RULE.

Here the three parts are connected or joined together, therefore the complement of AC is the *middle part;* and the complement of A and the complement of C are *extremes conjunct.*



rad × cos AC = cot A × cot $\overset{*}{\text{C}}$

∴ 10 + log cos AC − log cot A = log cot C.

or, cot∠A - - = 37° 25′ - 10·11633
: radius, sine of - 90 0 - 10·00000
: : cos AC - - = 78 20 - 9·30582
: cot∠C - - = 81 12 - 9·18949

Here angle C *is acute* (452).

BY RULE I.

sin CG : sin CI : : tan FG : tan DI. But CG is the complement of AC, CI radius, FG the complement of the angle A, and DI the measure of the angle C. Hence,

cos AC : rad : : cot A : tan C.

BY CONSTRUCTION.

See the construction of CASE I. (458).

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC = 61° 4′ 56″ ; The angle A = 61 50 29 } Required the angle C.

Answer. The angle C = 47° 54′ 20″ acute (452).

2. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC = 113° 55′ ; The angle A = 31 51 } Required the angle C.

Answer. The angle C = 104° 8′ obtuse (452).

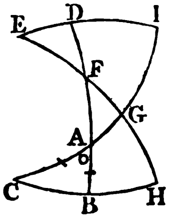
(461) CASE IV. *Given the hypothenuse and a leg, to find the angle adjacent to that given leg.*

Example. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC = 78° 20′ ; The side or leg AB = 75 25 } To find the angle A adjacent to AB.

BY NAPIER'S RULE.

The three parts are here connected; therefore the complement of A is the *middle part*, AB and the complement of AC are the *extremes conjunct*.



rad × cos $\overset{*}{\text{A}}$ = cot AC × tan AB.

∴ log cot AC + log tan AB − 10 = log cos A.

or, radius, sine of - 90° 0′ - 10·00000
: cot AC - - - = 78 20 - 9·31489
: : tan AB - - = 75 25 - 10·58474
: cos∠A - - - = 37 28 - 9·89963

Here the angle A *is acute* (453).

BY RULE I.

tan GI : tan DF : : sin EI : sin ED.

But GI = AC, DF = AB, EI is the radius, and ED the complement of the angle A. Hence,

tan AC : tan AB : : rad : cos∠A.

BY CONSTRUCTION. (*Plate V. Fig.* 9.)

1. With the chord of 60° describe the primitive circle, and through the centre P draw CP*e*, and *a*P*b* at right angles to it.

2. Set off the hypothenuse, by a scale of chords, from C to *m*, and draw the small circle *m*A*m* parallel to the right circle *a*P*b* (374).

3. Take the complement of AB (viz. 14° 35′) from the scale of semi-tangents, with which as a radius and the centre P describe a small circle parallel to the primitive (374) intersecting the circle *m*A*m* in A.

4. Through the three points CA*e* draw a great circle, then ABC is the triangle required.

To measure the required parts.

5. The angles A and C (379) are 37° 28′ and 81° 11′.

6. BC, applied to a scale of chords, will be 36° 34′.

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC = 61° 4′ 56″ / The side or leg AB = 40 30 20 } To find the angle A, adjacent to AB.

Answer. The angle A = 61° 50′ 30″ acute (453).

2. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC = 78° 20′ / The side or leg AB = 117 34 } To find the adjacent angle A.

Answer. The angle A = 113° 18′ obtuse (453).

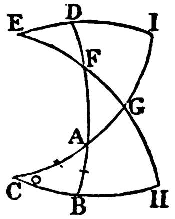
(462) CASE V. *Given the hypothenuse and a side or leg, to find the angle opposite to that given leg.*

Example. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC = 78° 20′ / The side AB = 75 25 } To find the angle C, opposite to AB.

BY NAPIER'S RULE.

The three parts are here disjoined by the intervention of the angle A; therefore AB is the *middle part*, the complement of AC and the complement of C are the *extremes disjunct.*



$$\text{rad} \times \sin AB = \sin AC \times \overset{*}{\sin} C.$$

$$\therefore 10 + \log \sin AB - \log \sin AC = \log \sin C.$$

or, sin AC - - - - = 78° 20′ - 9·99093
: radius, sine of - 90 0 - 10·00000
: : sin AB - - = 75 25 - 9·98578
: sin ∠ C - - = 81 11 - 9·99485

Here the angle C *is acute* (453).

BY RULE I.

sin AC : sin CG : : sin AB : sin GH.

But CG is radius, and GH is the measure of the angle C. Hence,

sin AC : rad : : sin AB : sin C.

BY CONSTRUCTION.

See the construction of CASE IV. (461).

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{l}\text{The hypothenuse AC} = 61° \ 4' \ 56'' \\ \text{The side or leg AB} = 40 \ 30 \ 20\end{array}\right\}$ Required the angle C, opposite to AB.

Answer. The angle C = 47° 54′ 19″ acute (453).

2. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{l}\text{The hypothenuse AC} = 78° \ 20' \\ \text{The side or leg AB} = 117 \ \ 34\end{array}\right\}$ Required the angle C, opposite to AB.

Answer. The angle C = 115° 9′ obtuse (453).

(463) CASE VI. *Given the hypothenuse and one side or leg, to find the other leg.*

Example. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{l}\text{The hypothenuse AC} = 78° \ 20' \\ \text{The side AB} = 75 \ \ 25\end{array}\right\}$ To find the other side BC.

BY NAPIER'S RULE.

Here the complement of AC is the *middle part*, because it is not joined either to AB or BC, the angles A and C coming between them; hence AB and BC are *extremes disjunct*, viz. not joined to the middle part.

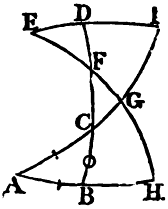
*

rad × cos AC = cos AB × cos BC.

∴ 10 + log cos AC − log cos AB = log cos BC.

or, cos AB - -	75° 25′	- =	9·40103
: radius, sine of	90 0	- -	10·00000
: : cos AC - -	78 20	- =	9·30582
: cos BC - -	36 34	- =	9·90479

Here BC *is acute* (453).



BY RULE I.

sin BH : sin CG : : sin FB : sin CF.

But BH is the complement of AB, CG the complement of AC, FC the complement of BC, and FB radius. Hence,

cos AB : cos AC : : rad : cos BC.

BY CONSTRUCTION.

See the construction of Case IV. (461).

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{ll}\text{The hypothenuse AC} = 61^\circ\ 4'\ 56'' \\ \text{The side, or leg, AB} = 40\ \ 30\ \ 20\end{array}\right\}$ Required the other leg BC.

Answer. BC = 50° 30′ 29″ acute (453).

2. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{ll}\text{The hypothenuse AC} = & 78^\circ\ 20' \\ \text{The side AB} = & 117\ \ 34\end{array}\right\}$ Required the other side BC.

Answer. BC = 115° 55′ obtuse (453).

(464) CASE VII. *Given a side and the angle adjacent, or joining to it, to find the side opposite to the given angle.*

Example. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{ll}\text{The side or leg AB} = 75^\circ\ 25' \\ \text{The adjacent angle A} = 37\ \ 25\end{array}\right\}$ To find the side BC, opposite to A.

BY NAPIER'S RULE.

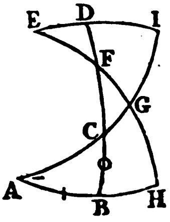
Here the three parts are joined together, because the right angle B is not regarded in the rule; therefore AB is the *middle part*, the complement of A, and BC are the *extremes conjunct.*

rad × sin AB = cot A × tan BC.

∴ 10 + log sin AB − log cot A = log tan BC.

or, cot A - -	= 37° 25′	-	10·11683
: rad, sine of -	90 0	-	10·00000
: : sin AB -	= 75 25	-	9·98578
: tan BC - -	= 36 31	-	9·86945

Here BC *is acute* (454).



BY RULE I.

sin AH : sin AB : : tan GH : tan BC.

But AH is radius, and GH is the measure of the angle A. Hence,

rad : sin AB : : tan ∠ A : tan BC.

BY CONSTRUCTION. (*Plate V. Fig.* 11.)

1. With the chord of 60 degrees describe the primitive circle, and through the centre P draw AP*a* and *mnr* at right angles to it.

2. Set one foot of your compasses on 90 degrees in the line of semi-tangents, extend the other towards the beginning of the scale, till the degrees between them be equal to the angle A 37° 25′, and apply this extent from *m* to *n*.

3. Through the three points A*na* draw a great circle.

4. Set off AB = 75° 25′ from A to B, by a scale of chords, and draw BP cutting the oblique circle in C, then ABC is the triangle required.

To measure the required parts.

5. BC measured by the scale of semi-tangents (as in Case I.) will be 36° 31′.

6. The angle C measured (379) will be 81° 11′.

7. The hypothenuse AC measured (376) will be 78° 20′.

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{l}\text{The leg AB} = 40^\circ\ 30'\ 20''\\ \text{The adjacent angle A} = 61\ \ 50\ \ 29\end{array}\right\}$ To find the leg BC, opposite to A.

Answer. BC = 50° 30′ 29″ acute (454).

2. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{l}\text{The side AB} = 54^\circ\ 22'\\ \text{The adjacent angle A} = 136\ \ 40\end{array}\right\}$ To find the leg BC, opposite to A.

Answer. BC = 142° 31′ obtuse (454).

(465) CASE VIII. *Given a side and its adjacent angle, to find the angle opposite to the given side.*

Example. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{l}\text{The side AB} = 75^\circ\ 25'\\ \text{Its adjacent angle A} = 37\ \ 25\end{array}\right\}$ To find the angle C, opposite to AB.

BY NAPIER'S RULE.

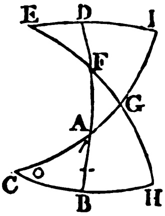
Here the complement of the angle C is the *middle part;* and the complement of A and AB are the *extremes disjunct*, being neither of them joined to the middle part.

$$\text{rad} \times \cos \overset{*}{\text{C}} = \sin \text{A} \times \cos \text{AB}.$$

$$\therefore \log \sin \text{A} + \log \cos \text{AB} - 10 = \log \cos \text{C}.$$

or, radius, sine of	- 90° 0′	-	10·00000
: sin ∠ A - -	=37 25	-	9·78362
: : cos AB - -	=75 25	-	9·40103
cos ∠ C - -	=81 12	-	9·18465

Here C *is acute* (454).



BY RULE I.

sin AD : sin AF : : sin ID : sin FG.

But AD is radius, AF the complement of AB, ID the measure of the angle A, and FG the complement of GH, which measures the angle C. Hence,

rad : cos AB : : sin A : cos C.

BY CONSTRUCTION.

See the construction of CASE VII. (464).

PRACTICAL EXAMPLES.

In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{l}\text{The side AB} = 40^\circ\ 30'\ 20''\\ \text{Its adjacent angle A} = 61\ \ 50\ \ 29\end{array}\right\}$ Required the angle C, opposite to AB.

Answer. The angle C = 47° 54′ 21″ acute (454).

2. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{ll}\text{The side AB} & =117^\circ\ 34' \\ \text{Its adjacent angle A} & =\ 31\ \ 51\end{array}\right\}\begin{array}{l}\text{Required the angle} \\ \text{C, opposite to AB.}\end{array}$$

Answer. The angle C = 104° 8′ obtuse (454).

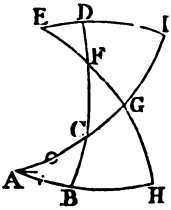
(466) Case IX. *Given a side and its adjacent angle, to find the hypothenuse.*

Example. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{ll}\text{The side AB} & =75^\circ\ 25' \\ \text{Its adjacent angle A} & =37\ \ 25\end{array}\right\}\begin{array}{l}\text{Required the hypo-} \\ \text{thenuse AC.}\end{array}$$

BY NAPIER'S RULE.

The three parts are here connected; therefore the complement of A is the *middle part;* AB and the complement of AC are the *extremes conjunct.*



*

rad × cos A = cot AC × tan AB.

∴ 10 + log cos A − log tan AB = log cot AC.

or, tan AB - - = 75° 25′ - 10·58474
: radius, sine of 90 0 - 10·00000
: : cos ∠ A - - = 37 25 - 9·89995
: cot AC - - 78 19 - 9·31521

Here AC *is acute* (454).

BY RULE I.

sin FH : sin FG : : tan BH : tan CG.

But FH is radius, FG is the complement of the angle A, BH the complement of AB, and CG the complement of AC. Hence,

rad : cos ∠ A : : cot AB : cot AC.

BY CONSTRUCTION.

See the construction of Case VII. (464).

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{ll}\text{The side AB} & =50^\circ\ 30'\ 30'' \\ \text{Its adjacent angle A} & =47\ \ 54\ \ 20\end{array}\right\}\begin{array}{l}\text{Required the} \\ \text{hypothenuse AC.}\end{array}$$

Answer. AC = 61° 4′ 56″ acute (454).

2. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{ll}\text{The side AB} & =117^\circ\ 34' \\ \text{Its adjacent angle A} & =\ 31\ \ 51\end{array}\right\}\begin{array}{l}\text{Required the} \\ \text{hypothenuse AC.}\end{array}$$

Answer. AC = 113° 55′ obtuse (454).

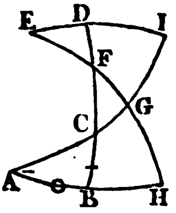
(467) Case X. *Given a side and its opposite angle, to find the side adjacent to that angle.*

Example. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{ll}\text{The side BC} & =36^\circ\ 31' \\ \text{Its opposite angle A} & =37\ \ 25\end{array}\right\}\begin{array}{l}\text{To find the side AB,} \\ \text{adjacent to A.}\end{array}$$

BY NAPIER'S RULE.

The right angle B having no concern in the rule, the three parts are joined together; therefore AB is the *middle part*, the complement of A, and BC are the *extremes conjunct*.



*

rad × sin AB = cot A × tan BC.

∴ log cot A + log tan BC − 10 = log sin AB.

or, radius, sine of - 90° 0′ - 10·00000
: cot ∠ A - - = 37 25 - 10·11633
: : tan BC - = 36 31 - 9·86947
: sin AB $\left\{\begin{matrix} 75° \ 25' \text{ or} \\ 104 \ \ 35 \end{matrix}\right\}$ - 9·98580

Here AB *is ambiguous* (455).

BY RULE I.

tan HG : tan BC : : sin AH : sin AB.

But HG is the measure of the angle A, and AH radius. Hence,

tan ∠ A : tan BC : : rad : sin AB.

BY CONSTRUCTION. (*Plate V. Fig.* 10.)

1. With the chord of 60 degrees describe the primitive circle, draw AP*a*, and *mn*P*r* at right angles to it.

2. Set one foot of your compasses on 90 degrees in the line of semi-tangents, extend the other towards the beginning of the scale till the degrees between the two points be equal to the angle A 37° 25′, and apply this extent from *m* to *n*.

3. Through the three points A*na* draw a great circle.

4. Take the complement of BC (viz. 53° 29′) from the scale of semi-tangents, with which as a radius and the centre P, describe a small circle *o*C parallel to the primitive (374), meeting the great circle A*na* in C.

5. Through C and P draw BCP, then ABC is the triangle required.

This problem is ambiguous, for ABC and *a*BC have the same data.

To measure the required parts.

6. The angle C (379) will be 81° 12′.

7. The side AC (376) is 78° 20′.

8. The side AB applied to a scale of chords will be 75° 25′.

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

Given $\left\{\begin{matrix} \text{The side BC} & = 40° \ 30' \ 20'' \\ \text{Its opposite angle A} & = 47 \ \ 54 \ \ 20 \end{matrix}\right\}$ To find the other side.

Answer. AB = 50° 30′ 30″ or 129° 29′ 30″, ambiguous (455).

2. In the right-angled spherical triangle ABC.

Given $\left\{\begin{matrix}\text{The side BC} & =51^\circ\ 30' \\ \text{Its opposite angle A} & =58\ \ 35\end{matrix}\right\}$ To find the other side.

Answer. AB = 50° 10′ or 129° 50′, ambiguous (455).

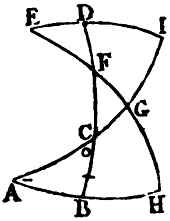
(468) CASE XI. *Given a side and its opposite angle, to find the adjacent angle.*

Example. In the right-angled spherical triangle ABC.

Given $\left\{\begin{matrix}\text{The side BC} & =36^\circ\ 31' \\ \text{Its opposite angle A} & =37\ \ 25\end{matrix}\right\}$ To find the angle C, adjacent to BC.

BY NAPIER'S RULE.

The complement of A is the *middle part;* and BC and the complement of C are the *extremes disjunct*, being neither of them joined to A, the middle part.



rad × cos A = sin $\overset{*}{\text{C}}$ × cos BC.

∴ 10 + log cos A − log cos BC = log sin A.

or, cos BC - = 36° 31′ - 9·90509

: radius, sine of 90 0 - - 10·00000

: : cos ∠ A - = 37 25 - 9·89995

: sin ∠ C - - $\left\{\begin{matrix}81 & 12 \text{ or} \\ 98 & 48\end{matrix}\right\}$ 9·99486

Here the angle C *is ambiguous* (455).

BY RULE I.

sin CF : sin CD : : sin FG : sin ID.

But CF is the complement of BC, CD radius, FG the complement of the angle A, and ID the measure of the angle C. Hence,

cos BC : rad : : cos ∠ A : sin C.

BY CONSTRUCTION.

See the construction of CASE X. (467).

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

Given $\left\{\begin{matrix}\text{The side BC} & =49^\circ\ 30'\ 20'' \\ \text{Its opposite angle A} & =47\ \ 54\ \ 20\end{matrix}\right\}$ Required the adjacent angle C.

Answer. Angle C = 61° 50′ 26″ or 118° 9′ 34″, ambiguous (455).

2. In the right-angled spherical triangle ABC.

Given $\left\{\begin{matrix}\text{The side BC} & =42^\circ\ 12' \\ \text{Its opposite angle A} & =48\ \ \ 0\end{matrix}\right\}$ To find the other angle C.

Answer. Angle C = 64° 35′ or 115° 25′, ambiguous (455).

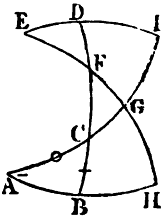
(469) CASE XII. *Given a side and its opposite angle to find the hypothenuse.*

Example. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{ll}\text{The side BC} & =36^\circ\ 31' \\ \text{Its opposite angle A} & =37\ \ 25\end{array}\right\}\begin{array}{l}\text{To find the hypothe-} \\ \text{nuse AC.}\end{array}$$

BY NAPIER'S RULE.

Here BC is the *middle part;* the complement of A, and the complement of AC, are the *extremes disjunct.*



$$\text{rad}\times\sin\text{BC}=\sin\overset{*}{\text{AC}}\times\sin\text{A}.$$

$$\therefore\ 10+\log\sin\text{BC}-\log\sin\text{A}=\log\sin\text{AC}.$$

or, sin ∠ A -	=37° 25′ -	9·78362
: radius, sine of	90 0 - -	10·00000
: : sin BC -	=36 31 - -	9·77456
: sin AC	78 20 or 101 40 -	9·99094

Here AC *is ambiguous* (455).

BY RULE I.

sin GH : sin BC : : sin AG : sin AC.

But GH is the measure of the angle A, and AG radius. Hence,

sin ∠ A : sin BC : : rad : sin AC.

BY CONSTRUCTION.

See the construction of CASE X. (467).

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{ll}\text{The side BC} & =40^\circ\ 30'\ 20'' \\ \text{Its opposite angle A} & =47\ \ 54\ \ 20\end{array}\right\}\begin{array}{l}\text{To find the hy-} \\ \text{pothenuse AC.}\end{array}$$

Answer. AC=61° 4′ 52″ or 118° 55′ 8″, ambiguous (455).

2. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{ll}\text{The side BC} & =11^\circ\ 30' \\ \text{Its opposite angle A} & =23\ \ 30\end{array}\right\}\begin{array}{l}\text{To find the hypo-} \\ \text{thenuse AC.}\end{array}$$

Answer. AC=30° or 150°, ambiguous (455).

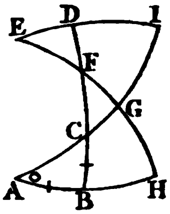
(470) CASE XIII. *Given the two sides, or legs, to find an angle.*

Example. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{l}\text{The side AB}=75^\circ\ 25' \\ \text{The side BC}=36\ \ 31\end{array}\right\}\text{To find the angle A.}$$

BY NAPIER'S RULE.

The three parts are here connected, because the right angle B is not regarded; therefore AB is the *middle part,* the complement of A, and BC are the *extremes conjunct.*



rad × sin AB = cot A × tan BC.

∴ 10 + log sin AB − log tan BC = log cot A.

tan BC - -	= 36° 31′	-	9·86947
: radius, sine of	90 0	-	10·00000
: sin AB - -	75 25	-	9·98578
: cot ∠ A - -	37 25	-	10·11631

Here A *is acute* (456).

BY RULE I.

sin AB : sin AH :: tan BC : tan GH.

But AH is radius, and GH is the measure of the angle A. Hence,

sin AB : rad :: tan BC : tan ∠ A.

BY CONSTRUCTION. (*Plate V. Fig.* 7.)

1. With the chord of 60 degrees describe the primitive circle, and through the centre P draw *a*B.

2. Set off BC = 36° 31′ from a scale of chords, and AB = 75° 25′ (377). Or set one foot of the compasses on 90 degrees on the line of semi-tangents, and extend the other towards the beginning of the scale, till the distance between the points be 75° 25′; this distance applied from B to A will give the point A.

3. From C through P draw CP*e*, and through the three points CA*e* describe a circle, then ABC is the triangle required.

To measure the required parts.

4. The angles C and A (379) will be found to be 81° 11′, and 37° 25′.

5. The hypothenuse AC (376) will be 78° 20′.

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

Given { The side AB = 50° 30′ 30″ / The side BC = 40 30 20 } To find the angle A.

Answer. A = 47° 54′ 19″, acute (456).

2. In the right-angled spherical triangle ABC.

Given { The side AB = 54° 22′ / The side BC = 142 31 } To find the angle A.

Answer. A = 136° 40′, obtuse (456).

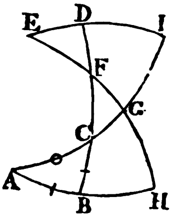
(471) CASE XIV. *Given the two sides, or legs, to find the hypothenuse.*

Example. In the right-angled spherical triangle ABC.

Given { The side AB = 75° 25′ / The side BC = 36 31 } To find the hypothenuse AC.

BY NAPIER'S RULE.

The complement of the hypothenuse AC is the *middle part*, being separated from the sides by the angles A and C; and AB and BC are the *extremes disjunct*.



*

rad × cos AC = cos AB × cos BC.

∴ log cos AB + log cos BC − 10 = log cos AC.

or, rad sine of	- 90° 0′	-	10·00000
: cos AB -	- =75 25	-	9·40103
: : cos BC -	- =36 31	-	9·90509
: cos AC -	- =78 20	-	9·30612

Here AC *is acute* (212).

BY RULE I.

sin FB : sin FC : : sin BH : sin CG.

But FB is radius, CF the complement of BC, BH the complement of AB, and CG the complement of AC. Hence,

rad : cos BC : : cos AB : cos AC.

BY CONSTRUCTION.

See the construction of CASE XIII. (470).

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{l}\text{The side AB} = 50° \; 30' \; 30'' \\ \text{The side BC} = 40 \;\; 30 \;\; 20\end{array}\right\}$ To find the hypothenuse AC.

Answer. AC = 61° 4′ 57″, acute (456).

2. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{l}\text{The side AB} = \;\; 54° \; 22' \\ \text{The side BC} = 142 \;\; 31\end{array}\right\}$ To find the hypothenuse AC.

Answer. AC = 117° 32′, obtuse (456).

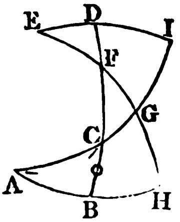
(472) CASE XV. *Given two angles, to find a side or leg.*

Example. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{l}\text{The angle A} = 37° \; 25' \\ \text{The angle C} = 81 \;\; 12\end{array}\right\}$ Required the side BC.

BY NAPIER'S RULE.

Here the complement of the angle A is the *middle part*, and BC and the complement of C are the *extremes disjunct*, being separated from the middle part by the hypothenuse AC.



*

rad × cos A = cos BC × sin C.

∴ 10 + log cos A − log sin C = log cos BC.

or, sin ∠ C -	=81° 12′	-	9·99486
: radius, sine of	90 0	-	10·00000
: : cos ∠ A -	=37 25	-	9·89995
: cos BC -	=36 31	-	9·90509

Here BC *is acute* (457).

BY RULE I.

sin ID : sin FG : : sin CD : sin CF.

But ID is the measure of the angle C, FG the complement of HG, which is the measure of the angle A, CD radius, and CF the complement of BC. Hence,

sin C : cos A : : rad : cos BC.

BY CONSTRUCTION. (*Plate V. Fig.* 12.)

1. With the chord of 60° describe the primitive circle, through the centre P draw CP*e*, and *r*P*m* at right angles to it.

2. Set one foot of your compasses on 90° in the line of semi-tangents, extend the other towards the left hand, till the degrees contained between the points of the compasses be equal to (81° 12′) the ∠ C, and apply this extent from *m* to *n*.

3. Through the three points C*ne* draw a great circle, and find its pole *p* (368).

4. With the tangent of the ∠ A 77° 25′, and centre P describe an arc; with the secant of the ∠ A, and centre *p* cross it in *o*; with *o* as a centre and radius *op*, describe the great circle AB*p*.

5. Then ABC is the triangle required, right-angled at B.

To measure the required parts.

6. The hypothenuse AC, measured by a scale of chords, =78° 20′.

7. The sides BC and AC (376) =36° 31′, and 75° 25′.

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC

Given { The angle A=47° 54′ 20″
The angle C=61 50 29 } Required the side AB.

Answer. AB=50° 30′ 30″, acute (457).

2. In the right-angled spherical triangle ABC.

Given { The angle A=104° 8′
The angle C= 31 51 } Required the side BC.

Answer. BC=117° 34′, obtuse (457).

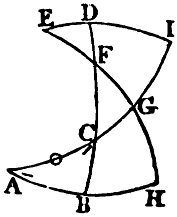
(473) CASE XVI. *Given two angles, to find the hypothenuse.*

Example. In the right-angled spherical triangle ABC.

Given { The angle A=37° 25′
The angle C=81 12 } To find the hypothenuse AC.

BY NAPIER'S RULE.

Here the complement of AC is the *middle part*, and the complements of A and C are the *extremes conjunct*



$$\text{rad} \times \cos \text{AC} = \cot \text{A} \times \cot \text{C}.$$

$$\therefore \log \cot \text{A} + \log \cot \text{C} - 10 = \log \cos \text{AC}.$$

or radius, sine of	90° 0′	-	10·00000
: cot ∠ A - -	=37 25	-	10·11633
: : cot ∠ C -	=81 12	-	9·18979
: cos AC - -	=78 19	-	9·30612

Here AC *is acute* (457).

BY RULE I.

tan ID : tan FG : : sin CI : sin CG.

But ID is the measure of the angle C, FG the complement of the angle A, CI radius, and CG the complement of AC. Hence,

tan ∠ C : cot ∠ A : : rad : cos AC.

BY CONSTRUCTION.

See the construction of CASE XV. (472).

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

$$\text{Given} \left\{ \begin{array}{l} \text{The angle A} = 47^\circ\ 54'\ 20'' \\ \text{The angle C} = 61\ \ 50\ \ 29 \end{array} \right\} \text{To find the hypothenuse AC.}$$

Answer. AC = 61° 4′ 55″, acute (457).

2. In the right-angled spherical triangle ABC.

$$\text{Given} \left\{ \begin{array}{l} \text{The angle A} = \ 31^\circ\ 51' \\ \text{The angle C} = 104\ \ \ 8 \end{array} \right\} \text{Required the hypothenuse AC.}$$

Answer. AC = 113° 55′ obtuse (457).

CHAP VII.

II. PRACTICAL RULES FOR SOLVING THE DIFFERENT CASES OF RECTILATERAL, OR QUADRANTAL, SPHERICAL TRIANGLES, WITH THEIR APPLICATION BY LOGARITHMS.

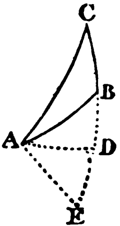
(474) 1. When one side of a spherical triangle is 90°, or a quadrant, it is called a quadrantal triangle.

2. If two sides of a quadrantal triangle be each 90°, the triangle will be isosceles, each of the angles at the base will be 90°, and the base itself will be the measure of the vertical angle. If the three sides be each 90°, the three angles will be each 90°.

3. A quadrantal spherical triangle may be changed into a right-angled spherical triangle, and the contrary (306).

RULE I.

(475) Subtract the angle opposite to the quadrantal side from 180°, and call the remainder the hypothenuse of a new triangle. The angle opposite to the quadrantal side must then be considered as a right angle, and the two remaining angles must be represented by the sides of the quadrantal triangle which are opposite to them.



The sides and angles of the right-angled spherical triangle, will represent the angles and sides of the quadrantal triangle.

Thus in the annexed figure, where AC is supposed to be a quadrant, the triangle ABC may be considered as a right-angled triangle, whose hypothenuse AC = 180° − ∠B, the base AB = ∠C; and the perpendicular BC = the ∠A; et contra.

RULE II.

(476) 1. In the triangle ABC, if one of the sides AC be a quadrant, and another side BC less than a quadrant.

Produce the side BC till it becomes a quadrant, and ADB will be a right-angled triangle; wherein AD is the measure of the angle C, DB is the complement of BC, and AB is the hypothenuse.

2. In the triangle AEC, where one of the sides AC is a quadrant, and another side CE greater than a quadrant.

Subtract 90° from CE, and the remainder will be the side DE of the right-angled triangle ADE; AE is the hypothenuse, and AD is the measure of the angle C.

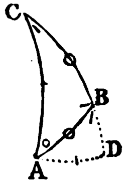
The same reasoning will apply whether AB and AE be less or greater than quadrants, for in either case ADB and ADE will be right-angled triangles.

(477) Case I. *Given a quadrantal side, its adjacent and opposite angle, to find the rest.*

Given	The quadrantal side AC =	90°	0′	Required the sides AB, BC and the ∠A.
	Its adjacent ∠C =	42	12	
	Its opposite ∠B =	115	20	

BY RULE II. (476)

Because AC and CD are quadrants, the angles CAD and CDA are each of them a right angle (474), and AD is the measure of the angle C.



The angles ABC and ABD, being supplements to each other, may be considered as the same angle, since they have the same sine, tangent, &c.

1. *To find* AB, *by Case XII. right-angled spherics.*

sin ABD : sin AD $=$ sin $\angle$ C : : rad : sin AB.

Here AB is acute $=48°$.

2. *To find* BD *and hence* BC, *by Case X. right-angled spherics.*

rad : cot ABD : : tan AD $=$ tan $\angle$ C : sin BD.

Here BD is acute $=25° \, 25'$, and its complement $=$ BC $=64° \, 35'$.

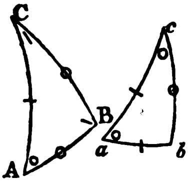
3. *To find the* $\angle$ BAD, *by Case XI. right-angled spherics.*

cos AD $=$ cos $\angle$ C : rad : : cos ABD $=$ cos $64° \, 40'$; sin BAD.

Here BAD is acute $=35° \, 17'$, and its complement $=$ BAC $=54° \, 43'$.

BY RULE I. (475)

Here the supplement of the angle B $=$ $64° \, 40'$, must represent the hypothenuse *ac*, the quadrantal side AC $=$ the right angle *b*. The angles C and A must represent the sides *ab* and *bc*, and the sides BC and AB must represent the angles *a* and *c*.



1. *To find the* $\angle$ *c, in the triangle abc, by Case V. right-angled spherics.*

sin *ac* $=$ sin suppt. $\angle$ B : rad : : sin *ab* $=$ sin $\angle$ C : sin $\angle$ *c* $=$ sin AB.

Here the $\angle$ *c* or AB is acute $=48°$.

2. *To find the* $\angle$ *a, in the triangle abc, by Case IV. right-angled spherics.*

rad : cot *ac* $=$ cot suppt. $\angle$ B : : tan *ab* $=$ tan $\angle$ C : cos $\angle$ *a* $=$ cos BC.

Here $\angle$ *a* or BC $=64° \, 35'$, acute.

3. *To find bc, in the triangle abc, by Case VI. right-angled spherics.*

cos *ab* $=$ cos $\angle$ C : rad : : cos *ac* $=$ cos suppt. $\angle$ B : cos *bc* $=$ cos $\angle$ A.

Here *bc* is acute $=54° \, 43' =$ $\angle$ A.

BY CONSTRUCTION. (*Plate V. Fig.* 13.)

1. With the chord of 60 degrees describe the primitive circle; and through the centre P draw *a*PD, and CP*e* at right angles to it.

2. Set one foot of your compasses on 90 degrees on the line of semi-tangents, extend the other towards the beginning of the scale, till the degrees between them be equal to the angle C $=42° \, 12'$, and apply this extent from D to A.

3. Through the three points CA*e* draw a great circle, then AC will represent the quadrantal side.

4. With the tangent of the complement of the angle B, and centre C, describe an arc; with the secant of the complement of the same angle, and centre A, cross it in *o*: with *o* as a centre, and radius *o*A, describe the great circle BA*b*. Then ABC is the triangle required.

To measure the required parts.

The sides AB and BC (376) will be 48° and 64° 35′, and the angle A = 54° 43′ (379).

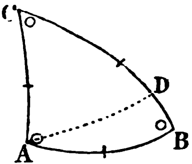
Note. The right-angled spherical triangle *abc* (Plate V. fig. 14.) into which the quadrantal triangle ABC is transformed, may be constructed and measured exactly in the same manner as Case IV. of right-angled spherical triangles was constructed and measured. See Plate V. fig. 9.

(478) Case II. *Given a quadrantal side, and the other two sides, to find the rest.*

Given { The quadrantal side AC = 90° 0′[†] } Required the
The side AB = 115 9 } angles A, B,
The side BC = 113 18 } and C.

BY RULE II. (476)

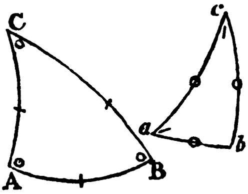
Make CD = AC, then CAD, CDA, and ADB are each of them a right angle, and AD measures the angle C (474); hence, if we take 90° from BC we get 23° 18′ = BD; and ADB is a right-angled triangle.



1. *To find the* ∠ B, *by Case IV. right-angled spherics.*
rad : cot AB : : tan BD : cos ∠ B = 78° 20′.
But ∠ B is obtuse (453), and therefore = 101° 40′.
2. *To find* AD, *by Case VI. right-angled spherics.*
cos BD : rad : : cos AB : cos AD = 62° 26′.
But AD is obtuse (453), and therefore = 117° 34′ = ∠ C.
3. *To find the* ∠ DAB, *by Case V. right-angled spherics.*
sin AB : rad : : sin BD : sin ∠ DAB = 25° 55′.
Here ∠ DAB is acute (453), and = 25° 55′.
∴ ∠ CAB = DAB + CAD = 25° 55′ + 90° = 115° 55′.

BY RULE I. (475)

Here the quadrantal side AC must represent the right angle *b*; AB the ∠ *c*; and BC the ∠ *a*; *ac* the supplement of the ∠ B, *ab* the ∠ C, and *bc* the ∠ A.



[†] This example was formed from Example 2. Case VI. of right-angled spherics. The ∠ B was made equal to the supplement of the hypothenuse, the two sides AB and BC were made equal to the angles of the right-angled triangle; and hence the legs of the right-angled triangle become angles in the quadrantal triangle. In a similar manner the other examples were made.

1. *To find the hypoth, ac, by Case XVI. right-angled spherics.*
rad : cot ∠ *a* = cot BC :: cot ∠ *c* = cot AB : cos *ac* = suppt. ∠ B = 78° 20′.

Here *ac* or suppt. ∠ B is acute (457); therefore ∠ B is obtuse = 101° 40′.

2. *To find ab, by Case XV. right-angled spherics.*
sin ∠ *a* = sin BC : rad :: cos ∠ *c* = cos AB : cos *ab* = cos ∠ C = 62° 26′.

Here *ab* = ∠ C is obtuse (457), and therefore = 117° 34′.

3. *To find bc, by Case XV. right-angled spherics.*
sin ∠ *c* = sin AB : rad :: cos ∠ *a* = cos BC : cos *bc* = cos ∠ A = 64° 5′.

Here *bc* = ∠ A is obtuse (457), and therefore = 115° 55′.

If the two preceding cases be thoroughly understood, there can be no difficulty in solving those which follow; for which reason they are given as practical exercises.

(479) CASE III. *Given a quadrantal side, and its two adjacent angles, to find the rest.*

Given
The quadrantal side AC = 90° 0′
The angle A = 54 43
The angle C = 42 12
Required AB, BC, and the angle B.

Answer. AB = 48°, BC = 64° 35′, and the angle B = 115° 20′.

(480) CASE IV. *Given a quadrantal side, one of the other sides, and the angle comprehended between them, to find the rest.*

Given
The quadrantal side AC = 90° 0′
The side AB = 115 9
The angle CAB = 115 55
Required BC, and the angles B and C.

Answer. BC = 113° 18′, the angle B = 101° 40′, C = 117° 34′.

(481) CASE V. *Given a quadrantal side, its adjacent angle, and a side opposite to that angle, to find the rest.*

Given
The quadrantal side AC = 90° 0′
Its adjacent angle C = 42 12
And the opposite side AB = 48 0
Required BC, and the angles A and B.

Answer. BC = 64° 35′, the angle A = 54° 43′, and B = 115° 20′.

But the required parts are ambiguous, and therefore are either acute or obtuse.

(482) CASE VI. *Given the quadrantal side, one of the other sides, and an angle opposite to the quadrantal side, to find the rest.*

Given
The quadrantal side AC = 90° 0′
The side AB = 115 9
The angle B = 101 40
Required BC, and the angles A and C.

Answer. BC = 113° 18′, the angle A = 115° 55′, and C = 117° 34′.

CHAP. VIII

III. PRACTICAL RULES FOR SOLVING ALL THE CASES OF OBLIQUE-ANGLED SPHERICAL TRIANGLES, WITH A PERPENDICULAR; AND THEIR APPLICATION BY LOGARITHMS.

RULE I.

(483) The rule Prop. xxv. (392) will solve ten cases; see also the rules in Scholium, art. (408).

RULE II.

(484) When the three sides are given, to find the angles.

Any one of the three sides may be called the base. Then,

The tangent of half the base,
Is to the tangent of half the sum of the sides,
As the tangent of half the difference between the sides,

Is to the tangent of the distance of a perpendicular from the middle of the base (411).

According as this distance is less or greater than half the base, the perpendicular falls within or without the triangle. When the sum of the two sides is less than 180°, the perpendicular falls nearest to the less side, when greater than 180° it falls nearest to the greater side; consequently the greater segment is joined to the greater side in the former case, and to the less side in the latter. The sum of half the base, and the fourth term found by the above proportion, gives the greater segment, their difference gives the less.

The triangle being thus divided into two right-angled triangles, the remaining parts must be found by the proper rules.

RULE III.

(485) When the three angles are given, to find the sides.

The cotangent of half the sum of the angles at the base,
Is to the tangent of half their difference,
As the tangent of half the vertical angle,

Is to the tangent of the excess of the greater of the two vertical angles (formed by a perpendicular), above half the aforesaid vertical angle (413).

If the sum of the base angles be less than 180°, the perpendicular and the less segment are nearest the greater base angle, if greater than 180° they are nearest the less base angle. The sum and difference of this fourth term, and half the vertical angle, gives the greater and less vertical angle formed by the perpendicular. The triangle being thus divided into two right-angled triangles, the remaining parts must be found by the preceding rules.

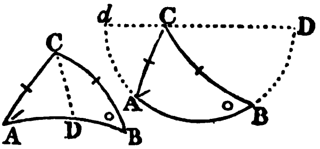
(486) CASE I. *Given two sides, and an angle opposite to one of them, to find an angle opposite to the other.*

$$\text{Given}\left\{\begin{array}{l}\text{The side AC}=80^\circ\ 19'\\ \text{The side BC}=63\ \ 50\\ \text{The}\quad\ \angle\text{A}=51\ \ 30\end{array}\right\}\text{Required the}\angle\text{B.}$$

DETERMINATION OF THE SPECIES.

A perpendicular in this case is unnecessary.

If AC+BC, A+(B *acute*), and A+(B *obtuse*), be each of the same species with respect to 180°, B is *ambiguous:* — But if only two of these sums be of the same species, that value of B must universally be taken which agrees with the sum of the sides; in all such cases B is *not* ambiguous (319).



SOLUTION.

sin BC : sin AC : : sin ∠ A : sin ∠ B = 59° 16′.

Here B is *ambiguous* = 59° 16′ or 120° 44′.

BY CASE† I. *page* 201.

Log sin B = *log* sin A + *log* sin b − *log* sin a.

Here ∠ B is *ambiguous* = 59° 16′ or 120° 44′, *vide* Table I. page 208.

EXAMPLE II.

$$\text{Given}\left\{\begin{array}{l}\text{The side AC}=\ 57^\circ\ 30'\\ \text{The side BC}=115\ \ 20\\ \text{The}\quad\ \angle\text{A}=126\ \ 37\end{array}\right\}\text{Required the}\angle\text{B.}$$

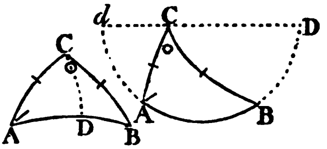
Answer. The ∠ B = 48° 30′, *not* ambiguous.

(487) CASE II. *Given two sides, and an angle opposite to one of them, to find the angle contained between these sides.*

$$\text{Given}\left\{\begin{array}{l}\text{The side AC}=80^\circ\ 19'\\ \text{The side BC}=63\ \ 50\\ \text{The}\quad\ \angle\text{A}=51\ \ 30\end{array}\right\}\text{Required the}\angle\text{C.}$$

DETERMINATION OF THE SPECIES.

1. If AC and the ∠ A be of the same species, the ∠ ACD is acute. The perpendicular CD is of the same species as the ∠ A. If BC and DC be of the same species, the ∠ BCD is acute.



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† In using the *Formulæ*, the three angles of the triangle are represented by A, B, C, and their opposite sides by a, b, c. The perpendicular is not regarded.

2. If the $\angle$ BCD be less than the $\angle$ ACD, and their sum less than 180°, then the $\angle$ ACB is *ambiguous;* but if their sum be *not* less than 180°, their difference is the *true* value of the $\angle$ ACB, *not* ambiguous.

If the $\angle$ BCD be *not* less than the $\angle$ ACD, and at the same time their sum be less than 180°, this sum is the *true* value of the $\angle$ ACB, *not ambiguous.*

SOLUTION.

1. In the triangle ADC, find the $\angle$ ACD. *

Thus $\text{rad} \times \cos \text{AC} = \cot \angle \text{A} \times \cot \angle \text{ACD}$.

Hence $\cot \angle \text{A} : \text{rad} :: \cos \text{AC} : \cot \text{ACD} = 78^\circ\ 4'$ acute.

2. In the triangle CDB, $\mathit{rad} \times \cos \overset{*}{\text{BCD}} = \mathit{tan}\ \text{DC} \times \cot \text{BC}$.

In the triangle ADC, $\mathit{rad} \times \cos \text{ACD} = \mathit{tan}\ \text{DC} \times \cot \text{AC}$.

Hence $\cot \text{AC} : \cot \text{BC} :: \cos \text{ACD} : \cos \text{BCD} = 53^\circ\ 28'$ acute.

Because the $\angle$ BCD is less than the $\angle$ ACD, and $\angle \text{BCD} + \angle \text{ACD}$ less than 180°, the $\angle$ ACB is *ambiguous; viz.* $\angle \text{ACB} = \angle \text{ACD} + \angle \text{BCD} = 131^\circ\ 32'$, or $\angle \text{ACB} = \angle \text{ACD} - \angle \text{BCD} = 24^\circ\ 36'$.

BY CASE II. *page* 201.

$\mathit{Log} \tan \varphi = \mathit{log} \cos b + \mathit{log} \tan \text{A} - 10$; hence $\varphi = 11^\circ\ 56'$ acute.

$\mathit{Log} \sin (\varphi + \text{c}) = \mathit{log} \tan b + \mathit{log} \sin \varphi - \mathit{log} \tan a = +9{\cdot}77481$,

from which we have $\varphi + \text{c} = 36^\circ\ 32'$ or $143^\circ\ 28'$.

Then $\angle \text{c} = (\varphi + \text{c}) - \varphi = 24^\circ\ 36'$ or $131^\circ\ 32'$.

This example is *ambiguous, vide* Table I. page 208.

EXAMPLE II.

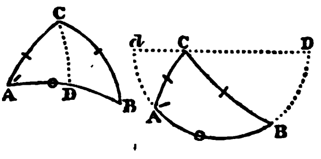
$$\text{Given} \left\{ \begin{array}{l} \text{The side AC} = 57^\circ\ 30' \\ \text{The side BC} = 115\ \ 20 \\ \text{The} \qquad \angle \text{A} = 126\ \ 37 \end{array} \right\} \text{Required the } \angle \text{c.}$$

Answer. $\angle \text{ACD} = 125^\circ\ 52'$ obtuse, $\angle \text{BCD} = 64^\circ\ 11'$ acute, and $\angle \text{ACB} = 61^\circ\ 41'$ *not* ambiguous.

(488) CASE III. *Given two sides, and an angle opposite to one of them, to find the other side.*

$$\text{Given} \left\{ \begin{array}{l} \text{The side AC} = 80^\circ\ 19' \\ \text{The side BC} = 63\ \ 50 \\ \text{The} \qquad \angle \text{A} = 51\ \ 30 \end{array} \right\} \text{Required the side AB.}$$

DETERMINATION OF THE SPECIES.



1. The segment AD is acute or obtuse, according as the $\angle$ A is of the same, or of different species with AC. DC is of the same species with the $\angle$ A.

DB is acute or obtuse, according as BC is of the same, or of different species with DC.

2. If DB be less than AD, and their sum less than 180°, then AB is *ambiguous;* but if their sum be *not* less than 180°, their difference is the *true* value of AB, *not* ambiguous.

If DB be *not* less than AD, and at the same time their sum be less than 180°, this sum is the *true* value of AB, *not* ambiguous.

SOLUTION.

1. In the triangle ADC, find the side AD. *

Thus $\text{rad} \times \cos \angle A = \cot AC \times \tan AD$.

$\therefore \cot AC : \text{rad} :: \cos \angle A : \tan AD = 74^\circ\ 40'$ acute. *

2. In the triangle CDB, $rad \times \cos BC = cos\ DC \times \cos DB$.

In the triangle ADC, $rad \times \cos AC = cos\ DC \times \cos AD$.

Hence, $\cos AC : \cos BC :: \cos AD : \cos DB = 46^\circ\ 6'$ acute.

Because DB is less than AD, and $AD + DB$ less than 180°, AB is *ambiguous;* viz. $AB = AD + DB = 120^\circ\ 46'$, or $AB = AD - DB = 28^\circ\ 34'$.

BY CASE III. *page* 202.

$Log \tan \varphi = log \cos A + log \tan b - 10$; hence $\varphi = 74^\circ\ 40'$.

$Log \cos (c - \varphi) = \log \cos a + \log \cos \varphi - \log \cos b = +9{\cdot}84091$, so that $c - \varphi = \pm 46^\circ\ 6'$.

$\therefore c = (c - \varphi) + \varphi = \pm 46^\circ 6' + 74^\circ 40' = 120^\circ 46'$ or $28^\circ 34' = AB$.

This example is *ambiguous, vide* Table I. page 208.

EXAMPLE II.

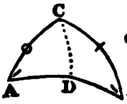
$$\text{Given}\left\{\begin{array}{l}\text{The side } AC = 57^\circ\ 30' \\ \text{The side } BC = 115\ \ 20 \\ \text{The } \angle A = 126\ \ 37\end{array}\right\}\text{Required the side } AB.$$

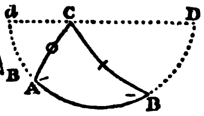
Answer. $AD = 136^\circ\ 53'$ obtuse, $DB = 54^\circ\ 27'$ acute, and $AB = AD - DB = 82^\circ\ 26'$ *not* ambiguous.

(489) CASE IV. *Given two angles, and a side opposite to one of them, to find the other opposite side.*

$$\text{Given}\left\{\begin{array}{l}\text{The } \angle A = 51^\circ\ 30' \\ \text{The } \angle B = 59\ \ 16 \\ \text{The side } BC = 63\ \ 50\end{array}\right\}\text{Required the side } AC.$$

DETERMINATION OF THE SPECIES.





A perpendicular in this case is unnecessary.

1. If BC + (AC *acute*), BC + (AC *obtuse*) and A + B, be each of the same species, with respect to 180°, AC is ambiguous; but, if only *two* of these sums be of the same kind, that value

of AC must universally be taken which agrees with the sum of the angles; in all such cases AC is *not* ambiguous.

SOLUTION.

$$\sin\angle A : \sin BC :: \sin\angle B : \sin AC = 80° \, 19'.$$

Because BC + (AC *acute*), BC + (AC *obtuse*) and A + B, are each less than 180°, AC is ambiguous, being either 80° 19′ or 99° 41′.

BY CASE IV. *page* 202.

Log sin b = *log* sin B + *log* sin a − *log* sin A.

Here b is *ambiguous* = 80° 19′ or 99° 41′, *vide* Table II. page 208.

EXAMPLE II.

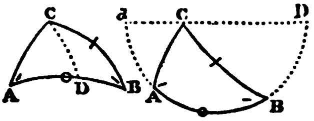
$$\text{Given}\left\{\begin{array}{ll}\text{The} & \angle A = 126° \; 37' \\ \text{The} & \angle B = \;\; 48 \;\; 30 \\ \text{The side} & BC = 115 \;\; 20\end{array}\right\}\text{Required the side AC.}$$

Answer. AC = 57° 30′ acute, and *not* ambiguous.

(490) CASE V. *Given two angles, and a side opposite to one of them, to find the side adjacent to these angles.*

$$\text{Given}\left\{\begin{array}{ll}\text{The} & \angle A = 51° \; 30' \\ \text{The} & \angle B = 59 \;\; 16 \\ \text{The side} & BC = 63 \;\; 50\end{array}\right\}\text{Required the side AB.}$$

DETERMINATION OF THE SPECIES.



1. DC is of the same species as the ∠ A.

The species of AD cannot be determined.

DB is acute, or obtuse, according as BC and DC are of the same, or of different species.

2. When the angles A and B are of the *same* species; if DB + (AD *acute*), and DB + (AD *obtuse*), be each less than 180°, AB is *ambiguous*; but if only one of these sums be less than 180°, *that* sum is the *true* value of AB, *not* ambiguous.

When the angles A and B are of *different* species; if both AD *acute*, and AD *obtuse*, be greater than DB, then AB is ambiguous; but if only one of them be greater, *that* value diminished by DB leaves AB *not* ambiguous.

SOLUTION.

1. In the triangle BDC, find the segment DB.

$$\text{Thus rad} \times \cos\angle B = \cot BC \times \tan \overset{*}{DB}.$$

$$\therefore \cot BC : \text{rad} :: \cos\angle B : \tan DB = 46° \, 8' \text{ acute.}$$

2. In the triangle ADC, $\textit{rad} \times \sin \overset{*}{AD} = \cot\angle A \times \textit{tan}\; DC$

In the triangle BDC, $rad \times \sin DB = \cot \angle B \times \tan DC$;

$\therefore \cot \angle B : \cot \angle A :: \sin DB : \sin AD = 74^\circ\ 42'$ or $105^\circ\ 18'$.

Because A and B are of the same species, and DB + (AD *acute*), DB + (AD *obtuse*), are each less than 180°, AB is *ambiguous*, viz. $AB = AD + DB = 120^\circ\ 50'$ or $151^\circ\ 26'$.

BY CASE V. *page* 202.

$Log \tan \varphi = log \cos B + log \tan a - 10$; hence $\varphi = 46^\circ\ 8'$.

$Log \sin (c-\varphi) = log \sin \varphi + log \tan B - log \tan A = +9{\cdot}98434$; hence $c - \varphi = 74^\circ\ 42'$ or $105^\circ\ 18'$.

$\therefore c = (c-\varphi) + \varphi = 120^\circ\ 50'$ or $151^\circ\ 26' = AB$.

This example is *ambiguous*, *vide* Table II. page 208.

EXAMPLE II.

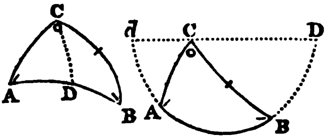
$$\text{Given}\left\{\begin{array}{ll}\text{The} & \angle A = 126^\circ\ 37' \\ \text{The} & \angle B = \ \ 48\ \ \ 30 \\ \text{The side} & BC = 115\ \ \ 20\end{array}\right\}\text{Required the side } AB.$$

Answer. $DB = 54^\circ\ 27'$ acute, $AD = 43^\circ\ 7'$ or $136^\circ\ 53'$, and $AB = 82^\circ\ 26'$ *not* ambiguous.

(491) CASE VI. *Given two angles, and a side opposite to one of them, to find the third angle.*

$$\text{Given}\left\{\begin{array}{ll}\text{The} & \angle A = 51^\circ\ 30' \\ \text{The} & \angle B = 59\ \ \ 16 \\ \text{The side} & BC = 63\ \ \ 50\end{array}\right\}\text{Required the angle } C.$$

DETERMINATION OF THE SPECIES.



1. DC is of the same species as the ∠ A.

The species of the ∠ ACD cannot be determined.

The ∠ BCD is acute, or obtuse, according as the side BC is of the same, or of different species with DC.

2. When the angles A and B are of the *same* species; if BCD + (ACD *acute*), and BCD + (ACD *obtuse*), be each less than 180°, ACB is *ambiguous;* but if only one of these sums be less than 180°, that sum is the *true* value of ACB, *not* ambiguous.

When the angles A and B are of *different* species; if both ACD *acute*, and ACD *obtuse*, be greater than BCD, then ACB is *ambiguous;* but if only one of them be greater, *that* value diminished by BCD leaves ACB *not* ambiguous.

SOLUTION.

1. In the triangle BDC, find the ∠ BCD. *

Thus $rad \times \cos BC = \cot \angle B \times \cot BCD$;

$\therefore \cot \angle B : rad :: \cos BC : \cot BCD = 53^\circ\ 26'$ acute.

2. In the triangle ADC, $rad \times \cos \angle A = \cos DC \times \sin \overset{*}{ACD}$.

In the triangle BDC, $rad \times \cos \angle B = \cos DC \times \sin BCD$.

Hence $\cos \angle B : \cos \angle A :: \sin BCD : \sin ACD = 78° 3' \text{ or } 101° 57'$.

Because A and B are of the same species, and BCD + (ACD *acute*), BCD + (ACD *obtuse*), are each less than 180°; ACB is *ambiguous* $= 131° 29'$ or $155° 23'$.

BY CASE VI. *page* 203.

$Log \cot \varphi = log \cos a + log \tan B - 10$; hence $\varphi = 53° 26'$.

$Log \sin (C - \varphi) = log \cos A + log \sin \varphi - log \cos B = +9{\cdot}99049$; hence $C - \varphi = 78° 3'$ or $101° 57'$.

$\therefore \angle C = (C - \varphi) + \varphi = 131° 29'$ or $101° 57'$.

This example is *ambiguous*, *vide* Table II. page 208.

EXAMPLE II.

$$\text{Given}\left\{\begin{array}{lrr}\text{The} & \angle A = 126° & 37' \\ \text{The} & \angle B = 48 & 30 \\ \text{The side} & BC = 115 & 20\end{array}\right\}\text{Required the } \angle C.$$

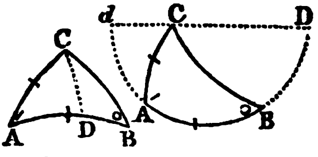
Answer. The $\angle BCD = 64° 11'$ acute, the $\angle ACD = 54° 8'$ or $125° 52'$ ambiguous, and the $\angle ACB = 61° 41'$ *not* ambiguous.

(492) CASE VII. *Given two sides, and an angle contained between them, to find an opposite angle.*

$$\text{Given}\left\{\begin{array}{lrr}\text{The side} & AC = 80° & 19' \\ \text{The side} & AB = 120 & 47 \\ \text{The} & \angle A = 51 & 30\end{array}\right\}\text{Required the } \angle B.$$

DETERMINATION OF THE SPECIES.

1. AD is acute, or obtuse, according as AC is of the same, or of different species with the ∠A.



2. The ∠B is of the *same*, or of *different* species with the ∠A, according as AD is *less*, or *greater* than AB.

SOLUTION.

1. In the triangle ADC, find AD.

Thus $rad \times \cos \angle A = \cot AC \times \tan \overset{*}{AD}$.

$\therefore \cot AC : rad :: \cos \angle A : \tan AD = 74° 40'$ acute.

Then $AB - AD = 46° 7' = DB$.

2. In the triangle BDC, $rad \times \sin DB = \tan DC \times \cot \angle \overset{*}{B}$.

In the triangle ADC, $rad \times \sin AD = \tan DC \times \cot \angle A$.

Hence, $\sin AD : \sin DB :: \cot \angle A : \cot \angle B = 59° 16'$ acute.

Because AD is less than AB, the ∠B is of the *same* species with the ∠A, and consequently it is acute.

BY CASE VII. *page* 204.

Log tan φ=*log cos* A+*log* tan $b-10$; hence $\varphi=74°\ 40'$.

Log cot B=log cot A+log sin $(c \sim \varphi)-\sin\varphi=+9{\cdot}77414$; hence ∠B=59° 16′.

EXAMPLE II.

$$\text{Given}\left\{\begin{array}{l}\text{The side AC} = \ \ 57°\ 30'\\ \text{The side AB} = \ \ 82\ \ 27\\ \text{The}\qquad \angle\text{A} = 126\ \ 37\end{array}\right\}\text{Required the } \angle\text{B}.$$

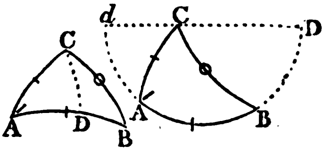
Answer. AD=136° 53′ obtuse, DB=54° 26′, and the ∠B= 48° 31′ acute.

(493) CASE VIII. *Given two sides, and the angle contained between them, to find the third side.*

$$\text{Given}\left\{\begin{array}{l}\text{The side AC} = \ \ 80°\ 19'\\ \text{The side AB} = 120\ \ 47\\ \text{The}\qquad \angle\text{A} = \ \ 51\ \ 30\end{array}\right\}\text{Required the side BC}.$$

DETERMINATION OF THE SPECIES.

1. AD is acute or obtuse, according as AC is of the *same*, or of *different* species with the ∠A.



2. CD is of the same species as the ∠A; therefore, according as the ∠A and DA are of the *same*, or of *different* species, BC is acute or obtuse.

SOLUTION.

1. In the triangle ADC, find AD. *

Thus rad × cos ∠A=cot AC × tan AD.

∴ cot AC : rad : : cos ∠A : tan AD=74° 40′ acute.

Then AB−AD=46° 7′=DB acute.

2. In the triangle BDC, *rad* × cos BC=*cos* DC × cos DB.

In the triangle ADC, *rad* × cos AC=*cos* DC × cos AD.

Hence cos AD : cos DB : : cos AC : cos BC=63° 50′ acute.

Because DB and the ∠A are both acute, BC is acute.

BY CASE VIII. *page* 204.

Log tan φ=*log* cos A+*log* tan $c-10$; hence $\varphi=133°\ 44'$.

Log cos a=*log* cos c+*log* cos $(b-\varphi)-\cos\varphi=+9{\cdot}64466$, and the corresponding are is 63° 51′=a or BC.

EXAMPLE II.

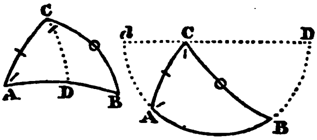
$$\text{Given}\left\{\begin{array}{l}\text{The side AC} = \ \ 57°\ 30'\\ \text{The side AB} = \ \ 82\ \ 27\\ \text{The}\qquad \angle\text{A} = 126\ \ 37\end{array}\right\}\text{Required the side BC}.$$

Answer. AD=136° 53′ obtuse, AD−AB=DB=54° 26′ acute, and BC=115° 21′ obtuse.

(494) CASE IX. *Given two angles, and the side adjacent to both of them, to find a side opposite to one of the given angles.*

$$\text{Given}\left\{\begin{array}{l}\text{The } \angle \text{A} = 51° \ 30'\\ \text{The } \angle \text{ACB} = 131 \ \ 30\\ \text{The side AC} = 80 \ \ 19\end{array}\right\}\text{Required the side BC.}$$

DETERMINATION OF THE SPECIES.



1. The ∠ ACD is acute or obtuse, according as the ∠ A is of the *same*, or of *different* species with the side AC.

2. The ∠ A is of the same species as CD; therefore according as the ∠ A and the ∠ BCD are of the *same*, or of *different* species, BC is acute or obtuse.

SOLUTION.

1. In the triangle ADC, find the ∠ ACD. *
   Thus rad × cos AC = cot ∠ A × cot ∠ ACD.
   ∴ cot ∠ A : rad : : cos AC : cot ACD = 78° 4′ acute.
   and ∠ BCD = ∠ ACB − ∠ ACD = 53° 26′ acute. *
2. In the triangle BDC, *rad* × cos BCD = *tan* DC × cot BC.
   In the triangle ADC, *rad* × cos ACD = *tan* DC × cot AC.

Hence cos ∠ ACD : cos ∠ BCD : : cot AC : cot BC = 63° 50′ acute.
Because the ∠ A is of the same species as the ∠ BCD, the side BC is acute.

BY CASE IX. *page* 205.

Log cot ϕ = *log* tan A + *log* cos $b - 10$; hence $\phi = 78° \ 4'$.
Log cot a = *log* cot b + *log* cos $(c - \phi)$ − *log* cos $\phi = +9{\cdot}69164$, and the corresponding arc is $63° \ 49' = a$ or BC.

EXAMPLE II.

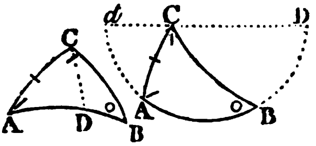
$$\text{Given}\left\{\begin{array}{l}\text{The } \angle \text{A} = 126° \ 37'\\ \text{The } \angle \text{ACB} = 61 \ \ 41\\ \text{The side AC} = 57 \ \ 30\end{array}\right\}\text{Required the side BC.}$$

Answer. ∠ ACD = 125° 52′ obtuse, ∠ BCD = ∠ ACD − ∠ ACB = 64° 11′ acute, and the side BC = 115° 20′ obtuse.

(495) CASE X. *Given two angles, and the sides adjacent to both of them, to find the other angle.*

$$\text{Given}\left\{\begin{array}{l}\text{The } \angle \text{A} = 51° \ 30'\\ \text{The } \angle \text{ACB} = 131 \ \ 30\\ \text{The side AC} = 80 \ \ 19\end{array}\right\}\text{Required the angle B.}$$

DETERMINATION OF THE SPECIES.



1. The $\angle$ ACD is acute or obtuse, according as the $\angle$ A is of the *same*, or of *different* species with the side AC.

2. The angles A and B are of the *same*, or of *different* species, according as the $\angle$ ACD is *less* or *greater* than the $\angle$ ACB.

SOLUTION.

1. In the triangle ADC, find the $\angle$ ACD. *
   Thus $\text{rad} \times \cos \text{AC} = \cot \angle \text{A} \times \cot \angle \text{ACD}$.
   $\therefore \cot \angle \text{A} : \text{rad} :: \cos \text{AC} : \cot \angle \text{ACD} = 78° \, 4'$ acute.
   and $\angle \text{BCD} = \angle \text{ACB} - \angle \text{ACD} = 53° \, 26'$ acute.

2. In the triangle BDC, $\textit{rad} \times \cos \overset{*}{\text{B}} = \textit{cos}\ \text{DC} \times \sin \text{BCD}$.
   In the triangle ADC, $\textit{rad} \times \cos \text{A} = \textit{cos}\ \text{DC} \times \sin \text{ACD}$.

Hence $\sin \angle \text{ACD} : \sin \angle \text{BCD} :: \cos \angle \text{A} : \cos \angle \text{B} = 59° \, 16'$ acute.

Because the $\angle$ ACD is less than the $\angle$ ACB, the $\angle$ B is of the same species as the $\angle$ A, viz. acute.

BY CASE X. *page* 205.

$\textit{Log} \cot \varphi = \textit{log} \tan \text{A} + \textit{log} \cos b - 10$; hence $\varphi = 78° \, 4'$.
$\textit{Log} \cos \text{B} = \textit{log} \cos \text{A} + \log \sin (\text{C} - \varphi) - \textit{log} \sin \varphi = +9{\cdot}70844$, and the corresponding arc is $59° \, 16' = \angle \text{B}$.

EXAMPLE II.

$$\text{Given} \left\{ \begin{array}{lrr} \text{The} \quad \angle \text{A} = & 126° & 37' \\ \text{The} \quad \angle \text{ACB} = & 61 & 41 \\ \text{The side AC} = & 57 & 30 \end{array} \right\} \text{Required the} \angle \text{B.}$$

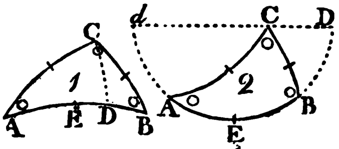
Answer. The $\angle \text{ACD} = 125° \, 52'$ obtuse, $\angle \text{BCD} = \angle \text{ACD} - \angle \text{ACB} = 64° \, 11'$ acute, and $\angle \text{B} = 48° \, 30'$ acute.

(496) CASE XI. *Given the three sides, to find the angles.*

$$\text{Given} \left\{ \begin{array}{lrr} \text{The side AC} = & 80° & 19' \\ \text{The side BC} = & 63 & 50 \\ \text{The side AB} = & 120 & 47 \end{array} \right\} \begin{array}{c} \text{Required the angles} \\ \text{A, B, and C.} \end{array}$$

The sum of the sides AC and BC being less than 180°, the perpendicular falls nearest to the less side BC; therefore if it falls without the triangle, the rule determines ED in fig. 2.

$$\begin{array}{llrrrcl} \tan \frac{1}{2}\,\text{AB} & = \tan & 60° & 23' & 30'' & = \ - & 10{\cdot}24544 \\ : \tan \frac{1}{2}\,(\text{AC}+\text{BC}) & = \tan & 72 & 4 & 30 & = \ - & 10{\cdot}49016 \\ :: \tan \frac{1}{2}\,(\text{AC}-\text{BC}) & = \tan & 8 & 14 & 30 & = \ - & 9{\cdot}16090 \\ : \tan \text{ED} & = \tan & 14 & 16 & 35 & = \ - & 9{\cdot}40562 \end{array}$$



Here ED is less than half the base, therefore the perpendicular falls *within* the triangle, and consequently the angles A and B are acute.

The segment AD = ½AB + ED = 74° 40′, and DB = 46° 7′.

In the triangle ADC, the segment AD and the hypothenuse AC are given to find the ∠ A = 51° 31′, and the ∠ ACD = 78° 3′.

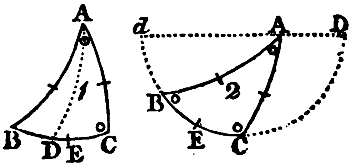
In the triangle BDC, the segment BD and the hypothenuse BC are given to find the ∠ B = 59° 17′, and the ∠ BCD = 53° 25′.

Lastly, ∠ ACB = ∠ ACD + ∠ BCD = 131° 28′.

OR THUS, *Let* BC *be considered as the base.*

Then the sum of the sides AB and AC being greater than 180°, the perpendicular falls nearest to the greater side AB; therefore, if it falls without the triangle, the rule determines E*d* in fig. 2.

tan ½ BC = tan 31° 55′ = - 9·79438
: tan ½ (AB + AC) = tan 100 33 = - 10·72992
:: tan ½ (AB − AC) = tan 20 14 = - 9·56654
: tan ED or E*d* = tan 72 32 = - 10·50208



Here the fourth proportional is greater than half the base, therefore the perpendicular falls *without* the triangle.

The segment *d*B = E*d* − ½ BC = 40° 37′; the segment BD = ½ BC + supplement of E*d* = 139° 23′.

In the triangle A*d*B, or ADB, the ∠ B may be found = 59° 17′. And in the triangle ACD, or AC*d*, the ∠ C may be found = 131° 28′.

EXAMPLE II.

Given { The side AC = 57° 30′ / The side BC = 115 20 / The side AB = 82 28 } Required the angles A and B.

Answer. ∠ A = 126° 36′, and ∠ B = 48° 32′.

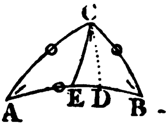
(497) CASE XII. *Given the three angles to find the sides.*

Given { The ∠ A = 51° 30′ / The ∠ B = 59 16 / The ∠ C = 131 30 } Required the sides AC, BC, and AB.

Let the angles A and B, which are of the same species, be considered as base angles, then the perpendicular will fall within the triangle.

And since the sum of the two angles A and B is less than 180°, the sum of the sides AC and BC is less than 180°, therefore the perpendicular falls nearest to B, the greater angle.

$\cot \frac{1}{2}(A+B) = \cot 55° 23' = \quad - \quad 9{\cdot}83903$
$: \tan \frac{1}{2}(B-A) = \tan 3\ 53 = \quad - \quad 8{\cdot}83175$
$:: \tan \frac{1}{2} \angle ACB = \tan 65\ 45 = \quad - \quad 10{\cdot}34634$
$: \tan \angle ECD = \tan 12\ 19 = \quad - \quad 9{\cdot}33906$
Then $\angle ACD = \frac{1}{2} \angle ACB + \angle ECD = 78° 4'$,
and $\angle BCD = \frac{1}{2} \angle ACB - \angle ECD = 53° 26'$.



In the triangle ADC, the ∠ A and the ∠ ACD are given, to find AC = 80° 19′, and AD = 74° 41′.

In the triangle BDC, the ∠ B and the ∠ BCD are given, to find BC = 63° 50′, and DB = 46° 7′.

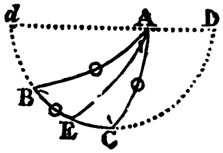
Lastly, AB = AD + DB = 120° 48′.

OR THUS, *Let* BC *be considered as the base.*

The base angles being here of different species, the perpendicular falls without the triangle.

And because the sum of the two angles B and C is greater than 180°, the perpendicular falls nearest to the less angle B, or greatest side AB, consequently the rule determines the ∠ EA*d*.

$\cot \frac{1}{2}(B+C) = \cot 95° 23' = 8{\cdot}97421$
$: \tan \frac{1}{2}(C-B) = \tan 36\ 7 = 9{\cdot}86312$
$:: \tan \frac{1}{2} \angle BAC = \tan 25\ 45 = 9{\cdot}68336$
$: \tan \angle EAd = \tan 75\ 1 = 10{\cdot}57227$



Then $\angle CAd = \frac{1}{2} \angle BAC + \angle EAd = 100° 46'$; and hence the ∠ CAD, being the supplement, = 79° 14′ and ∠ BAD = 130° 44′.

In the triangle A*d*C, or ADC, the side AC will be found = 80° 19′, and the side *d*C or DC may likewise be found.

In the triangle A*d*B, or ADB, the side AB will be found = 120° 48′, and the side *d*B, or DB, may likewise be found.

Lastly, 180° − (DC + *d*B), or *d*C − *d*B, or DB − DC = BC = 63° 50′.

EXAMPLE II.

Given { The ∠ A = 48° 31′
The ∠ B = 62 52
The ∠ C = 125 20 } Required the sides AC, BC, and AB.

Answer. AC = 83° 9′, BC = 56° 42′, and AB = 114° 29′.

CHAP. IX.

IV. PRACTICAL RULES FOR SOLVING ALL THE DIFFERENT CASES OF OBLIQUE-ANGLED SPHERICAL TRIANGLES, WITHOUT A PERPENDICULAR, WITH THEIR APPLICATION BY LOGARITHMS.

CASE I.

(498) When two sides and an angle opposite to one of them are given, to find the rest.

RULE.

1. *To find the other opposite angle.*

Sine of the side opposite to the given angle,
Is to sine of the given angle,
As sine of the other given side,
Is to sine of its opposite angle.

To the angle found by this proportion, and its supplement, add the given angle. Then, if each of these sums be of the same species with respect to 180°, as the sum of the given sides, the problem is ambiguous; that is, the angle thus found may be either acute or obtuse.

But, if only *one* of these sums be of the same species with the sum of the sides, that value of the angle, found by this proportion, must be taken, whether it be acute or obtuse, which, when added to the given angle, agrees with the sum of the sides. In this case the problem is not ambiguous.

2. *To find the angle contained between the given sides.*

Find the angle opposite to the other given side, by the first part of the rule, and note whether it be acute, or obtuse, or ambiguous.

Then sine of half the difference between the two given sides,
Is to sine of half their sum,
As tangent of half the difference between their opposite angles,
Is to cotangent of half the angle * contained between the given sides (428).

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* Since a side, or an angle, of any spherical triangle is always less than 180°, the *half* of any side or angle must *always* be acute. The ambiguity therefore ascribed to Case I. and II. arises from the first proportion in each case; if the angle, or side, found by these proportions be ambiguous, the remaining parts of the triangle will necessarily be ambiguous; but if the angle, or side, found by

3. *To find the third side.*

Find the angle opposite to the other given side, by the first part of the rule, and note whether it be acute, obtuse, or ambiguous.

Then, sine of half the difference between the two angles,
Is to sine of half their sum,
As tangent of half the difference between the two given sides,
Is to tangent of half the required side (429).

CASE II.

(499) When two angles of an oblique-angled spherical triangle, and a side opposite to one of them are given, to find the rest.

RULE.

1. *To find the other opposite side.*

Sine of the angle opposite to the given side,
Is to sine of the given side,
As the sine of the other given angle,
Is to the sine of its opposite side.

To the side found by this proportion, and its supplement, add the *given* side. Then if each of these sums be of the same species with respect to 180°, as the sum of the given angles, the problem is ambiguous; that is, the side thus found may be either acute or obtuse.

But, if only *one* of these sums be of the same species as the sum of the given angles, that value of the side, found by this proportion, must be taken, which when added to the given side agrees with the sum of the angles. In this case the problem is not ambiguous.

2. *To find the side adjacent to the two given angles.*

Find the side opposite to the other given angle, by the first part of the rule, and note whether it be acute, obtuse, or ambiguous.

Then, sine of half the difference between the two given angles,
Is to sine of half their sum,

---

these proportions be determinate, the remaining parts of the triangle will also be determinate.

The ambiguous parts derived from the first proportion, in Case I. or II. are always supplements of each other; but the remaining parts of the triangle, when ambiguous, are not supplements of each other, as is obvious both from the constructions and calculations following.

As tangent of half the difference between the two sides,
Is to tangent of half their side (429).

3. *To find the third angle.*

Find the side opposite to the other given angle, by the first part of the rule, and note whether it be acute, obtuse, or ambiguous.

Then, sine of half the difference between the two sides containing the required angle,
Is to sine of half their sum,
As tangent of half the difference between the other two angles,
Is to cotangent of half the required angle (428).

CASE III.

(500) When two sides and the included angle, of an oblique-angled spherical triangle, are given, to find the rest.

RULE.

1. *To find the other two angles.*

Cosine of half the sum of the two given sides,
Is to cosine of half their difference,
As cotangent of half the included angle,
Is to tangent of half the sum of the other two angles,

Half the sum of these two angles must be of the same species as half the sum of the given sides.

Secondly.

Sine of half the sum of the two given sides,
Is to sine of half their difference,
As cotangent of half the included angle,
Is to tangent of half the difference between the other two angles (428).

Half the difference between these angles is always acute.

Lastly.

Half the sum of the two angles increased by half their difference, gives the angle opposite to the greater side, and diminished by the same, leaves the angle opposite to the less side (113).

2. *To find the third side.*

Find the two required angles by the first part of the rule.

Then, sine of half the difference between these angles,
Is to sine of half their sum,

As tangent of half the difference between the given sides,
Is to tangent of half the third side (429).

OR, *without finding the other two angles.*

To the sum of the logarithmic sines of the given sides, add double the logarithmic sine of half the contained angle, and reject 40 from the index. Take the difference between twice the natural number corresponding to this logarithm, and the natural cosine of the difference between the given sides; the remainder will be the natural cosine of the side required, which will be acute or obtuse, according as the double natural number is *less* or *greater* than the natural cosine of the difference between the given sides (438).

CASE IV

(501) When two angles of an oblique-angled spherical triangle, and the side adjacent to both of them, are given, to find the rest.

RULE.

1. *To find the other two sides.*

Cosine of half the sum of the two given angles,
Is to cosine of half their difference,
As tangent of half the adjacent side,
Is to tangent of half the sum of the other two sides.

Half the sum of these sides, must be of the same species as half the sum of the given angles.

Secondly.

Sine of half the sum of the two given angles,
Is to sine of half their difference,
As tangent of half the adjacent side,
Is to tangent of half the difference between the other two sides (429).

Half the difference between these sides is always acute.

Lastly.

Half the sum of the two sides increased by half their difference, gives the side opposite to the greater angle, and diminished by the same, leaves the side opposite to the less (113).

2. *To find the third angle.*

Find the two required sides by the first part of the rule.

Then, sine of half the difference between these sides,
Is to sine of half their sum,
As tangent of half the difference between the given angles,
Is to cotangent of half the third angle (428).

OR, *without finding the other two sides.*

To the sum of the logarithmic sines of the given angles, add double the logarithmic cosine of half the given side, and reject 40 from the index. Take the difference between twice the natural number corresponding to this logarithm, and the natural cosine of the difference between the given angles; the remainder will be the natural cosine of the angle required. This angle is acute or obtuse, according as the double natural number is *greater* or *less*, than the cosine of the difference between the given angles (439).

CASE V.

(502) When the three sides of an oblique-angled spherical triangle are given, to find the angles.

RULE I.

From half the sum of the three sides subtract the side opposite to the required angle, and note the half sum and remainder. Then add together,

The logarithmic cosecants of each of the sides containing the required angle, rejecting the indices, and the sines of the above half sum and remainder; half the sum of these four logarithms is the logarithmic cosine of half the angle sought (423).

RULE II.

Add all the three sides together, from the half sum subtract each side containing the required angle, and note the remainders. Then add together,

The logarithmic cosecants of each of the sides containing the required angle, rejecting the indices, and the sines of the above-noted remainders; half the sum of these four logarithms is the logarithmic sine of half the angle sought (422).

RULE III.

From half the sum of the three sides subtract each side separately. Then add together,

The logarithmic cosecants of half the sum of the sides, and of the difference between that half sum and the side opposite to the angle required, rejecting the indices, and the logarithmic sines of the difference between the half sum and each side containing the required angle; half the sum of these four logarithms is the logarithmic tangent of half the angle sought (424).

CASE VI.

(503) When the three angles of an oblique-angled spherical triangle are given, to find the sides.

RULE I.

Add all the three angles together, take the difference between the half sum and the angle opposite to the side sought, and note the half sum and remainder. Then add together,

The logarithmic cosecants of each of the angles adjacent to the required side, rejecting the indices, and the cosines of the above half sum and remainder; half the sum of these four logarithms is the logarithmic sine of half the side sought (425).

RULE II.

Take the supplements of each of the angles, and use the remainders as sides in a new triangle.

Find the angles of this triangle, by any of the rules in Case V., the supplements of which will be the sides sought (304).

(504) CASE I. *Given two sides of an oblique spherical triangle, and an angle opposite to one of them, to find the rest.*

In the oblique spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{l}\text{The side AC}=80^\circ\ 19'\\ \text{The side BC}=63\ \ 50\\ \text{The}\quad\ \angle A=51\ \ 30\end{array}\right\}\ \text{Required the rest.}$$

BY CONSTRUCTION. (*Plate V. Fig.* 15.)

1. With the chord of 60 degrees describe the primitive circle; through the centre P draw CP*e*, and *a*P*r* at right angles to it.

2. Set off the side AC$=80^\circ\ 19'$ from C to A, by the scale of chords.

3. Through A draw the great circle A*b*B*n*, making an angle of $51^\circ\ 30'$ with the primitive (369).

4. Set off the side BC$=63^\circ\ 50'$ by a scale of chords, from C to *m*, and draw the parallel circle *mb*B*m* (374). Through the points *b*, B, where it cuts the oblique circle A*b*B*n*, and the point C, draw the great circles C*be*, CB*e*.

5. Then A*b*C or ABC is the triangle required, each having the same *data*, which shows this example to be *ambiguous*.

To measure the required parts.

6. The side A*b* (376)=28° 33′, and AB=120° 48′.
7. The ∠ AC*b* (379)=24° 37′, and ∠ ACB=131° 29′; the ∠ A*b*C=120° 44′, and the ∠ ABC=59° 16′.

BY CALCULATION.

I. *To find the* ∠ B.

sin BC	=63°	50′	- -	9·95304
: sin ∠ A	=51	30	- -	9·89354
: : sin AC	=80	19	- -	9·99377
: sin ∠ B	=59	16	- -	9·93427

Because AC + BC, A + (B *acute*), and A + (B *obtuse*) are each of the same species with respect to 180°, the ∠ B is *ambiguous* (486), being=59° 16′ or its supplement 120° 44′.

II. *To find the* ∠ C.

sin ½ (AC ∼ BC)	= 8°	14′	30″	9·15639
: sin ½ (AC + BC)	=72	4	30	9·97839
: : tan ½ (A ∼ B)	= 3	53	0	8·83175
: cot ½ ∠ C	=65	44	44	9·65375
			2	
	131	29	28= ∠ C	

Had the obtuse value of the ∠ B been used, viz. 120° 44′, the ∠ C by the same method would have been 24° 37′ 28″.

III. *To find the side* AB.

sin ½ (A ∼ B)	= 3°	53′	0″	8·83075
: sin ½ (A + B)	=55	23	0	9·91538
: : tan ½(AC ∼ BC)	= 8	14	30	9·16090
: tan ½ AB	=60	23	48	10·24553
			2	
	120	47	36=AB.	

If the obtuse value of the ∠ B had been used, the side AB by the same process would have been 28° 32′ 40″.

EXAMPLE II.

In the oblique spherical triangle ABC.

Given { The side AC= 57° 30′
The side BC=115 20
The ∠ A =126 37 } Required the rest.

BY CONSTRUCTION. (*Plate V. Fig.* 16.)

1. With the chord of 60° describe the primitive circle, through the centre P draw CP*e*, and *a*P*r* at right angles to it.

2. Set off the side AC=57° 30′ from C to A, by the scale of chords.

3. Through A draw the great circle AB*n*, making an angle of 126° 37′ with the primitive (369).

4. Set off the side BC=115° 20′ by a scale of chords, from C to *m*, and draw the parallel circle *m*B*m* (374). Through the point B, where it cuts the oblique circle AB*n*, and the point C, draw the great circle CB*e*.

5. Then ABC is the triangle required; and though it has similar *data* to the former example, *none of the parts are ambiguous.*

To measure the required parts.

6. The side AB (376) =82° 26′.
7. The ∠ B (379)=48° 30′, and the ∠ C=61° 41.

BY CALCULATION.

I. *To find the* ∠ B.

sin BC = 115° 20′ - - 9·95609
: sin ∠ A = 126 37 - - 9·90452
: : sin AC = 57 30 - - 9·92603
: sin ∠ B = 48 30 - - 9·87446

Because AC + BC, and A + (B *acute*) only, are of the same species with respect to 180°, the ∠ B is acute, and *not* ambiguous (486).

II. *To find the* ∠ C.

sin ½ (AC ∼ BC) = 28° 55′ 0″ 9·68443
: sin ½ (AC + BC) = 86 25 0 9·99915
: : tan ½ (A ∼ B) = 39 3 30 9·90927
: cot ½ ∠ C = 30 50 20 10·22399
2

61 40 40 = ∠ C.

III. *To find the side* AB.

sin ½ (A ∼ B) = 39° 3′ 30″ 9·79942
: sin ½ (A + B) = 87 33 30 9·99960
: : tan ½ (AC ∼ BC) = 28 55 0 9·74226
: tan ½ AB = 41 12 51 9·94244
2

82 25 42 = AB.

PRACTICAL EXAMPLES.

1. In the oblique spherical triangle ABC.

Given { The side AC = 40° 0′ 0″ / The side BC = 70 0 0 / The ∠ A = 130 3 11 } *Answer.* { ∠ B = 31° 34′ 27″ / ∠ C = 30 28 12 / AB = 38 30 0 }

Required the other parts.

2. In the oblique spherical triangle ABC.

Given { The side AC = 119° 5′ / The side BC = 79 13 / The ∠ A = 58 8 } *Answer.* { ∠ B = 130° 56′ / ∠ C = 50 12 / AB = 62 42 }

Required the other parts.

3. In the oblique spherical triangle ABC.

Given { The side AC = 30° 0′ / The side BC = 24 4 / The ∠ A = 36 8 } *Ans.* { ∠ B = 46° 18′ or 133° 42′ / ∠ C = 104 0 or 11 23 / AB = 42 9 or 7 51 }

Required the other parts.

(505) CASE II. *Given two angles of an oblique spherical triangle, and a side opposite to one of them, to find the rest.*

In the oblique spherical triangle ABC.

Given { The ∠ A = 51° 30′ / The ∠ B = 59 16 / The side BC = 63 50 } Required the rest.

BY CONSTRUCTION. (*Plate V. Fig.* 17.)

1. With the chord of 60 degrees describe the primitive circle, through the centre P draw BPe, and DPE at right angles to it.

2. Set one foot of your compasses on 90 degrees, on the line of semi-tangents, extend the other towards the beginning of the scale, till the degrees between them be equal to the angle

B = 59° 16′, and apply this extent from E to *n* (369); and through the three points B*ne* draw a great circle.

3. Set off the side BC = 63° 50′, taken from a scale of chords, from B to *m*, and draw the parallel circle *mcm* (374), cutting the oblique circle, B*ne* in C.

4. With the tangent of the angle A = 51° 30′, and P as a centre, describe an arc; and with the secant of the same angle, and C as a centre, cross it in *o*.

5. With the centre *o*, and radius *o*C, draw the great circle *a*CA. Then ABC is the triangle required.

To measure the required parts.

6. AB measured by a scale of chords will be 120° 47′, or 151° 27′ *ambiguous.*

7. AC will be 80° 19′, or 99° 41′ (376) *ambiguous.*

8. The ∠ C = 131° 29′, or 155° 22′ (379) *ambiguous.*

BY CALCULATION.

I. *To find the side* AC.

sin ∠ A	= 51° 30′	-	9·89354
: sin BC	= 63 50	- -	9·95504
: : sin ∠ B	= 59 16	-	9·93427
: sin AC	= 80 19	- -	9·99377

Because BC + (AC *acute*), BC + (AC *obtuse*), and A + B, are each of the same species, with respect to 180°, AC is *ambiguous* (489), being either 80° 19′ or 99° 41′.

II. *To find the side* AB.

sin ½ (A ∼ B)	= 3° 53′	0″		8·83075
: sin ½ (A + B)	= 55	23	0	9·91538
: : tan ½ (AC ∼ BC)	= 8	14	30	9·16090
: tan ½ AB	= 60	23	48	10·24553
			2	
	120	47	36 = AB.	

If the obtuse value of AC (= 99° 41′) had been used, the side AB, by the same process, would have been = 151° 27′ 8″.

III. *To find the* ∠ C.

sin ½ (AC ∼ BC)	= 8°	14′	30″	9·15639
: sin ½ (AC + BC)	= 72	4	30	9·97839
: : tan ½ (A ∼ B)	= 3	53	0	8·83175
: cot ½ ∠ C	= 65	44	44	9·65375
			2	
	131	29	28 = ∠ C.	

If the obtuse value of the side AC had been used, the ∠ C, by the same method, would have been = 155° 22′ 22″.

EXAMPLE II.

In the oblique spherical triangle ABC.

Given { The ∠ A = 126° 37′ } Required the rest.
The ∠ B = 48 30
The side BC = 115 20

BY CONSTRUCTION. (*Plate V. Fig.* 18.)

1. With the chord of 60 degrees describe the primitive circle, through the centre P draw BP*e*, and DPE at right angles to it.

2. Draw the great circle BC*e* making an angle of 48° 30′ with the primitive (369).

3. Set off the side BC = 115° 20′ from B to *m*, and draw the parallel circle *mcm* (374), cutting the oblique circle BC*e* in C.

4. With the tangent of the complement of the angle A = 53° 23′, and centre P, describe an arc; and with the secant of the same angle and centre C, cross it in *o*.

5. With the centre *o*, and radius *o*C, draw the great circles A*cb*, *acb*. Then ABC is the triangle required; and *none* of the parts are ambiguous.

To measure the required parts.

6. AB, measured by a scale of chords, = 82° 26′.
7. AC (376) = 57° 30′.
8. The ∠ C (379) = 61° 41′.

BY CALCULATION.

I. *To find the side* AC.

sin ∠ A = 126° 37′ - 9·90452
: sin BC = 115 20 - - 9·95609
: : sin ∠ B = 48 30 - 9·87446
: sin AC = 57 30 - - 9·92603

Because BC + (AC *acute*), and A + B only, are of the same species with respect to 180°, the side AC is acute, and *not* ambiguous (489).

II. *To find the side* AB.

sin ½ (A ∼ B) = 39° 3′ 30″ 9·79942
: sin ½ (A + B) = 87 23 30 9·99960
: : tan ½ (AC ∼ BC) = 28 55 0 9·74226
: tan ½ AB = 41 13 0 9·94244
2

82 26 0 = AB.

III. *To find the* ∠ C.

sin ½ (AC ∼ BC) = 28° 55′ 0″ 9·68443
: sin ½ (AC + BC) = 86 25 0 9·99915
: : tan ½ (A ∼ B) = 39 3 30 9·90927
: cot ½ ∠ C = 30 50 21 10·22399
2

61 40 42 = ∠ C.

PRACTICAL EXAMPLES.

1. In the oblique spherical triangle ABC.

Given	The ∠ A = 121° 36′ 20″	*Ans.*	AC = 50° 10′ 33″
	The ∠ B = 42 15 13		AB = 40 0 4
	The side BC = 76 35 36		∠ C = 34 14 59

Required the other parts.

2. In the oblique spherical triangle ABC.

Given	The ∠ A = 34° 14′ 59″	*Ans.*	AC = 50° 10′ 27″ or 129° 49′ 33″
	The ∠ B = 42 15 13		AB = 76 35 42 or 167 5 54
	The side BC = 40 0 4		∠ C = 121 36 24 or 168 43 34

Required the other parts.

3. In the oblique spherical triangle ABC.

Given	The ∠ A = 36° 8′	*Ans.*	AC = 30° 0′ or 150° 0′
	The ∠ B = 133 42		AB = 7 51 or 137 51
	The side BC = 24 4		∠ C = 11 23 or 76 0

Required the other parts.

(506) Case III. *Given two sides of an oblique spherical triangle, and the angle contained between them, to find the rest.*

Given	The side AC = 80° 19′ The side AB = 120 47 The ∠A = 51 30	Required the rest.

BY CONSTRUCTION (*Plate V. Fig.* 19.)

1. With the chord of 60 degrees describe the primitive circle, through the centre P draw CP*e*, and *a*P*r* at right angles to it.

2. Set off the side AC = 80° 19′ by a scale of chords, and through the point A, draw the great circle AB*n*, making an angle of 51° 30′ with the primitive (369).

3. Set off the supplement of AB = 59° 13′ from *n* to *m*, and draw the parallel circle *m*B*m* (374), cutting the oblique circle AB*n* in B. Through the three points C, B, *e*, draw a great circle, then ABC is the triangle required.*

To measure the required parts.

4. BC (376) = 63° 50′.

5. The angles B and C (379) = 59° 16′ and 131° 29′.

BY CALCULATION.

I. *To find the angles* B *and* C.

$\cos\frac{1}{2}(AB+AC)$	= 100°	33′	0″	9·26267	$\sin\frac{1}{2}(AB+AC)$	= 100°	33′	0″	9·99260
$:\cos\frac{1}{2}(AB\sim AC)$	= 20	14	0	9·97234	$:\sin\frac{1}{2}(AB\sim AC)$	= 20	14	0	9·53888
$::\cot\frac{1}{2}\angle A$	= 25	45	0	10·31664	$::\cot\frac{1}{2}\angle A$	= 25	45	0	10·31664
$:\tan\frac{1}{2}(B+C)$	= 84	37	23	11·02631	$:\tan\frac{1}{2}(B\sim C)$	= 36	6	16	9·86292

Because $\frac{1}{2}$ (AB + AC) is obtuse, $\frac{1}{2}$ (B + C) is obtuse, and = 95° 22′ 37″.

Hence 95° 22′ 37″ + 36° 6′ 16″ = 131° 28′ 53″ = ∠C,
and 95° 22′ 37″ − 36° 6′ 16″ = 59° 16′ 21″ = ∠B.

II. *To find* BC.

1. *By using the angles* B *and* C.

$\sin\frac{1}{2}(B\sim C)$	= 36°	6′	16″	9·77031
$:\sin\frac{1}{2}(B+C)$	= 95	22	37	9·99808
$::\tan\frac{1}{2}(AB\sim AC)$	= 20	14	0	9·56654
$:\tan\frac{1}{2}BC$	= 31	54	45	9·79431
			2	
	63	49	30 = BC.	

2. *Without the angles* B *and* C.

log sin AC	= 80° 19′ -	9·99377
log sin AB	= 120 47 -	9·98405
log sin $\frac{1}{2}\angle A$	= 25 45 -	9·63794 9·63794
sum of logs	-	−1·20370
nat. no	- - -	·15984
		2
		·31968
nat. cos (AB ∼ AC)	= 40° 28′	·76078
nat. cos BC	= 63° 49′ 32″ diff.	·44110

Here BC is acute.

---

* A perpendicular may be drawn from the vertical angle C upon the base AB, by finding *p* the pole of the oblique circle AB*n* (368), and drawing a great circle CD*p*, through *p* and the point C (372).

PRACTICAL EXAMPLES.

1. In the oblique spherical triangle ABC.

	Given		Answer.
Given	The side AC = 57° 30′	Answer.	∠B = 48° 30′
	The side AB = 82 27		∠C = 61 42
	The ∠A = 126 37		BC = 115 21

Required the other parts.

2 In the oblique spherical triangle ABC.

	Given		Answer.
Given	The side AC = 30° 0′	Answer.	∠B = 46° 18′
	The side AB = 42 9		∠C = 104 0
	The ∠A = 36 8		BC = 24 4

Required the other parts.

3. In the oblique spherical triangle ABC.

	Given		Ans.
Given	The side AC = 50° 10′ 27″	Ans.	∠B = 42° 15′ 12″
	The side AB = 40 0 14		∠C = 34 14 58
	The ∠A = 121 36 24		BC = 76 35 36

Required the other parts.

(507) CASE IV. *Given two angles, and the side adjacent to both of them, to find the rest.*

Given { The ∠A = 51° 30′
The ∠C = 131 30
The side AC = 80 19 } Required the rest.

BY CONSTRUCTION. (*Plate V. Fig.* 20.)

1. With the chord of 60 degrees describe the primitive circle, through the centre P draw CP*e*, and *a*P*r* at right angles to it.

2. Set off the side AC = 80° 19′ by a scale of chords, and through the point A, draw the great circle AB*n*, making an angle of 51° 30′ with the primitive (369). In the same manner draw the great circle CB*e* through C, making an angle BC*r* = 48° 30′, the supplement of C. Then ABC is the triangle required.*

To measure the required parts.

3. AB and BC (376) = 120° 48′, and 63° 50′.

4. The ∠B (379) = 59° 16′.

BY CALCULATION.

I. *To find the sides* AB *and* BC.

cos ½ (A + C)	= 91° 30′	0″	8·41792	sin ½ (A + C)	= 91° 30′	0″	9·99985
: cos ½ (A ∼ C)	= 40 0	0	9·88425	: sin ½ (A ∼ C)	= 40 0	0	9·80607
: : tan ½ (AC)	= 40 9	30	9·92625	: : tan ½ AC	= 40 9	30	9·92625
: tan ½ (AB + BC)	= 87 41	0	11·39258	: tan ½ (AB ∼ BC)	= 28 29	0	9·73447

Because ½ (∠A + ∠C) is obtuse, ½ (AB + BC) is obtuse and = 92° 19′.

Hence 92° 19′ + 28° 29′ = 120° 48′ = AB.
and 92° 19′ − 28° 29′ = 63° 50′ = BC.

---

* A perpendicular CD may be drawn, if required, by the note Case III.

II. *To find the* ∠B.

1 *By using the sides* AB *and* BC.

sin ½ (AB ∼ BC) = 28° 29′ - 9·67843
: sin ½ (AB + BC) = 92 19 - 9·99964
: : tan ½ (A ∼ C) = 40 0 - 9·92381
· cot ½ ∠B = 29 38 - 10·24502
2

59 16 = ∠B.

2. *Without the sides* AB *and* BC.

log sin A = 51° 30 0″ - 9·69354
log sin C = 131 30 0 - 9·87446
log cos ½ AC = 40 9 30 {9·88325, 9·88325

{sum of logs - - —1·53450
{nat. no. - - - - ·34237
2

·68474
nat. cos (A ∼ C) = 80° 0′ - ·17365

nat. cos ∠B = 59 16 diff. ·51109

PRACTICAL EXAMPLES.

1. In the oblique spherical triangle ABC.

Given {The ∠A = 126° 37′ | The ∠C = 61 41 | The side AC = 57 30
Answer. {AB = 82° 26′ | BC = 115 20 | ∠B = 48 30

Required the other parts.

2. In the oblique spherical triangle ABC.

Given {The ∠A = 121° 36′ 20″ | The ∠C = 42 15 13 | The side AC = 40 0 10
Ans. {AB = 50° 10′ 27″ | BC = 76 35 33 | ∠B = 34 15 4

Required the other parts.

3. In the oblique spherical triangle ABC.

Given {The ∠A = 36° 8′ | The ∠C = 104 0 | The side AC = 30 0
Answer {AB = 42° 9′ | BC = 24 4 | ∠B = 46 18

Required the other parts.

(508) CASE V. *Given the three sides to find an angle.*

Given {The side AC = 80° 19′ | The side BC = 63 50 | The side AB = 120 47} Required the angles.

BY CONSTRUCTION (*Plate V. Fig.* 21.)

1. With the chord of 60 degrees describe the primitive circle, through the centre P draw CPE, and *aPr* at right angles to it.

2. Set off the side BC = 63° 50′ from the scale of chords, and draw BPE.

3. Set off the side AC = 80° 19′, by a scale of chords, from C to *m*, and draw the parallel circle *mAm* (374).

4. Set off the supplement of AB - 59° 13′, by a scale of chords, from E to *n*, and draw the parallel circle *nAn* (374).

5. Through C and A, and B and A, draw the great circles CA*e* and BAE. Then ABC is the triangle required.*

To measure the required parts.

6. The angles A, B, and C (379) will be 51° 31′, 59° 17′, and 131° 29′.

BY CALCULATION.

To find the ∠ A *by Rule* I.

cosec AC - - - -	= 80° 19′ -	·00623	reject indices.†
cosec AB - - -	=120 47 -	·06595	
sin ½ (AC+BC+AB) -	=132 28 -	9·86786	
sin ½ (AC+BC+AB)−BC	= 68 38 -	9·96907	
		2)19·90911	
cos ½ ∠ A	=25° 45′ 20″	9·95456	
	2		
	51 30 40 = ∠ A.		

To find the ∠ B *by Rule* II.

cosec BC - - -	= 63° 50′ -	·04696	reject indices.
cosec AB - -	=120 47 -	·06595	
sin ½ (AC+BC+AB)−BC	= 68 38 -	9·96907	
sin ½ (AC+BC+AB)−AB	= 11 41 -	9·30643	
		2)19·38841	
sin ½ ∠ B	=29° 38′ 24″ -	9·69421	
	2		
	59 16 48 = ∠ B.		

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[*] A perpendicular CD may be drawn from the vertical ∠ C upon the base AB, by the note to Case III.

[†] Or take the sines of AC and AB, add them together and subtract the sum from 20; the remainder will be the same as the sum of these cosecants without the indices.

To find the ∠C *by Rule* III.

cosec $\frac{1}{2}$ (AC+BC+AB) ÷	=132° 28′	-	·13214	reject indices.
cosec $\frac{1}{2}$ (AC+BC+AB)−AB=	11 41	-	·69357	
sin $\frac{1}{2}$ (AC+BC+AB)−BC=	68 38	-	9·96907	
sin $\frac{1}{2}$ (AC+BC+AB)−AC=	52 9	-	9·89742	
			2)20·69220	
tan $\frac{1}{2}$∠C=	65° 44′ 18″	-	10·34610	
	2			
	131 28 36 =∠C.			

PRACTICAL EXAMPLES.

1. In the oblique spherical triangle ABC.

Given		Answer.	
	The side AC= 57° 30′		∠A=126° 35′
	The side BC=115 20		∠B= 48 32
	The side AB= 82 28		∠C= 61 44

Required the angles.

BY CONSTRUCTION.* (*Plate V. Fig.* 22.)

1. Describe the primitive circle with the chord of 60°, on which set off AB=82° 28′, and from A and B through the centre P draw AP*e* and BP*r*.

2. Set off the side AC=57° 30′, by a scale of chords, from A to *m*, and draw the parallel circle *mcm* (374).

3. Set off the supplement of the side BC=64° 40′ from *r* to *n*, and draw the parallel circle *ncn* (374).

4. Through A, and the point of intersection C, draw the oblique circle AC*e*; and through B, and the same point, draw BC*r*.

5. Then ABC is the triangle required; and *cd*, or CD, are perpendiculars on the base AB produced.

2. In the oblique spherical triangle ABC.

Given		Ans.	
	The side AC=50° 10′ 30″		∠A= 34° 15′ 4″
	The side BC=40 0 10		∠B= 42 15 12
	The side AB=76 35 36		∠C=121 36 20

Required the angles.

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* The construction of the figure to this example is not essentially different from the construction of the figure to the preceding example, and is introduced here to show in what manner the general figures, Case XI. &c. of oblique spherics with a perpendicular, were formed.— Compare the two figures in this Case (V.) with those in Case XI., referred to above.

3. In the oblique spherical triangle ABC.

Given		Answer.	
	The side AC = 30° 0′		∠A = 36° 8′
	The side BC = 24 4		∠B = 46 18
	The side AB = 42 9		∠C = 104 0

Required the angles.

(509) CASE VI. *Given the three angles, to find the sides.*

Given		
	The ∠A = 51° 30′	Required the sides.
	The ∠B = 59 16	
	The ∠C = 131 30	

BY CONSTRUCTION. (*Plate V. Fig.* 23.)

1. With the chord of 60° describe the primitive circle, through the centre P draw CP*e* and *a*P*r* at right angles to it.

2. Draw the great circle CB*e*, making an angle of 48° 30′ (the supplement of C) with the primitive (369), and find its pole *p* (368).

3. With the semi-tangent of the ∠A = 51° 30′ and centre P, describe the small circle *oso*.

4. Through *e* and *p* draw *epm*; make *mw* = 59° 16′, the ∠B; draw *ew* cutting *ar* in *n*; bisect *nn* in *v*; with *v* as a centre, and radius *vn*, draw the small circle *nsn*, cutting *oso* in *s*.

5. Through *s* and P draw *bs*P*b*, and AP*d* at right angles to it; through A and *s* draw A*sx*, make *xy* an arc of 90°, and draw A*y* cutting *bs*P*b* in *t*.

6. Through the three points A*td* draw a great circle. Then ABC is the triangle required.

To measure the required parts.

7. AC = 80° 19′, BC = 63° 50′, and AB = 120° 47′ (376).

BY CALCULATION.

To find the side AC, *by Rule* I.

cosec ∠C	- -	= 131° 30′	-	·12554	reject indices.
cosec ∠A	- -	= 51 30	-	·10646	
cos $\frac{1}{2}$ (A + B + C)	-	= 121 8	-	9·71352	
cos $\frac{1}{2}$ (A + B + C) − B		= 61 52	-	9·67350	

2)19·61902

sin $\frac{1}{2}$ AC = 40° 9′ 36″ 9·80951

2

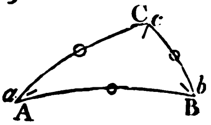
80 19 12 = AC.

By the same rule the other sides may be found.

OR, *To find the sides* BC, *by Rule* II.

$180^\circ - \angle A = bc$, $180^\circ - \angle B = ac$, and $180^\circ - \angle C = ab$.

cosec ab - -	$= 48^\circ$	$30'$	·12554	reject indices.
cosec ac - -	$= 120$	44	·06573	
sin $\frac{1}{2}(bc+ac+ab)$	$= 148$	52	9·71352	
sin $\frac{1}{2}(bc+ac+ab)-bc$	$= 20$	22	9·54161	



2)19·44640

cos $\frac{1}{2}\angle a = 58^\circ\ 5'$ 9·72320
2

$\angle a = 116\ \ 10$ its suppt. $= 63^\circ\ 50' =$ BC.

To find the side AB, *by Rule* II.

cosec $\frac{1}{2}(bc+ac+ab)$ -	$= 148^\circ$	$52'$	·28648	reject indices.
cosec $\frac{1}{2}(bc+ac+ab)-ab$	$= 100$	22	·00715	
sin $\frac{1}{2}(bc+ac+ab)-bc$	$= 20$	22	9·54161	
sin $\frac{1}{2}(bc+ac+ab)-ac$	$= 28$	8	9·67350	

2)19·50874

tan $\frac{1}{2}\angle c = 29^\circ\ 35'\ 52''$ - 9·75437
2

$\angle c = 59\ \ 11\ \ 44$ its suppt. $= 120^\circ\ 48'\ 16'' =$ AB.

PRACTICAL EXAMPLES.

1. In the oblique spherical triangle ABC.

Given	Answer.
The $\angle$ A $= 126^\circ\ 35'$	AC $= 57^\circ\ 30'$
The $\angle$ B $= 48\ \ 32$	BC $= 115\ \ 20$
The $\angle$ C $= 61\ \ 44$	AB $= 82\ \ 28$

Required the sides.

2. In the oblique spherical triangle ABC.

Given	Answer.
The $\angle$ A $= 34^\circ\ 15'\ 4''$	AC $= 50^\circ\ 10'\ 30''$
The $\angle$ B $= 42\ \ 15\ \ 12$	BC $= 40\ \ 0\ \ 10$
The $\angle$ C $= 121\ \ 36\ \ 20$	AB $= 76\ \ 35\ \ 36$

Required the sides.

3. In the oblique spherical triangle ABC.

Given	Answer.
The $\angle$ A $= 36^\circ\ 8'$	AC $= 30^\circ\ 0'$
The $\angle$ B $= 46\ \ 18$	BC $= 24\ \ 4$
The $\angle$ C $= 104\ \ 0$	AB $= 42\ \ 9$

Required the sides.

CHAP. X.

ASTRONOMICAL DEFINITIONS AND INTRODUCTORY PROBLEMS.

I. *Astronomical Definitions.*

(510) *The Celestial Sphere* is that apparent concave in which the sun, moon, stars, and all the heavenly bodies seem to be situated.

(511) *The axis* of the celestial sphere is an imaginary line passing through the centre of the earth, about which all the heavenly bodies appear to have a diurnal revolution.*

(512) *The poles* of the celestial sphere are the extremities of its axis, the one called the north pole, the other the south pole; and the points lying directly opposite to them, on the terrestrial sphere, are the north and south poles of the earth.†

(513) *The equator* or *equinoctial* is a great circle which divides the heavens into two hemispheres, the northern and southern; it is so called, because, when the sun appears in it, the days and nights all over the world are equal, viz. twelve hours each. This happens twice in the year, about the 20th of March and the 23rd of September; the former is called the *vernal* equinox, the latter the *autumnal* equinox.

(514) *The ecliptic* is a great circle in which the sun makes his apparent annual progress; it cuts the equinoctial in an angle of 23° 28′ nearly ‡, called the obliquity of the ecliptic; and the points of intersection are called the equinoctial points.

The ecliptic is divided into twelve equal parts, called signs:

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* Although the earth's *real* motion on its axis from west to east is the cause of day and night; and its motion in its orbit, or path round the sun, is the cause of the variation of the seasons of the year: yet as all appearances and places of the celestial bodies will be the same, whether the earth moves and the celestial sphere is at rest, or the earth is at rest and the celestial sphere in motion, astronomers, for the ease of calculation, assume the earth as a point at rest in the centre of the celestial sphere, and ascribe to the heavenly bodies that motion which they appear to have to a spectator on the earth.

† The pole star is not directly in the point which is the true north pole of the heavens; its mean right ascension, on January 1, 1840, being 1° 2′ 0″·68, and annual variation + 16″·50; also its mean declination is 88° 27′ 21″·9 N., and annual variation + 19″·32. (*Nautical Almanac*). Its mean polar distance is therefore 1° 32′ 38″·1 on Jan. 1, 1840.

‡ The angle which the ecliptic makes with the equinoctial is a variable quantity, and is equal to half the difference between the greatest and least meridian altitudes of the sun at any place, supposing the sun to have the greatest declination when on the meridian.

each sign contains 30 degrees. Their names and characters are as follow: —

♈ Aries
♉ Taurus
♊ Gemini
♋ Cancer
♌ Leo
♍ Virgo
♎ Libra
♏ Scorpio
♐ Sagittarius
♑ Capricornus
♒ Aquarius
♓ Pisces.

The first six signs lie on the north of the equinoctial, and are called northern signs; the six following lie on the south side of the equinoctial, and are called southern signs. The sun continues about a month in one of these signs, and goes through nearly a degree in a day.

(515) *The Zodiac* is a space which extends about 8 degrees on each side of the ecliptic, like a belt or girdle, within which the motions of all the greater planets are performed.

(516) *The Nodes* are the points where the orbits or paths of the planets round the sun intersect the ecliptic. That where the planet ascends from the south towards the north of the ecliptic is called the north or ascending node, and is marked thus ☊; the other the south or descending node, and is marked thus ☋. The names and characters of the planets are as follow: —

☉ The Sun
☿ Mercury
♀ Venus
⊕ The Earth
☽ The Moon
♂ Mars
⚶ Vesta
⚵ Juno
⚳ Ceres
⚴ Pallas
♃ Jupiter
♄ Saturn
♅ Herschel, or Georgian.

When two planets are referred to the same point of the ecliptic, they are said to be in *conjunction;* and those that are referred to opposite points of the ecliptic are said to be in *opposition*, or 180 degrees apart. If they be three signs, or 90 degrees distant, they are in a *quartile* aspect. If two signs, or 60 degrees, a *sextile* aspect. The astronomical marks are as follow: —

☌ Conjunction when planets are in the same point of the ecliptic.
✱ Sextile when 2 Signs dist.
□ Quartile when 3 Signs dist.
△ Trine when 4 Signs dist.
☍ Opposition when 6 Signs dist.

The conjunctions and oppositions are also called the *syzygies*, and the quartile aspects the *quadratures;* these terms are applied chiefly to the moon.

(517) *The horizon* is a great circle which separates the visible half of the heavens from the invisible.

This horizon is distinguished by the *sensible* and *rational* horizon, when applied to the earth. The *sensible horizon* is the boundary of the spectator's view at sea or land; and a plane parallel to this circle, passing through the earth's centre, is called the rational horizon.

(518) *The cardinal points* are the *east, west, north* and *south*,

points of the horizon. The mariner's compass, which is divided into 32 points, each 11° 15′ (153), is a representation of the horizon.

(519) *The Zenith* is a point in the celestial sphere directly over the head of the spectator, being the elevated pole of the horizon.

(520) *The Nadir* is a point in the celestial sphere directly under the feet of the spectator, and is diametrically opposite to the zenith; being the depressed pole of the horizon.

(521) *Azimuth* or *vertical circles*, are great circles passing through the zenith and nadir. They cut the horizon at right angles. The altitudes of the heavenly bodies are measured on these circles.

(522) *The prime vertical* is that azimuth circle which passes through the east and west points of the horizon.

(523) *Meridians* are great circles passing through the poles of the world, and cutting the equinoctial at right angles. They are also called hour circles; and upon the terrestrial sphere, circles of longitude.

(524) *Circles of celestial longitude* are great circles passing through the poles of the ecliptic, and cutting it at right angles.

(525) *The latitude* of any object in the heavens, is an arc of a circle of longitude contained between the centre of that object and the ecliptic.

(526) *The latitude* of any place on the earth, is the elevation of the pole above the horizon, and the *complement* of the latitude, is the distance of the pole from the zenith. Or the latitude is the distance of the zenith of the place from the equinoctial, on the celestial sphere.

(527) *The declination* of any celestial object, is an arc of a meridian contained between the centre of that object and the equinoctial.

(528) *Parallels of declination* are small circles parallel to the equinoctial.

(529) *The altitude* of any object in the heavens, is an arc of an azimuth or vertical circle, contained between the centre of the object and the horizon.

(530) *Parallels of altitude* are small circles parallel to the horizon.

(531) *Parallels of celestial latitude* are small circles parallel to the ecliptic.

(532) *The tropics* are small circles parallel to the equinoctial, at 23° 28′ from it, and touch the ecliptic in the points of cancer and capricorn; they are the limits of the sun's progress to the north and south of the equinoctial.

(533) *The zenith distance* of any celestial object is the arc of a vertical circle, contained between the centre of that object and the zenith, being the complement of the altitude.

(534) *The polar distance* of any object in the heavens, is an arc of a meridian contained between the centre of that object and the pole of the equinoctial.

(535) *The amplitude* of any celestial object is an arc of the horizon, contained between the centre of the object when rising or setting, and the east or west point of the horizon.

(536) *The azimuth* of any object in the heavens, is an arc of the horizon, contained between an azimuth or vertical circle (passing through the object), and the north or south point of the horizon.

(537) *The right ascension* of an object, is the distance between the first point of aries and a meridian passing through the object, reckoned on the equinoctial. It is so called, because in a right sphere, this meridian will coincide with the horizon when the object is rising. Or, we may define it to be the angle at the pole formed between a meridian passing through the first point of aries, and a meridian passing through the object.

(538) *The oblique ascension* of an object, is the distance of the equinoctial point aries from the horizon when the object is rising. Or, it is that degree of the equinoctial which rises with the object in an oblique sphere.

(539) *The oblique descension* is the distance of the first point of aries from the horizon when the object is setting. Or, it is that degree of the equinoctial which sets with the object in an oblique sphere.

(540) *The ascensional*, or *descensional difference*, is the difference between the right and oblique ascension or descension; and with respect to the sun, it is the time he rises before six, when his declension is of the same name as the latitude, or sets before six, when the declination and latitude have contrary names.

(541) *The equinoctial colure* is a great circle passing through the pole and the equinoctial points aries and libra.

(542) *The solstitial colure* is a great circle passing through the pole and the points ♋ and ♑, called solstitial points; because when the sun is near these points he seems to have nearly the same altitude at noon for several days, and therefore apparently stops or stands still.

(543) *The arctic circle* is a parallel of declination at the distance of 23° 28′ from the north pole, or 66° 32′ from the equinoctial. It is generally called the north polar circle.

(544) *The antarctic circle*, called likewise the south polar circle, is the same distance from the south pole as the arctic circle is from the north pole.

(545) *Apparent noon*, the time when the sun comes to the meridian, or 12 o'clock, as shown by a sun-dial.

(546) *True*, or *mean noon*, twelve o'clock, as shown by a well regulated chronometer, so adjusted as to go 24 hours in a mean solar day.*

(547) *The equation of time* at noon, is the interval between true and apparent noon.

(548) *A sidereal year* is the interval of time from the sun's leaving any fixed star till he returns to it again, and consists of $365^d\ 6^h\ 9^m\ 11^s{\cdot}5$ of mean solar time.

(549) *A tropical* or *solar year* is the interval of time from the sun's leaving one tropic, or equinox, till he returns to it again, and consists of $365^d\ 5^h\ 48^m\ 51^s{\cdot}6$ of mean solar time.

(550) *The nonagesimal degree*, or *medium cœli*, is the 90th degree of the ecliptic, reckoned from its intersection with the eastern part of the horizon at any time. Its altitude is equal to the distance between the zenith and the pole of the ecliptic; or it is measured by the angle which the ecliptic makes with the horizon, at any elevation of the pole.

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* A *mean solar day* is a period not marked out by any observable phenomena, but an artificial interval of time. The time elapsed from the sun's leaving the meridian on any day till it returns to the same meridian the next day is called a *true solar day*, and is subject to a continual variation, arising from the obliquity of the ecliptic, and the unequal motion of the earth in its orbit.

A clock, or chronometer, therefore, which measures time by equal motion, cannot be so adjusted as to keep time exactly with the sun, or always to show 12 o'clock when the sun is on the meridian; to correct these irregularities, the year is divided into as many imaginary days, each of 24 hours in length, as there are real days in the year measured by the sun's return to the meridian; one of these imaginary days is called a *mean solar day*, and a clock adjusted so as to go 24 hours in one of these days, is said to be regulated to *mean solar time.*

The year thus consists of as many *mean solar days* as *true solar days;* the clock being just as much before the sun, on some days of the year, as the sun is before the clock on others. The difference, which is the *equation of time*, is given in pages I. and II. of the Nautical Almanac for every day in the year. The time shown by the clock is called *true* or *mean* time, and the time shown by the sun is called *apparent* time.

If a clock be adjusted to go 24 hours, from the passage of any fixed star over the meridian till it returns to it again, its rate of going at any time may be determined by comparing it with the transit of that fixed star. A clock thus regulated is said to be adjusted to *sidereal* time. Here nature affords a standard exceeding in exactness any imitation that can be produced by art: there is no irregularity in the earth's diurnal motion, its diurnal revolution on its axis being uniformly performed in 24 hours of sidereal time $= 23^h\ 56^m\ 4^s$ of mean solar time

(551) *The Crepusculum*, or *twilight*, is that faint light which we perceive before the sun rises, and after he sets. It is produced by the rays of light being refracted in their passage through the earth's atmosphere, and reflected from the different particles thereof.

(552) *A constellation* is a collection of stars on the surface of the celestial sphere, circumscribed by the outlines of some assumed figure; as a *ram*, a *dragon*, a *bear*, &c.

This division is necessary, in order to direct a person to any part of the heavens, where any particular star is situated.

(553) *The diurnal and nocturnal arcs.* In all places of the earth, except the two poles, the horizon cuts the equinoctial into two equal parts. In all places situated on the equator, the horizon cuts all the parallels of declination into two equal parts, and here the sun and all the stars are 12 hours above the horizon, and 12 hours below. In places between the equator and the elevated pole, the parallels of declination are unequally divided; the greater arc being above the horizon, and the less arc below. In all places between the equator and the depressed pole, the parallels of declination are unequally divided; the greater arc being below the horizon, and the less arc above.

In all cases, the arcs which are above the horizon are called w urnal arcs, and those below, nocturnal arcs. Or, the parallel, ehitch the sun, moon, or stars, describe from their rising to toding, is called the *diurnal* arc; and that parallel which each is them describes, from the setting to the rising, is called the *nocturnal* arc.

II. *Introductory Astronomical Problems.*[*]

PROBLEM I.

(554) *To turn degrees, minutes, and seconds, of the equator into time.*

Rule. Multiply the quantity by 4, divide the degrees of the product by 60, and the quotient will be hours, the remainder minutes, and the other parts of the product seconds, &c. of the corresponding time.

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[*] These are the same as in the former editions, being extracted from the general examples, and from the notes upon them. Those which depend upon the Nautical Almanac have been recalculated and adapted to the year 1840.

EXAMPLE.

Turn 25° 15′ 16″ of the equator into time.

$$\begin{array}{r} 25^\circ\ 15'\ 16'' \\ 4 \\ \hline 60)101\quad 1\quad 4 \\ \hline \textit{Answer.}\quad 1^h\ 41^m\ 1^s\ 4^t \\ \hline \end{array}$$

Also 77° 2′ 10″ of longitude = $5^h\ 8^m\ 8^s\ 40^t$ of time, and 124° 16′ 30″ of the equator = $8^h\ 17^m\ 6^s$ of time.

PROBLEM II.

(555) *To turn time into degrees, minutes, and seconds.*

RULE. Multiply the hours by 60, and add the odd minutes, if any, to the product, one fourth of which will be degrees, multiply the remainder by 60, and add the odd seconds, if any, to the product, one fourth of which will be minutes, &c.

EXAMPLE.

Find the number of degrees, &c. corresponding to $1^h\ 41^m\ 1^s\ 4^t$.

$$\begin{array}{r} 1^h\ 41^m\ 1^s\ 4^t \\ 60 \\ \hline 4)101 \\ \hline \textit{Answer.}\quad 25^\circ\ 15'\ 16'' \\ \hline \end{array}$$

Also $3^h\ 4^m\ 28^s$ of time = 46° 7′ of longitude, and $8^h\ 17^m\ 6^s$ of time = 124° 16′ 30″ of longitude.

PROBLEM III.

(556) *Given the time under any known meridian, to find the corresponding time at Greenwich.*[*]

RULE. Turn the longitude of the place under the known meridian into time (554): add this time to the time at the given place if the longitude be west, or subtract it if east, and the sum or remainder will be the time at Greenwich. If the sum exceed 24 hours, subtract 24 hours from it, the remainder will show the time at Greenwich on the *following* day: if the longitude, when turned into time, cannot be subtracted from the time at the given place, add 24 hours to the time at the given place before you subtract, the remainder will show the time on the *preceding* day.

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[*] Since the earth makes one revolution on its axis from west to east in 24 hours, the sun must *apparently* make one revolution round the earth from east to west in the same time. Now, the longitudes of all places on the earth are reckoned on the equator, which is divided into 360 degrees, and the whole of it passes the sun in 24 hours; it follows that every 15° of motion is *one* hour in time, every degree 4 minutes, &c. (as in Prob. I. and II.) Hence, a place one degree eastward of Greenwich will have noon, and every hour of the day, *four* minutes sooner than at Greenwich; and a place one degree westward of Greenwich will have noon, and every hour of the day, *four* minutes later.

EXAMPLE I.

Find the time at Greenwich, on the 12th of August, when it is $7^h\ 25^m$ at a place in longitude 97° 45′ West.

Time at the given place $7^h\ 25^m$
Long. 97° 45′, in time = 6 31 W.

Time at Greenwich - - 13 56 or 56 minutes past 1 in the morning on the 13th of August.*

EXAMPLE II.

Find the time at Greenwich, on the 1st of May, when it is $22^h\ 40^m$ at a place in longitude 160° W.

Time at the given place $22^h\ 40^m$
Long. 160°, in time - = 10 40 W.

Sum 33 20
24

Time at Greenwich - - 9 20 on May 2d.

EXAMPLE III.

Find the time at Greenwich, on the 8th of April, when it is $16^h\ 26^m$ at a place in longitude 98° 45′ East.

Time at the given place $16^h\ 26^m$
Long. 98° 45′, in time = 6 35 E.

Time at Greenwich - - 9 51 on the 8th of April.

EXAMPLE IV.

Find the time at Greenwich, on the 4th of June, when it is $5^h\ 26^m$ at a place in longitude 120° East.

Time at the given place + $24^h = 29^h\ 26^m$
Long. 120°, in time - - - = 8 0 E.

Time at Greenwich - - - 21 26 on the 3d of June.

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* The astronomical day begins at noon, and is counted forward to 24 hours, or the succeeding noon, when the next day begins, being 12 hours later than the civil day, which commences at the preceding midnight; thus August 12th, at $13^h\ 56^m$ astronomical time, is August 13th at $1^h\ 56^m$ in the morning, according to civil reckoning.

PRACTICAL EXAMPLES.

1. What Greenwich time answers to noon at a place in 60° East longitude?

Answer. 20 hours on the preceding day.

2. What Greenwich time answers to noon at a place in longitude 60° West?

Answer. 4 hours.

3. Find the time at Greenwich when it is $19^{h}\ 42^{m}$ at a place in 28° 30′ E. longitude.

Answer. $17^{h}\ 48^{m}$.

PROBLEM IV.

(557) *Given the time at Greenwich, to find the corresponding time under any known meridian.*

RULE. Turn the longitude of the place under the known meridian into time (554): add this time to the time at Greenwich if the longitude be east, or subtract it if west, and the sum or remainder will be the time under the known meridian. If the sum exceed 24 hours, subtract 24 hours from it, the remainder will show the time at the given meridian on the *following* day: if the longitude, when turned into time, cannot be subtracted from the given time at Greenwich, add 24 hours to the time at Greenwich before you subtract, the remainder will show the time on the *preceding* day.

EXAMPLE I.

When it is $9^{h}\ 51^{m}$ at Greenwich, on the 8th of April, what hour is it in longitude 98° 45′ East?

Time at Greenwich - $9^{h}\ 51^{m}$
Long. 98° 45′, in time = 6 35 E.

Time in long. 98° 45′ E. = 16 26, on April 8th.

EXAMPLE II.

When it is $21^{h}\ 26^{m}$ at Greenwich on the 3d of June, what hour is it at a place in 120° East longitude?

Time at Greenwich - - $21^{h}\ 26^{m}$
Longitude 120° in time = 8 0 E.

Sum 29 26
24

Time in longitude 120° E. = 5 26, on June 4th.

EXAMPLE III.

When it is $13^h\ 56^m$ at Greenwich, on the 12th of August, find what hour it is at a place in longitude 97° 45′ West.

Time at Greenwich - $13^h\ 56^m$
Longitude 97° 45′ in time = 6 31 W.

Time in long. 97° 45′ W. = 7 25 on Aug. 12th.

EXAMPLE IV.

When it is $9^h\ 20^m$ at Greenwich, on the 2d of May, what hour is it at a place in longitude 160° West?

Time at Greenwich - - $+24^h = 33^h\ 20^m$
Longitude 160°, in time - - =10 40 W.

Time in longitude 160° W. - - =22 40 on the 1st of May, or 40 minutes past 10 in the morning on the 30th of April.

PRACTICAL EXAMPLES.

1. When may an immersion of an eclipse of the first satellite of Jupiter be observed at Berlin, in longitude 3° 34′ 22″ E. which, by the Nautical Almanac, happens on the 14th of April 1840, at $10^h\ 46^m\ 18^s$ Greenwich mean time?
Answer. $11^h\ 39^m\ 54^s$.

2. What is the expected time of the beginning of the Lunar eclipse, which happens on August 12th, 1840, at $17^h\ 58^m$, Greenwich mean time, in longitude 76° 49′ 30″ West?
Answer. $12^h\ 50^m\ 42^s$.

PROBLEM V.

(558) *To reduce the declination of the sun, as given in the Nautical Almanac, to any other meridian, and to any given time of the day.*

Rule. The corresponding time at Greenwich being ascertained (556), take the sun's declination and its daily change, from the *Nautical Almanac*, for the mean or apparent noon of the given day, as it may be required; then as 24 hours : this change : : the time from noon at Greenwich : the variation of the sun's declination in that time. This variation must be added to the sun's declination at noon, or subtracted from it, according as the declination is increasing or decreasing.

Note. By a similar process the sun's longitude, right ascension, and equation of time, may be determined for any given time, or at any given place; and also the right ascension and declination of a planet.

EXAMPLE I.

Required the sun's declination at mean noon, on the 12th of October 1840, at Glasgow, longitude 4° 15′ W.

First, 4° 15′ = 17 minutes, the time by which the clocks at Glasgow are *slower* than at Greenwich: hence when it is noon at Glasgow, it is $0^h\ 17^m$ at Greenwich.

⊙'s declination at mean noon October 12th, Naut. Alm. is	7° 32′ 2″·5
⊙'s declination at mean noon October 13th, Naut. Alm. is	7° 54′ 32″·1
Increase of declination in 24 hours - - -	22′ 29″·6

Then 24^h : 22′ 29″·6 :: 17^m : 15″·9 the increase of the sun's declination in 17 minutes of time; consequently when it is noon at Glasgow, the sun's declination is (7° 32′ 2″·5 + 15″·9 =) 7° 32′ 18″·4 South.

EXAMPLE II.

What is the sun's right ascension, June 5th, 1840, at $13^h\ 48^m$ mean time, in longitude 63° 10′ E.?

	h	m	s	
Time at the given place - -	13^h	48^m	0^s	
Longitude 63° 10′ in time -	4	12	40	E.
Time at Greenwich - - - - -	9	35	20	

	h	m	s
⊙'s right ascension at mean noon June 5th is	4^h	54^m	3^s·83
⊙'s right ascension at mean noon June 6th is	4	58	11 ·04
Increase of right ascension in 24 hours -		4	7 ·21

Then 24^h : $4^m\ 7^s$·21 :: $9^h\ 35^m\ 20^s$: $1^m\ 38^s$·77

	h	m	s
⊙'s right ascension at noon, June 5th -	4^h	54^m	3^s·83
Variation of the right ascension, add -		1	38 ·77
⊙'s right ascension at $13^h\ 48^m$ in long. 63° 10′ E	4	55	42 ·60

PRACTICAL EXAMPLES.

1. Required the sun's declination on the 25th of August 1840, at $8^h\ 20^m$ mean time, in longitude 48° West.

The declination at Greenwich mean noon (*Nautical Almanac*) being 10° 40′ 18″·0 N., and on the 26th, 10° 19′ 25″·4 N.

Answer. 10° 30′ 16″·1 N.

2. Required the sun's right ascension at apparent noon, on the 25th of May 1840, in longitude 124° East.

The right ascension at Greenwich, apparent noon (*Naut. Alm.*), being $4^{h}\ 9^{m}\ 8^{s}\cdot 83$, and on the 24th, $4^{h}\ 5^{m}\ 6^{s}\cdot 64$.

Answer. $4^{h}\ 7^{m}\ 45^{s}\cdot 41$.

3. Required the sun's declination January 24th, 1840, at $18^{h}\ 40^{m}$ mean time, in longitude 132° East.

The declination at Greenwich mean noon (*Naut. Alm.*) being 19 21′ 25″·8 S., and on the 25th, 19° 6′ 59″·7 S.

Answer. 19° 15′ 29″·7 S.

4. Required the right ascension of Jupiter on the 16th May, 1840, at $18^{h}\ 48^{m}$ mean time, in longitude 68° West.

The right ascension at Greenwich mean noon (*Naut. Alm.*) being $14^{h}\ 41^{m}\ 25^{s}\cdot 43$., and on the 17th, $14^{h}\ 40^{m}\ 56^{s}\cdot 78$.

Answer. $14^{h}\ 40^{m}\ 57^{s}\cdot 58$.

PROBLEM VI.

(559) *To reduce the semidiameter or horizontal parallax of the moon, as given in the Nautical Almanac, to any other meridian, and to any given time of the day.*

RULE. Find the time at Greenwich corresponding to the time at the given place (556). Take the change of the element in 12 hours from the Nautical Almanac. Then as 12 hours : this change : : the time at Greenwich : the variation in that time. This variation must be added to the moon's semidiameter or horizontal parallax (at noon or midnight) if increasing, or subtracted if decreasing.

EXAMPLE.

Required the moon's semidiameter and horizontal parallax on the 6th of January 1840, in longitude 11° 45′ West, at $15^{h}\ 45^{m}$ mean time.

Time at the given place - - - $15^{h}\ 45^{m}$
Longitude 11° 45′ West, in time = $0\ 47$ W.

Time at Greenwich - - = $16^{h}\ 32^{m}$, or $4^{h}\ 32^{m}$ past midnight.

☽ 's semidiam. at midnight, 6th	= 15′ 17″·6	hor. paral. -	= 56′ 7″·3
☽ 's semidiam. at noon - 7th	= 15 21 ·5	hor. paral. -	= 56 21 ·6
Increase in 12 hours	= 0 3 ·9	increase in 12 hours	= 0 14 ·3
12^{h} : 3″·9 : : $4^{h}\ 32^{m}$:	0′ 1″·5	12^{h} : 14″·3 : : $4^{h}\ 32^{m}$:	0′ 5″·4
☽ 's semidiam. at midnight	= 15 17 ·6	hor. par. at midnight	= 56 7 ·3
☽ 's semidiam. required	= 15 19 ·1	hor. par. required -	= 56 12 ·7

PRACTICAL EXAMPLES.

1. Required the moon's semidiameter and horizontal parallax on the 19th of May, 1840, in longitude 38° 40′ E. at $11^h\ 15^m$ mean time.

The moon's semidiameter at Greenwich (*Naut. Alm.*) at noon and midnight being 14′ 44″·7 and 14′ 46″·4; and the horizontal parallax at the same times 54′ 6″·6 and 54′ 12″·9.

Answer. The time at Greenwich is $8^h\ 40^m\ 20^s$, ☽'s semidiameter = 14′ 45″·9, and horizontal parallax = 54′ 11″·2.

2. Required the moon's semidiameter and horizontal parallax at 6^h A. M., January 17th, 1840, in longitude 15° East.

The moon's semidiameter at Greenwich (*Naut. Alm.*) at midnight on the 16th, and noon on the 17th, are 16′ 21″·4, and 16′ 19″·8, and the horizontal parallax at the same times 60′ 1″·6 and 59′ 55″·7.

Answer. The time at Greenwich is Jan. $16^d\ 17^h$, ☽'s semidiameter 16′ 20″·7, and horizontal parallax = 59′ 59″·1.

PROBLEM VII.

(560) *To reduce the moon's right ascension and declination, as given in the Nautical Almanac, to any other meridian, and to any given time of the day.*

RULE. Find the time at Greenwich corresponding to the time at the given place (556). Take the right ascension and declination for the given hour, and the change to the succeeding one from the *Nautical Almanac.* Then 60 minutes : this change : : minutes and seconds in the Greenwich time : the change corresponding to the minutes and seconds, to be added to or subtracted from the element at the given hour, according as it is increasing or decreasing.

EXAMPLE.

Required the moon's right ascension and declination, Jan. 8th, 1840, at $8^h\ 45^m$ mean time, in longitude 60° East.

Time at the given place	- -	8^h 45^m
Long. 60° E. in time	- - -	4 0 E
Time at Greenwich	- - -	4 45

Rt. Asc. at 4^h -	22^h	15^m	$57^s{\cdot}17$	Dec. at 4^h -	11°	19′	39″·4	S.
——— at 5^h -	22	17	57 ·44	Dec. at 5^h -	11	5	11 ·7	S.
		2	0 ·27			14	27 ·7	
60^m : $2^m\ 0^s{\cdot}27$: 45^m :		1	30 ·20	60^m : 14′ 27″·7 : : 45^m :	10	50	·8	
Rt. Asc. at 4^h -	22	15	57 ·17	Dec. at 4^h - -	11	19	39 ·4	
Rt. Asc. required -	22	17	27 ·37	Dec. required -	11	8	48 ·6	S.

PRACTICAL EXAMPLES.

1. Required the moon's right ascension and declination on the 10th of January, 1840, at $17^h\ 47^m$ mean time, in longitude 162° W.

	☽'s Rt. Asc.	☽'s Dec.
Jan. $11^d\ 4^h$	$0^h\ 39^m\ 40^s{\cdot}83$	7° 27′ 1″·3 N.
5	0 41 44·56	7 42 52 ·2 N.

Answer. The time at Greenwich is Jan. $11^d\ 4^h\ 35^m$, the moon's right ascension $0^h\ 40^m\ 53^s{\cdot}01$, and declination 7° 36′ 16″·0 N.

2. Required the moon's right ascension and declination on the 9th of May, 1840, at $18^h\ 47^m\ 34^s$ mean time, in longitude 62° E.

	☽'s Rt. Asc.	☽'s Dec.
May $9^d\ 14^h$	$10^h\ 35^m\ 17^s{\cdot}06$	8° 33′ 14″·7 N.
15	10 37 13·46	8 18 39 ·7 N.

Answer. The time at Greenwich is $14^h\ 39^m\ 34^s$, the moon's right ascension $10^h\ 36^m\ 33^s{\cdot}82$, and declination 8° 23′ 37″·7 N.

3. What is the moon's declination on December 15, 1840, at $14^h\ 49^m$ mean time, in longitude 114° E.

	☽'s Dec.
Dec. $15^d\ 7^h$	0° 14′ 38″·5 N.
8	0 0 35 ·8 S.

Answer. The time at Greenwich is $7^h\ 13^m$, and moon's declination 0° 11′ 20″·4 N.

PROBLEM VIII.

(561) *To find the apparent time at which a star culminates, or passes the meridian of a given place on a given day.*

RULE. From the star's right ascension (Tab. VIII.), corrected for the annual variation, and increased, if necessary, by 24 hours, subtract the sun's right ascension at apparent noon (Page I. *Nautical Almanac*), and the remainder is the required time *nearly*. To this add the longitude in time, if west, but subtract it if east, and the result will be the corresponding Greenwich time. Multiply the hourly difference of the sun's right ascension (from the same page of the *Naut. Alm.*) by the Greenwich time, and subtract the product from the time *nearly*, found before, and the remainder will be the required time *still more nearly*.

The operations must be repeated if greater accuracy be required; but this will seldom be necessary.

EXAMPLE.

At what time will *Arcturus* pass the meridian of Petersburgh in longitude 30° 19′ E., on the 1st of December, 1840?

★'s Rt. Asc. Jan. 1, 1840 (Tab. VIII.) + 24^h =	38^h	8^m	22^s	(+ $2^s{\cdot}73$ an. var)
$2^s{\cdot}73 \times 0{\cdot}9$ =			+ 2	
★'s Rt. Asc. at given time - - -	38	8	24	
⊙'s Rt. Asc. at noon, Dec 1, 1840 -	16	31	0	(+ $10^s{\cdot}83$ hourly diff.)
Required time nearly - - -	21	37	24	
Longitude 30° 19′ E. in time - -	2	1	16	
Greenwich time - - - -	19	36	8	= $19^h{\cdot}6$
$10^s{\cdot}83 \times 19{\cdot}6$ =	–	3	32	
Required time - -	21	33	52	

PRACTICAL EXAMPLES.

1. At what time will *Aldebaran* culminate at Greenwich on the 20th November, 1840?

The right ascension of *Aldebaran* being $4^h\ 26^m\ 48^s$ (Tab. VIII.) the sun's right ascension on November 20th, at noon (*Naut. Alm.*), $15^h\ 44^m\ 7^s$, and hourly diff. $10^s{\cdot}5$.

Answer. $12^h\ 40^m\ 28^s$.*

2. At what time will *Fomalhaut* culminate at the Cape of Good Hope, in longitude $1^h\ 13^m\ 55^s$ E, on April 14, 1840?

The right ascension of *Fomalhaut* being $22^h\ 48^m\ 49^s$ (Tab. VIII.); ⊙'s right ascension, April 14th, at noon (*Naut. Alm.*), being $1^h\ 31^m\ 6^s$, and hourly diff. $9^s{\cdot}23$.

Answer. $21^h\ 14^m\ 38^s$.

3. At what time, on the 11th of October, 1840, will *Sirius, the Dog Star*, culminate at Genoa, in longitude 8° 54′ E.?

The right ascension of *Sirius* being $6^h\ 38^m\ 8^s$ (Tab. VIII.); ⊙'s right ascension, 11th October (*Naut. Alm.*), at noon, $13^h\ 7^m\ 14^s$, and hourly diff. $9^s{\cdot}24$.

Answer. $17^h\ 28^m\ 18^s$.

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* The motion of clocks or watches may be examined, and their errors rectified, by the culminating of the stars. For instance, *Aldebaran* is on the meridian of Greenwich, November 20th, at $12^h\ 40^m\ 28^s$ *apparent* time, from which take the equation of time, $13^m\ 58^s$, as directed in page I. of the month in the Nautical Almanac, and the remainder $12^h\ 26^m\ 3^s$ is the mean time which the clock ought to show when the star is on the meridian.

4. At what time, on the 1st of December, 1840, will *Castor* culminate at Edinburgh, in longitude $0^h\ 12^m\ 44^s$ W.?

The right ascension of *Castor* being $7^h\ 24^m\ 26^s$ (Tab. VIII.); ☉'s right ascension (*Naut. Alm.*) December 1st, at noon, $16^h\ 31^m\ 0^s$, and hourly diff. $10^s{\cdot}83$.

Answer. $14^h\ 50^m\ 42^s$.

PROBLEM IX.

(562) *To find the apparent time at which the moon, or planet, culminates, or passes the meridian of a given place, on a given day.*

RULE. The meridian passage of the moon and planets, in mean time, for the meridian of Greenwich, are given to the nearest tenth of a minute in the Nautical Almanac, and from these we may readily deduce an approximate time for the passage of the object over any other meridian, sufficiently exact for the purposes usually required.

1. *For the Moon's Meridian Passage.* Take the difference of the times of two consecutive transits; then as 24^h : this difference : : longitude in time of the proposed meridian: a correction, to be added to the time of the Greenwich passage if the longitude be west, but subtracted if east, and the result will be the mean time of passage at the given meridian. To this apply the equation of time for the nearest noon, and we shall have the apparent time required.

2. *For a Planet's Meridian Passage.* Take the difference of the times of two consecutive transits, and considering this difference as an acceleration or retardation of the meridian passage while the planet has passed over 24^h of geographical longitude, take the proportional part of it for the proposed longitude, for a correction to be applied to the meridian passage at Greenwich, bearing in mind that in east longitudes the passage precedes that at Greenwich, when times are accelerated, and follows it when they are retarded; and the contrary in west longitudes. Apply the equation of time as before.

EXAMPLE I.

Required the apparent time of the moon's passing the meridian 3^h west of Greenwich, on January 25, 1840.

The moon passes the meridian of Greenwich on Jan. 25, at $17^h\ 33^m{\cdot}9$, and on Jan. 26, at $18^h\ 17^m{\cdot}7$; the diff. is $0^h\ 43^m{\cdot}8$.

Moon's mer. pass. at Greenwich	17^h 33^m·9
24^h : 43^m·8 : : 3^h :	5 ·5
Moon's mer. pass. at place, mean time	17 39 ·4
Equation of time	— 12 ·8
Moon's mer. pass. at place, appt. time	17 26 ·6

EXAMPLE II.

Required the apparent time at which Venus will pass the meridian 2^h east of Greenwich on May 15, 1840.

Venus passes the meridian of Greenwich, on May 15 at 22^h 42^m·7, and on May 16, at 22^h 43^m·5; the difference is 0^m·8.

Venus' mer. pass. at Greenwich	22^h 42^m·7
24^h : 0^m·8 : : 2^h :	— 0 ·1
Venus' mer. pass. at place, mean time	22 42 ·6
Equation of time	+ 3 ·9
Venus' mer. pass. at place, appt. time	22 46 ·5

PRACTICAL EXAMPLES.

1. At what time will the moon pass the meridian of Petersburgh, in longitude 2^h 1^m 16^s East, on March 2, 1840?

The moon passes the meridian of Greenwich on March 2, at 23^h 39^m·9, and on March 4, at 0^h 26^m·8; and the equation of time is — 12^m·1.

Answer. 23^h 23^m·9 apparent time.

2. At what time will Jupiter pass the meridian of St. Domingo, in longitude 69° 59′ 30″ West, on September 16, 1840?

Jupiter passes the meridian of Greenwich on Sept. 16, at 3^h 12^m·1, and on Sept. 17, at 3^h 8^m·8; and the equation of time is + 5^m·3. As the passages are retarded, and the longitude is west, the correction must be subtracted.

Answer. 3^h 16^m·7 apparent time.

PROBLEM X.

(563) *Given the observed altitude of a fixed star, to find its true altitude.*

RULE. From the observed altitude, subtract the refraction (Table IV.). If the star be observed at sea, subtract the dip of the horizon (Table V.). See art. (183).

EXAMPLE I.

The observed altitude of *Spica Virginis* was 20° 39′ 40″, the error* of the quadrant 19″ subtractive, and the height of the eye 18 feet above the level of the sea; required the correct altitude.

Observed altitude of *Spica Virginis*		20° 39′ 40″
Refraction (Table IV.) - - -	=	−2 30
		20 37 10
Dip for 18 feet (Table V.) - -	=	−4 3
		20 33 7
Index error of the quadrant -		−19
True altitude of the star - -		20 32 48

Or, 20° 39′ 40″ − (2′ 30″ + 4′ 3″ + 19″) = 20° 32′ 48″. *Answer.*

2. Suppose the observed altitude of *Regulus* to be 45° 13′ 15″, the height of the eye 14 feet above the level of the sea, and the error of the quadrant 5′ 6″ additive, required the true altitude.

Answer. 45° 13′ 50″.

PROBLEM XI.

(564) *Given the observed altitude of the sun's lower or upper limb*†, *to find the true altitude of its centre.*

RULE. To the observed altitude apply the semidiameter (taken from page II. of the month in the *Nautical Almanac*) by addition or subtraction, according as the lower or upper limb has been observed; from this result subtract the refraction ‡ (Table IV.), and then add the parallax in altitude § (Table VI.). If the altitude be taken at sea, the dip of the horizon (Table V.) must be deducted, the last result will be the true altitude of the sun's centre.

EXAMPLE I.

On the 13th of March, 1840, if the altitude of the sun's lower limb, observed at sea, be 18° 40′; required the true central altitude, the height of the eye being 22 feet.

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* Observations taken with a quadrant are liable to errors, arising from the bending and elasticity of the index, and the resistance it meets with in turning round its centre. These errors, though they cannot in all cases be avoided, may be pretty accurately allowed for by a correct observer.

† The sun's upper limb is the upper edge of its face, or the upper extremity of the vertical diameter; and the lower limb is the lower edge, or the lower extremity of the vertical diameter.

‡ Art. (165), et seq. § Art. (184).

Observed altitude of the ☉'s lower limb = 18° 40′ 0″
☉'s semidiameter (*Naut. Alm.*) - - = + 16 6
18 56 6
Refraction for 19° of altitude (Table IV.) = −2 44
18 53 22
☉'s parallax in altitude (Table VI.) = + 8
18 53 30
Dip for 22 feet (Table V.) - - - - = −4 28
True altitude of the ☉'s centre - - = 18 49 2

Or, (16′ 6″ + 8″) − (2′ 44″ + 4′ 28″) = 9′ 2″ and 18° 40′ + 9′ 2″ = 18° 49′ 2″. *Answer.*

NOTE. The index error, if any, of the quadrant must be applied to the observed altitude, previous to the other corrections.

2. On the 3d of September, 1840, suppose at sea the altitude of the sun's upper limb to be 28° 31′ 30″, the index error 40″ additive, height of the eye 12 feet, and the sun's semidiameter 15′ 53″; what is the true altitude of the sun's centre?
Answer. 28° 11′ 20″.

PROBLEM XII.

(565) *Given the observed altitude of the moon's lower or upper limb, to find the true altitude of its centre.*

RULE. Find the moon's semidiameter and horizontal parallax for the time and place of observation (559), and increase the semidiameter by the augmentation answering to the moon's altitude (Table VII.).

To the observed altitude* apply the augmented semidiameter by addition or subtraction, according as the lower or upper limb has been observed; and, if the observation has been made at sea, subtract the dip of the horizon (Table V.); the result will be the *apparent altitude* of the moon's centre.

To the cosine of the apparent altitude of the moon's centre, add the logarithm of the horizontal parallax in seconds (found above), the sum, rejecting 10 from the index, will be the logarithm of the moon's parallax in altitude in seconds (185), from which take the refraction of the moon in altitude (Table IV.), and the remainder will be the *moon's correction*, which, added to the apparent altitude, will give the true altitude.

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* Here, as in the preceding problem, the index error of the quadrant, if any, must be applied to the observed altitude, previous to the other corrections.

EXAMPLE I.

On the 18th of July 1840, in longitude 113° 10′ W. at $20^h\ 45^m\ 30^s$, if the observed altitude of the moon's lower limb be 45° 22′ 3″, error of the quadrant 58″ subtractive, and the eye 21 feet above the level of the sea; what is the true central altitude?

Time at the given place - - -	20^h	45^m	30^s
Longitude 113° 10′ W. in time -	= 7	32	40 W.
	28	18	10
	24		
Time at Greenwich on the 19th of July	= 4	18	10

☽'s semidiameter at noon, 19th = 15 30″, Horizontal parallax - 56′ 51″
☽'s semidiameter at midnight, 19th = 15 34 , Horizontal parallax - 57 9

Difference 0 4 Difference 0 18

12^h : 4″ : : $4^h\ 18^m\ 10^s$:	0′ 1″
☽'s semidiameter at noon -	15 30
☽'s semidiam. at $4^h\ 18^m\ 10^s$ =	15 31
☽'s augmentation (Tab. VII) =	11
☽'s aug. semidiameter -	= 15 42
☽'s observed altitude	= 45° 22′ 3″
Error of the quadrant	= −58
	45 21 5
Semidiameter -	= +15 42
	45 36 47
Dip (Table V.) -	= −4 22
App. altitude ☽'s centre	= 45 32 25
☽'s correction -	= 38 57
True alt. ☽'s centre	= 46 11 22

12^h : 18″ : : $4^h\ 18^m\ 10^s$:	0′ 6″
Horizontal parallax at noon -	56 51
Horizontal par. at $4^h\ 18^m\ 10^s$ =	56 57
	60
In seconds =	3417
Cos ☽'s app. alt. 45° 32′ 25″ =	9·84535
Horizontal parallax 3417″ log =	3·53364
Parallax in altitude 2393″ log =	3·37899
Then 2393″ - -	= 39 53″
☽'s refraction - -	= −0 56
☽'s correction -	= 38 57

2. On the 22d of April, 1840, in longitude 105° E. of Greenwich, at $13^h\ 30^m$, if the observed altitude of the moon's upper limb be 23° 48′, and the eye 18 feet above the level of the sea; required the moon's true central altitude.

The moon's semidiameter and horizontal parallax at noon (*Naut. Alm.*), being 14′ 49″ and 54′ 21″; and at midnight, 14′ 51″ and 54′ 30″.

Answer. The time at Greenwich is $6^h\ 30^m$; ☽'s aug. semidiameter 14′ 57″, horizontal parallax 54′ 26″; apparent altitude of the ☽'s centre 23° 29′ 0″; the ☽'s correction 47′ 45″, and the true altitude of her centre 24° 16′ 45″.

PROBLEM XIII.

(566) *Given the sun's meridian altitude, to find the latitude of the place of observation.*

RULE. Reduce the sun's declination to the meridian of the place of observation * (558).

Subtract the sun's corrected altitude (564) from 90°, the remainder is the zenith distance.

If the sun be south of the observer when the altitude is taken, call the zenith distance north; but, if north, call it south.

Then, if the zenith distance and declination have the same name, their sum is the latitude; but, if they have contrary names, their difference is the latitude, and it is always of the same name with the greater of the two quantities.

EXAMPLE I.

On the 17th of October, 1840, in longitude 52° W., suppose the meridian altitude of the sun's lower limb to be 28° 40′, the observer at sea, the sun to the south of him, and the eye 14 feet above the surface of the water. Required the latitude of the place of observation.

Obser. alt. ⊙'s lower limb	28° 40′ 0″
Semidiameter - -	+ 16 5
	28 56 5
Refraction (Table IV.)	−1 42
	28 54 23
⊙'s paral. in alt. (Table VI.)	+ 8
	28 54 31
Dip for 14 feet (Table V.)	−3 34
True altitude ⊙'s centre	28 50 57
Zenith distance - -	61 9 3 N.

Time at given place -	$0^h\ 0^m$
Longitude 52° W. - -	3 28
Time at Greenwich - -	3 28
⊙'s decl. Oct. 17, at noon	9° 23′ 8″ S.
⊙'s decl. Oct. 18, at noon	9 45 2
Increase in 24 hours -	21 54
$24^h : 21' 54' :: 3^h\ 28^m :$	3′ 10″
⊙'s declin. 17th at noon	9° 23′ 8″ S.
⊙'s reduced decl. -	9 26 18 S.
Zenith distance .	61 9 3 N.
True latitude - -	51 42 45 N.

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* The same rule, with a little variation, will answer for a planet; and for the moon, correcting for the declination as at (560).—See *Dr. Mackay's* Navigation, 1st edition, page 151—155.

2. On the 18th of March, 1840, in longitude 106° E., suppose the meridian altitude of the sun's lower limb, when north of the observer, to be 74° 56′, and the height of the eye 16 feet above the level of the sea; required the latitude.

The sun's semidiameter (*Naut. Alm.*) being 16′ 5″; the declination at noon on the 17th 1° 11′ 37″ S., and on the 18th 0° 47′ 56″ S.

Answer. The sun's true altitude = 75° 8′ 3″; time at Greenwich $16^h\ 56^m$ on the 17th, sun's reduced declination 0° 54′ 54″ S., latitude 15° 46′ 51″ S.

3. On the 28th of September, 1840, suppose the meridian altitude of *Arcturus* to be 36° 18′ north of the observer, and the eye 20 feet above the level of the sea; required the latitude.

Answer. 33° 46′ 41″ S.

CHAP. XI.

THE APPLICATION OF RIGHT-ANGLED SPHERICAL TRIANGLES TO ASTRONOMICAL PROBLEMS.

(567) The celestial sphere is represented by Plate III. Fig. 1.

1. Let the circular space, *South*, *Zenith*, *North*, *Nadir*, represent the brazen meridian* of a celestial or terrestrial globe, having its north pole elevated above the horizon.

2. Imagine the globe to be cut in halves by the brass meridian, and the semi-globe to be of transparent glass with the circles of the sphere drawn on it. Now if a sheet of paper be put upon the section, and a light be placed in the point aries (the eye being in libra), the shadows of all the most useful circles of the sphere will form a plane figure similar to Figure 1. Plate III.

3. NAS will present the axis of the globe, N the north pole, S the south pole.

4. ÆAQ the equator.

5. HO the horizon.

6. *Zenith* A the quadrant of altitude screwed on the zenith, and passing through aries. Or, *Zenith* A *Nadir*, the prime vertical passing through aries A.

7. ♑ ⊙A⊙ ♋ the ecliptic, *n* its north pole, *m* its south pole.

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* It is here presumed that the learner has some knowledge of the globes, and of the circles described thereon. It would be a very good exercise to solve the succeeding problems both by the globes and by calculation.

8. ♋ ♋ the tropic of cancer, or any parallel ot the sun's north declination.

9. ♑ ♑ the tropic of capricorn, or any parallel of the sun's south declination.

10. ON the elevation of the north pole above the horizon, equal to Æ *zenith*, the latitude of the place.

11. HÆ the elevation of the equator above the horizon, equal to *zenith* N, the complement of the latitude.

12. T⊙W a parallel circle 18 degrees above the horizon, or the boundary between twilight and dark night.

13. *Zenith* SEC, *Zenith* G*d*, &c. azimuth, or vertical circles.

14. NSDS, NG ⊙ CWS, &c. meridians or circles of terrestrial longitude.

15. *n*VX*m*, *n*CV*m*, &c. circles of celestial longitude.

NOTE. The several triangles in this general figure, and the lines which form them, are explained at the head of each problem, to which the triangle is applied.

PROBLEM I. (*Plate III. Fig.* 1.)

(568) *Given the obliquity of the ecliptic and the sun's longitude, to find his right ascension and declination.*

In the right-angled spherical triangle AG⊙, or AB⊙.

1. A⊙ = sun's longitude, an arc of the ecliptic from aries.

2. AG, or AB = sun's right ascension, an arc of the equinoctial; the point B always comes to the meridian with the sun.

3. G⊙, or B⊙ = sun's declination, an arc of a meridian passing through the sun's place. This arc likewise comes to the meridian with the sun.

4. GA⊙, or BA⊙ = the obliquity of the ecliptic, or the angle formed between the equinoctial and the ecliptic.

5. $A\bar{\odot}G$, or A⊙B = the angle formed by the ecliptic, and the meridian passing through the sun's place.

Any two of these *five* quantities being given, the rest may be found.

EXAMPLE.

On the 17th of May, 1840, the sun's longitude at mean noon is 56° 35′ 20″; required his declination, right ascension, and the angle formed between the ecliptic and the meridian passing through the sun's place, the obliquity of the ecliptic being 23° 27′ 44″.

1. *To find the sun's declination, in the triangle* AB⊙.

$$\text{rad} \times \sin \overset{*}{B}\odot = \sin BA\odot \times \sin A\odot.$$

$$\therefore \log \sin BA\odot + \log \sin A\odot - 10 = \log \sin B\odot.$$

or, rad : sin A⊙ = 56° 35′ 20″ : : sin BA⊙ = 23° 27′ 44″ : sin B⊙ = 19° 24′ 40″.

2. *To find the sun's right ascension, in the triangle* $AB\odot$.

$$\text{rad} \times \cos BA\odot = \cot A\odot \times \tan \overset{*}{AB}.$$

$$\therefore\ 10 + \log \cos BA\odot - \log \cot A\odot = \log \tan AB.$$

or, $\cot A\odot = 56^\circ\ 35'\ 20'' : \text{rad} :: \cos BA\odot = 23^\circ\ 27'\ 44'' : \tan AB = 54^\circ\ 16'\ 46'' = 3^h\ 37^m\ 7^s{\cdot}1$.

3. *To find the angle* $A\odot B$ *between the ecliptic and meridian*†.

$$\text{rad} \times \cos A\odot = \cot BA\odot \times \cot \overset{*}{A\odot B}.$$

$$\therefore\ 10 + \log \cos A\odot - \log \cot BA\odot = \log \cot A\odot B.$$

or, $\cot BA\odot = 23^\circ\ 27'\ 44'' : \text{rad} :: \cos A\odot = 56^\circ\ 35'\ 20'' : \cot A\odot B = 76^\circ\ 33'\ 31''$.

(569) Astronomers reckon the sun's longitude and right ascension from aries quite round the globe. Hence, though at equal distances from the equinoctial points, aries and libra, the sun may have the same quantity of declination and of the same name, viz. from aries to libra, *north;* and from libra to aries, *south;* yet the longitudes and right ascensions differ materially at these points where the declinations agree. Hence,

If the sun be in the second quadrant of the ecliptic, subtract the longitude from 180°; use the remainder to find the right ascension; and take the right ascension, when found, from 180°.

If the sun be in the third quadrant, subtract 180° from his longitude; use the remainder to find his right ascension; and add 180° to the right ascension when found.

If the sun be in the fourth quadrant of the ecliptic, subtract his longitude from 360°; use the remainder to find the sun's right ascension; and take the right ascension, when found, from 360°.

In the first quadrant of the ecliptic between the 20th March and 21st June, the sun's declination is north, and increasing; and in the third quadrant, between the 22nd September and 21st December, the sun's declination is south and increasing.—In the second quadrant of the ecliptic, from 21st June to 22nd September, the sun's declination is north, and decreasing; and in the fourth quadrant, from 21st December to the 20th March, the sun's declination is south and decreasing.

PRACTICAL EXAMPLES.

1. On the noon of the 17th of May, 1840, the obliquity of the ecliptic = 23° 27′ 44″, the sun's declination = 19° 24′ 40″ N. and increasing; required his longitude, right ascension, and

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† The finding of this angle is of no other use than as an exercise of right-angled spherical triangles.

the angle formed between the ecliptic and the meridian passing through the sun's place.

Ans. The sun's longitude =56° 35′ 20″
The sun's right ascension =54 16 46 =$3^h\ 37^m\ 7^s{\cdot}1$
The angle A⊙B =76 33 31

The answers here would have been ambiguous, had it not been shown that the sun was in the first quadrant of the ecliptic.

This problem is useful for calculating tables of the sun's longitude. For the latitude of the place being accurately determined, and a quadrant fixed in the plane of the meridian, the observation of the sun's meridian altitude gives his declination, or distance from the equator; whence his longitude for any day is readily obtained.

2. At mean noon on the 5th of August, 1840, the obliquity of the ecliptic =23° 27′ 44″, the sun's right ascension =$9^h\ 2^m\ 4^s{\cdot}4$, or 135° 31′ 6″; required the sun's longitude, declination, and the angle formed between the ecliptic and the sun's meridian.

Ans. The sun's longitude =133° 2′ 51″
Declination = 16 54 56 N.
The angle A⊙B =106 30 11

The sun's right ascension being greater than 90°, and less than 180°, he is in the second quadrant of the ecliptic.

3. On the 10th of November, 1840, at apparent noon, the sun's right ascension is $15^h\ 2^m\ 54^s{\cdot}7$ or 225° 43′ 40″, and his declination 17° 15′ 50″ S.; required his longitude and the obliquity of the ecliptic.

Ans. The sun's longitude =228° 11′ 39″
Obliquity of the ecliptic = 23 27 43

The sun's right ascension being greater than 180°, and less than 270°, he is in the third quadrant of the ecliptic.

4. On the 1st of February, 1840, at mean noon, the sun's longitude is 311° 46′ 13″, his declination =17° 16′ 28″ S.; required his right ascension and the obliquity of the ecliptic.

Ans. The sun's right ascen. =314° 14′ 9″ or $20^h\ 56^m\ 57^s$
Obliquity of the ecliptic = 23 27 45

PROBLEM II. (*Plate III. Fig.* 1.)

(570) *Given the latitude of the place, and the sun's declination*†, *to find its amplitude, ascensional difference, and the time of his rising and setting.*

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† This problem may be applied to the rising or setting of a planet, or of a star. But where great exactness is required, the declination of the sun or planet

In the right-angled spherical triangle ABS or AGC.

1. AB or AG = the sun's, or star's, ascensional difference.
2. BS or GC = the sun's, or star's, declination.
3. AS or AC = the amplitude.
4. BAS, measured by the arc QO; or CAG, measured by the arc HÆ, = the complement of the latitude.
5. ASB or ACB = the angle formed between the horizon and the meridian passing through the sun.

Any two of the above five quantities being given, the rest may be found.

(571) *The same things may be found from the right-angled triangle* SON *or* CHS.

For, in the triangle SON, SO is the complement of AS; SN is the complement of BS; ON is the complement of the arc QO, which measures the angle BAS; and the angle SNO, measured by the arc BQ, is the complement of the ascensional difference. In the same manner the triangle CHS may be compared with the triangle AGC.

EXAMPLE I.

Given the latitude of London 51° 32′ N. on the longest day when the sun has 23° 28′ N. declination; required the sun's amplitude, ascensional difference, time of rising, setting, the length of the day and night, &c.

1. *In the triangle* ABS *to find the amplitude* AS.

$$\text{rad} \times \sin \text{BS} = \sin \text{BAS} \times \sin \text{AS}.$$

$$\therefore 10 + \log \sin \text{BS} - \log \sin \text{BAS} = \log \sin \text{AS}.$$

or, sin BAS = 38° 28′ : rad :: sin BS = 23° 28′ : sin AS = 39° 48′; viz. Cosine of the latitude is to radius, as the sine of the sun's declination is to the sine of the amplitude.

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must be calculated as near to the time of rising and setting as possible; especially for the moon, on account of her swift and irregular motion. Likewise, the declination of the sun near the equinoxes alters considerably in the compass of an hour. Hence what is generally called a parallel of the sun's declination, and drawn as such in the projection of the sphere, is not strictly a parallel circle, but a spiral line.

When the latitude of the place and the declination of the object have the same name, the right ascension diminished by the ascensional difference, leaves the oblique ascension, and increased by the same gives the oblique descension.

But when the latitude of the place and declination have contrary names; the right ascension increased by the ascensional difference gives the oblique ascension; and being diminished by the ascensional difference leaves the oblique descension.

This part of the problem is useful in navigation, for finding the variation of the compass. — It is evident, by comparing the triangle ABS with the triangle AGC, that the amplitude is always of the same name with the declination, whether north or south.

2. *To find the ascensional difference* AB.

$$\text{rad} \times \sin \overset{*}{\text{AB}} = \cot \text{BAS} \times \tan \text{BS}.$$

$$\therefore \log \cot \text{BAS} + \log \tan \text{BS} - 10 = \log \sin \text{AB}.$$

or, $\text{rad} : \cot \text{BAS} = 38° \, 28' :: \tan \text{BS} = 23° \, 28' : \sin \text{AB} = 33° \, 7'$; viz. Radius is to the tangent of the latitude, as tangent of the sun's declination is to the sine of the ascensional difference.

(572) By comparing the triangle ABS with the triangle AGC, it is evident that when the sun's declination and the latitude of the place are of the same name, the ascensional difference in time subtracted from six hours gives the time of sun-rising, and added to six hours gives the time of setting: but when the latitude of the place and the sun's declination are of contrary names, the ascensional difference added to six hours gives the time of sun-rising, and subtracted gives the time of setting.

In the present case the sun rises at $3^h \, 47^m \, 32^s$, and sets at $8^h \, 12^m \, 28^s$; double the time of rising gives the length of the night, and double the time of setting, gives the length of the day.

When the sun's declination is equal to, or greater than, the complement of the latitude; viz. when Q ♋ is equal to or exceeds QO (a circumstance that can never happen in latitudes lower than 66° 32′) the parallel of declination ♋ ♋ will not cut the horizon HO, and consequently the sun will never set at these times. The same observations hold true with stars whose co-declination or polar distance CN, is equal to, or less than, the latitude of the place, or elevation of the pole ON, and in the same hemisphere.

EXAMPLE II.

Required the amplitude, time of rising, culminating, and setting, of *Arcturus* at Greenwich, latitude 51° 28′ 39″ N. on the 1st of December, 1840, its right ascension being $14^h \, 8^m \, 24^s$†, and declination 20° 0′ 48″ N.

The sun's right ascension at apparent noon December 1st (*Naut. Alm.*) being $16^h \, 31^m \, 0^s$, and hourly difference $10^s \cdot 83$.

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† See Table VIII., at the end of the Book.

1. *To find the amplitude* AS, *in the triangle* ABS.

The star's declination being north, the triangle ABS must be used, wherein BAS = the co-lat. and BS the declination are given, to find AS.

$$\text{rad} \times \sin \text{BS} = \sin \overset{*}{\text{AS}} \times \sin \text{BAS}.$$

$$\therefore\ 10 + \log \sin \text{BS} - \log \sin \text{BAS} = \log \sin \text{AS}.$$

or, sin BAS = 38° 31′ 21″ : rad : : sin BS = 20° 0′ 48″ : sin AS = 33° 19′ 57″.

The amplitude is always of the same name as the declination; and since the variation of a star's declination is extremely small[†], the same star may be considered as having constantly the same amplitude, or to rise and set in the same point of the horizon, during the whole year, in the same latitude.

2. *To find the ascensional difference* AB, *in the triangle* ABS.

$$\text{rad} \times \sin \overset{*}{\text{AB}} = \cot \text{BAS} \times \tan \text{BS}.$$

$$\therefore\ \log \cot \text{BAS} + \log \tan \text{BS} - 10 = \log \sin \text{AB}.$$

or, rad : cot BAS = 38° 31′ 21″ : : tan BS = 20° 0′ 48″ : sin AB = 27° 13′ 43″ = $1^h\ 48^m\ 55^s$, and $6^h + 1^h\ 48^m\ 55^s = 7^h\ 48^m\ 55^s$ the arc BADÆ, or half the time of the star's continuance above the horizon.[‡]

3. *Find the time of the* ✱*'s culminating*, Prob. VIII. (561).

		h	m	s
Then,	From the time of the ✱'s culm. (561)	21	33	30
	Take the semi-diurnal arc - - -	7	48	55
	Time of the ✱'s rising, December 1st	13	44	35
And,	To the time of the ✱'s culminating (561)	21	33	30
	Add the semi-diurnal arc - - - -	7	48	55
	Sum	29	22	25

Then $29^h\ 22^m\ 25^s - 24^h = 5^h\ 22^m\ 25^s$, time of setting, December 2nd.

[†] The amplitude of this star on the same day, in the year 1822, was 33° 30′ 50″, making a variation of only 10′ 53″ in its amplitude during 18 years. In the fourth edition of this work (which was published in the year 1820) the places of the stars were corrected to the year 1822.

[‡] When the declination of the star and the latitude of the place are of the same name, the ascensional difference added to 6 hours gives the semi-diurnal arc; but when of contrary names, the ascensional difference subtracted from 6 hours, leaves the semi-diurnal arc.

(573) On account of the small change in the declinations of the stars, the same star, in any latitude, may be considered as having the same ascensional difference during the year; consequently the diurnal difference of the same star's rising, culminating, and setting, in the same latitude, is *nearly* equal to the diurnal difference of the sun's right ascension.

The sun's mean daily motion is $3^{\mathrm{m}}\ 56^{\mathrm{s}}\cdot555$, the daily difference between the rising, southing, and setting of any fixed star, in the same latitude.

PRACTICAL EXAMPLES.

1. Given the sun's amplitude = 39° 48′ N., his declination = 23° 28′ N.; to find the latitude of the place, time of sun rising and setting, and the length of the day and night.

Answer.
The latitude = 51° 32′ N.
Sun rises at $3^{\mathrm{h}}\ 47^{\mathrm{m}}\ 32^{\mathrm{s}}$.
Sun sets at $8^{\mathrm{h}}\ 12^{\mathrm{m}}\ 28^{\mathrm{s}}$.
Length of day = $16^{\mathrm{h}}\ 24^{\mathrm{m}}\ 56^{\mathrm{s}}$, length of night = $7^{\mathrm{h}}\ 35^{\mathrm{m}}\ 4^{\mathrm{s}}$.

2. Required at what time *Sirius, the Dog Star*, will rise, culminate, and set at Greenwich, lat. 51° 28′ 39″ N. on the 18th of December 1840.

The right ascension of *Sirius* being $6^{\mathrm{h}}\ 38^{\mathrm{m}}\ 8^{\mathrm{s}}$, and declination 16° 30′ 12″ S.; the sun's right ascension at apparent noon (*Naut. Alm.*) being $17^{\mathrm{h}}\ 45^{\mathrm{m}}\ 42^{\mathrm{s}}$, and hourly diff. $11^{\mathrm{s}}\cdot11$.

Answer.
Ascensional difference = 21° 50′ 58″ = $1^{\mathrm{h}}\ 27^{\mathrm{m}}\ 24^{\mathrm{s}}$.
Semi-diurnal arc = $4^{\mathrm{h}}\ 32^{\mathrm{m}}\ 36^{\mathrm{s}}$.
Sirius will be on the meridian at $12^{\mathrm{h}}\ 50^{\mathrm{m}}\ 3^{\mathrm{s}}$.
———— rise at $8^{\mathrm{h}}\ 17^{\mathrm{m}}\ 27^{\mathrm{s}}$, in the evening.
———— set at $5^{\mathrm{h}}\ 22^{\mathrm{m}}\ 39^{\mathrm{s}}$ next morning.

Star's declination being south, the triangle AGC is used.

3. Required the amplitude, time of rising, setting, and culminating of *Aldebaran* at Greenwich, on the 20th of November 1840.

The right ascension of *Aldebaran* being = $4^{\mathrm{h}}\ 26^{\mathrm{m}}\ 48^{\mathrm{s}}$, declination = 16° 11′ 1″ N., the sun's right ascension at apparent noon (*Naut. Alm.*) being $15^{\mathrm{h}}\ 44^{\mathrm{m}}\ 7^{\mathrm{s}}$, and hourly diff. $10^{\mathrm{s}}\cdot5$.

Answer.
Ascensional difference = 21° 22′ 51″ = $1^{\mathrm{h}}\ 25^{\mathrm{m}}\ 31^{\mathrm{s}}$.
Semi-diurnal arc $7^{\mathrm{h}}\ 25^{\mathrm{m}}\ 31^{\mathrm{s}}$.
Aldebaran culminates at $12^{\mathrm{h}}\ 40^{\mathrm{m}}\ 28^{\mathrm{s}}$.
———— rises at $5^{\mathrm{h}}\ 14^{\mathrm{m}}\ 57^{\mathrm{s}}$ evening.
———— sets at $8^{\mathrm{h}}\ 5^{\mathrm{m}}\ 59^{\mathrm{s}}$ next morning.
The amplitude = 26° 35′ 2″ towards the N.

4. Given the latitude = 51° 32′ N., the sun's amplitude =

39° 48′ N. of the east; required the sun's declination, ascensional difference, time of rising, setting, &c.

Answer.
Sun's declination $=23^\circ$ 28′ N.
Ascensional diff. $=33$ 7
Sun rises at 3^h 47^m 32^s
Sun sets at 8 12 28

PROBLEM III. (*Plate III. Fig.* 1.)

(574) *The latitude of the place, and the sun's (or a star's) declination being given, to find the altitude and azimuth, &c. at six o'clock.*

In the right-angled spherical triangle ADS.

1. AD = the complement of DO the sun's or star's azimuth from the north.
2. DS = the altitude at six o'clock.
3. AS = the declination.
4. SAD = NO the elevation of the pole, or latitude of the place.

Any two of the above quantities being given, the rest may be found.

The triangle S *Zenith* N is complemental to the triangle ADS; it therefore follows, that the several things concerned in the problem may be determined in this triangle, right-angled at N.

As the meridian of the place, with respect to the sun, is called the 12 o'clock hour circle, so that circle NSAS which cuts it at right angles, being 6 hours from the meridian, is called the 6 o'clock hour circle.

The sun and stars are on the eastern half of this circle 6 hours before they come to the meridian, and on the western half 6 hours after they have passed the meridian.

EXAMPLE I.

In latitude 51° 32′ North, when the sun has 19° 39′ North declination; required his altitude and azimuth at 6 o'clock.

1. *To find the altitude* DS, *in the triangle* ADS.

$$\textbf{rad} \times \textbf{sin } \overset{*}{\textbf{DS}} = \textbf{sin A} \times \textbf{sin AS}.$$

$$\therefore \textbf{log sin A} + \textbf{log sin AS} - 10 = \textbf{log sin DS}.$$

$$\textbf{or, rad : sin AS} = 19^\circ\, 39' \textbf{ :: sin SAD} = 51^\circ\, 32' \textbf{ : sin DS} = 15^\circ\, 16'.$$

2. *To find* AD *the co-azimuth, in the triangle* ADS.

$$\textbf{rad} \times \textbf{cos A} = \textbf{cot AS} \times \textbf{tan } \overset{*}{\textbf{AD}}.$$

$$\therefore 10 + \textbf{log cos A} - \textbf{log cot AS} = \textbf{log tan AD}.$$

$$\textbf{or, cot AS} = 19^\circ\, 39' \textbf{ : rad :: cos SAD} = 51^\circ\, 32' \textbf{ : tan AD} =$$

12° 31′ 23″. Hence, 90° − 12° 31′ 23″ = 77° 28′ 37″ = OD the azimuth from the north.

EXAMPLE II.

At what time will *Arcturus* appear upon the six o'clock hour line at Greenwich, latitude 51° 28′ 39″ N., and what will its altitude and azimuth be on the first of April, 1840?

Its right ascension being $14^{h}\ 8^{m}\ 23^{s}$, and declination 20° 1′ N. (Table VIII.), and the sun's right ascension at appt. noon on the first of April (*Naut. Alm.*) being $0^{h}\ 43^{m}\ 32^{s}$, and hourly diff. $9^{s}\cdot 1$.

1. *To find the time of the star's culminating.*

By Problem VIII. (561) *Arcturus* will culminate at $13^{h}\ 22^{m}\ 49^{s}$.

Consequently it will be upon the six o'clock hour circle at $(13^{h}\ 22^{m}\ 49^{s} - 6^{h} =)\ 7^{h}\ 22^{m}\ 49^{s}$ in the evening, in the eastern hemisphere; and at $(13^{h}\ 22^{m}\ 49^{s} + 6^{h} =)\ 19^{h}\ 22^{m}\ 49^{s}$, or $7^{h}\ 22^{m}\ 49^{s}$ the next morning, in the western hemisphere.

2. *To find the altitude* DS, *in the triangle* ADS.

$$\text{rad} \times \sin \overset{*}{\text{DS}} = \sin \text{SAD} \times \sin \text{AS}.$$

$$\therefore \log \sin \text{SAD} + \log \sin \text{AS} - 10 = \log \sin \text{DS}.$$

or, rad : sin AS = 20° 1′ :: sin SAD = 51° 28′ 39″ : sin DS = 15° 32′.

3. *To find the azimuth* AD *from the east, in the triangle* ADS.

$$\text{rad} \times \cos \text{SAD} = \cot \text{AS} \times \tan \overset{*}{\text{AD}}.$$

$$\therefore 10 + \log \cos \text{SAD} - \log \cot \text{AS} = \log \tan \text{AD}.$$

or, cot AS = 20° 1′ : rad :: cos SAD = 51° 28′ 39″ : tan AD = 12° 47′ 2″. Hence OD = 77° 12′ 58″ the azimuth from the north.

(575) On account of the small change in the right ascension and declination of a star, it may, without material error, be said to have the same altitude and azimuth every time it arrives at the six o'clock hour circle; and the difference of the times it arrives there, may be considered as equal to the diurnal difference of the sun's right ascension, viz. about four minutes (573).

PRACTICAL EXAMPLES.

1. At what time will *Aldebaran* appear upon the six o'clock hour circle at Greenwich, latitude 51° 28′ 39″ N., and what will its altitude and azimuth be on the 20th November 1840?

The right ascension being $4^h\ 26^m\ 48^s$, and declination 16° 11′ 1″ N. (Table VIII.); also the sun's right ascension at appt. noon on the 20th Nov. (*Naut. Alm.*) $=15^h\ 44^m\ 7^s$, and hourly diff. $10^s{\cdot}5$.

Answer. Aldebaran culminates at $12^h\ 40^m\ 28^s$ (561), therefore it is on the six o'clock hour circle in the eastern hemisphere at $6^h\ 40^m\ 28^s$ in the evening, and in the western hemisphere at $6^h\ 40^m\ 28^s$ the next morning.

Its altitude at each time being 12° 35′ 43″, and the azimuth 79° 45′ 15″ from the North.

2. Required the sun's altitude and azimuth at 6 o'clock, in latitude 51° 32′ N. when his declination is 15° 23′ North?

Answer. Altitude = 11° 59 17″, azimuth 80° 17′ 17″ from the North.

3. Given the sun's declination 19° 39′ North, his altitude at 6 equal 15° 16′; required the azimuth and the latitude?

Answer. Azimuth = 77° 28′ 37″ from the North, latitude = 51° 32′ North.

4. Given the sun's declination = 19° 39′ North, his azimuth at 6 in the morning N. 77° 28′ 37″ E.; required his altitude and the latitude?

Answer. Altitude = 15° 16′, latitude 51° 32′ North.

5. At what time will *Regulus* appear upon the six o'clock hour circle NSAS at Greenwich, latitude 51° 28′ 39″ North, and what will its altitude and azimuth be on the 6th February, 1840?

The right ascension being $9^h\ 59^m\ 51^s$, and declination 12° 44′ 45″ N. (Table VIII.), also the sun's right ascension at apparent noon on the same day (*Naut. Alm.*) $=21^h\ 17^m\ 14^s$, and hourly diff. $10^s{\cdot}02$.

Answer. Regulus culminates at $12^h\ 40^m\ 30^s$ (561), is on the 6 o'clock hour circle, at $6^h\ 40^m\ 30^s$ in the evening in the eastern hemisphere, and at $6^h\ 40^m\ 30^s$ the next morning in the western hemisphere. Its altitude = 9° 56′ 23″, and its azimuth = 81° 58′ 51″ from the North.

6. At what time will *Castor* appear upon the 6 o'clock hour circle at Greenwich, latitude 51° 28′ 39″ N., and what will its altitude and azimuth be on the 1st of December 1840?

The right ascension being $7^h\ 24^m\ 26^s$ and declination 32° 13′ 49″ N. (Table VIII.), also the sun's right ascension on the 1st of December at app. noon (*Naut. Alm.*) $=16^h\ 31^m\ 0^s$, and hourly diff. $10^s{\cdot}83$.

Answer. Castor culminates at $14^h\ 50^m\ 45^s$ (561), will be on the six o'clock hour circle at $8^h\ 50^m\ 45^s$ in the evening, in the eastern hemisphere, and at $8^h\ 50^m\ 45^s$ next morning in the western hemisphere; altitude = 24° 39′ 41″, azimuth = 68° 33′ 39″ from the north.

PROBLEM IV. (*Plate III. Fig.* 1.)

(576) *The latitude of a place, and the declination of the sun (or of a star) being given, to find the altitude, and the time when it will be due east and west.*

In the right-angled spherical triangle AGS.

1. AG = the time from six o'clock when the sun is due east or west. Or it is the complement of the time from a star's culminating.

2. AS = the altitude of the sun, or a star, above the horizon, when on the prime vertical ZSAN.

3. GS = the sun's or star's declination, of the same name as the latitude.

4. The angle GAS = the latitude of the place; for the elevation HÆ of the equinoctial is always equal to the complement of the latitude.

Any two of these quantities being given, the rest may be found.

The triangle SN *Zenith*, right-angled at *Zenith*, is complemental to the triangle AGS.

EXAMPLE I.

In latitude 51° 32′ North, when the sun's declination is 19° 39′ North, required his altitude, and the time when he will appear due east or west.

1. *To find the altitude* AS *in the triangle* AGS.

$$\text{rad} \times \sin \text{GS} = \sin \text{GAS} \times \sin \overset{*}{\text{A}}\text{S}.$$

$$\therefore\ 10 + \log \sin \text{GS} - \log \sin \text{GAS} = \log \sin \text{AS}.$$

or, $\sin \text{GAS} = 51° 32' : \sin \text{GS} = 19° 39' :: \text{rad} : \sin \text{AS} = 25° 26'$.

2. *To find* AG *the hour from six, in the triangle* AGS.

$$\text{rad} \times \sin \overset{*}{\text{A}}\text{G} = \tan \text{GS} \times \cot \text{GAS}.$$

$$\therefore\ \log \tan \text{GS} + \log \cot \text{GAS} - 10 = \log \sin \text{AG}.$$

or, $\text{rad} : \tan \text{GS} = 19° 39' :: \cot \text{GAS} = 51° 32' : \sin \text{AG} = 16° 28' 49''$.

Now $16° 28' 49'' = 1^h\ 5^m\ 55^s$ the time from six, hence the sun will be east at $7^h\ 5^m\ 55^s$ in the morning, or west at $4^h\ 54^m\ 5^s$ in the afternoon.

(577) As the declination increases, the altitude and hour from six increase, and when the declination vanishes, the sun appears in the horizon upon the prime vertical. When the latitude is equal to 90°, the sun's altitude upon the prime vertical is equal to his declination.

When the latitude of the place is less than the declination, the sun never appears on the prime vertical.

EXAMPLE II.

At what time will *Arcturus* appear upon the prime vertical, or be due east or west from Greenwich, latitude 51° 28′ 39″ N., on the first of April, 1840; and what will be its altitude?

Its right ascension being $14^h\ 8^m\ 23^s$, and declination 20° 1′ 0″ North (Table VIII.), the sun's right ascension on the same day at app. noon (*Naut. Alm.*) being $0^h\ 43^m\ 32^s$, and hourly diff. $9^s{\cdot}1$.

1. *To find the altitude* AS *in the triangle* AGS.

$$\text{rad} \times \sin \text{GS} = \sin \text{GAS} \times \sin \overset{*}{\text{AS}}.$$

$$\therefore\ 10 + \log \sin \text{GS} - \log \sin \text{GAS} = \log \sin \text{AS}.$$

or, sin GAS = 51° 28′ 39″ : rad : : sin GS = 20° 1′ : sin AS = 25° 56′ 44″.

2. *To find* AG, *the complement of the time from the star's culminating, in the triangle* AGS.

$$\text{rad} \times \sin \overset{*}{\text{AG}} = \tan \text{GS} \times \cot \text{GAS}.$$

$$\therefore\ \log \tan \text{GS} + \log \cot \text{GAS} - 10 = \log \sin \text{AG}.$$

or, rad : tan GS = 20° 1′ : : cot GAS = 51° 28′ 39″ : sin AG = 16° 51′ 30″.

Hence 90° − 16° 51′ 30″ = 73° 8′ 30″, which reduced to time = $4^h\ 52^m\ 34^s$, the time from the star's culminating. The star comes to the meridian at $13^h\ 22^m\ 49^s$ (561); therefore $13^h\ 22^m\ 49^s - 4^h\ 52^m\ 34^s = 8^h\ 30^m\ 15^s$, the time in the evening when the star will appear due east; and $13^h\ 22^m\ 49^s + 4^h\ 52^m\ 34^s = 18^h\ 15^m\ 23^s$, or $6^h\ 15^m\ 23^s$ next morning, when the star will appear due west.

(578) The height of the same star upon the prime vertical in any place, is always nearly the same, for the reasons already assigned (575); and the difference of the times of its coming to the prime vertical, will be equal to the difference of the times of its culminating, which is nearly equal to the diurnal difference of the sun's right ascension.

PRACTICAL EXAMPLES.

1. The sun's declination being 19° 39′ north, and having 25° 26′ altitude upon the prime vertical; required the latitude of the place, and the hour of the day.

Answer. Latitude=51° 32′ N., time $7^h\ 5^m\ 55^s$ or $4^h\ 54^m\ 5^s$, according as the observation was made in the forenoon or afternoon.

2. At what time will *Aldebaran* appear due east or west at Greenwich, latitude 51° 28′ 39″ N. on the 20th of November 1840, and what will its altitude be at that time?

Its right ascension being $4^h\ 26^m\ 48^s$ and declination 16° 11′ 1″ N. (Tab. VIII.), the sun's right ascension at app. noon 20th November (*Naut. Alm.*) $15^h\ 44^m\ 7^s$, and hourly diff. $10^s{\cdot}5$.

Answer. *Aldebaran* culminates at $12^h\ 40^m\ 28^s$ (561), is due east at $7^h\ 33^m\ 54^s$, due west at $17^h\ 47^m\ 2^s$, and its altitude=20° 52′ 13″. The arc AG=13° 21′ 28″.

3. The sun's declination being 19° 39′ N., and he was observed to be due west at $4^h\ 54^m$ in the afternoon; required his altitude, and the latitude of the place.

Answer. Altitude=25° 26′, latitude=51° 32′ N.

4. At what time will *Regulus* appear due east or west at Greenwich, latitude 51° 28′ 39″ N. on the 6th of February, 1840, and what will be its altitude at that time?

Its right ascension being $9^h\ 59^m\ 51^s$, and declination 12° 44′ 45″ N. (Tab. VIII.), the right ascension of the sun at app. noon on the 6th of February (*Naut. Alm.*) being $21^h\ 17^m\ 14^s$, and hourly diff. $10^s{\cdot}02$.

Answer. *Regulus* culminates at $12^h\ 40^m\ 30^s$, is due east at $7^h\ 22^m$, due west at $17^h\ 59^m$, and its altitude=16° 22^s 46″. The arc AG=10° 22′ 26″.

5. In latitude 51° 32′ North, the sun's altitude when on the prime vertical was 25° 26′; required his declination, and the hour of the day.

Answer. Declination=19° 39′ North.
Time $7^h\ 5^m\ 55^s$ or $4^h\ 54^m\ 5^s$, according as the observation was made in the forenoon or afternoon.

PROBLEM V. (*Plate III. Fig.* 1.)

(579) *Given the latitude of the place, and the sun's altitude, when on the equinoctial, to find his azimuth and the hour of the day.*

In the right-angled spherical triangle A*d*G.

1. AG=the complement of ÆG the hour from noon, being an arc of the equinoctial, on which G represents the sun's place between noon and 6 o'clock.

2. A*d*=the complement of H*d* the sun's azimuth from the meridian, or south point of the horizon.

3. *d*G=the sun's altitude.

4. GA*d* = the complement of the latitude, being measured by the arc HÆ the elevation of the equator above the horizon.

Any two of these quantities being given, the rest may be found.

This triangle is complemental to the triangle GÆ *Zenith*; for GZ is the complement of *d*G the altitude, ÆG is the hour from noon, ÆZ the latitude, and the angle ÆZG, measured by the arc H*d*, is the sun's azimuth from the meridian.

EXAMPLE.

In latitude 51° 32′ North, when the sun was on the equinoctical, his altitude was 22° 15′; required his azimuth, and the hour of the day.

1. *To find* A*d the complement of the azimuth in the triangle* A*d*G.

$$\text{rad} \times \sin \overset{*}{\text{A}d} = \tan d\text{G} \times \cot \text{GA}d.$$

$$\therefore \log \tan d\text{G} + \log \cot \text{GA}d - 10 = \log \sin \text{A}d.$$

or, rad : tan *d*G = 22° 15′ : : cot GA*d* = 38° 28′ : sin AD = 31° 0′.

Hence the azimuth = 59° from the south towards the east or west, according as the time is before or after noon.

2. *To find* AG, *the hour from* 6, *in the triangle* A*d*G.

$$\text{rad} \times \sin d\text{G} = \sin \overset{*}{\text{AG}} \times \sin \text{GA}d.$$

$$\therefore 10 + \log \sin d\text{G} - \log \sin \text{GA}d = \log \sin \text{AG}.$$

or, sin GA*d* = 38° 28′ : rad : : sin *d*G = 22° 15′ : sin AG = 37° 29′ 46″, which reduced into time = $2^{h}\ 29^{m}\ 59^{s}$; this added to, and subtracted from, six hours, gives $8^{h}\ 29^{m}\ 59^{s}$, or $3^{h}\ 30^{m}\ 1^{s}$, according as the sun is in the eastern or western hemisphere.

PRACTICAL EXAMPLES.

1. In latitude 51° 32′ N., when the sun has no declination, what is his altitude and azimuth at three hours and a half from noon?

Answer. The altitude = 22° 15′, and azimuth = S. 59° E. or W.

2. At the time of the equinox the sun's altitude was found to be 22° 15′, and its azimuth S. 59° E.; required the hour of the day, and the latitude of the place?

Answer. Time $8^{h}\ 30^{m}$, and latitude 51° 32′ North.

3. In latitude 51° 32′ North, the sun being in the equinox, there is given the angle (AG*d*) which the vertical circle passing

through the sun forms with the equinoctial = 57° 46′; to find the hour of the day and the altitude and azimuth of the sun.

Answer. Time $8^h\ 29^m\ 53^s$, or $3^h\ 30^m\ 7^s$, altitude 22° 14′, and azimuth S. 59° 2′ E. or W.

PROBLEM VI. (*Plate III. Fig.* 2.)

(580) *The difference of longitudes between two places, both in one parallel of latitude being given, to find the distance between them.*

Before we give any examples to this problem, it will be necessary to remark, that all the meridians on the terrestrial globe are great circles meeting each other in the poles of the equator. Therefore the meridional distances of places vary as the latitudes; that is, the distance between the meridians PE and PW is greater on the equator WOQE (supposing WOQE to be a part of the equator) than it is on the parallel of latitude MN, and greater at MN than at *l*L, &c.

Longitude is always counted on the equator, where a degree is reckoned 60 geographical miles, hence a degree of longitude in the parallel MN must be less than 60 miles. This shows that what is called meridional distance in navigation, is always less than the longitude, except the ship sail on the equator.

Since all the meridians PW, PO, PQ, &c. cut the equator at right angles (291), they must necessarily cut all the parallels of latitude at right angles.

The arcs of the equator WO, OQ, &c. are the measures of the spherical angles WPO, OPQ, &c. at the pole (286).

If W*m* or O*n* represent the latitude of two places on the same parallel, then *m*P or *n*P will be their co-latitudes.

The arcs WO and *mn* are similar (297.) and (299); hence *mr* : *mn* : : CW : WO, viz. CW : *mr* : : WO : *mn*. But *m*V is the sine of the latitude W*m*, *mr* = VC is its cosine, and CW is the radius of the sphere. Therefore,

Radius : cosine of *any latitude* : : the length of any arc on the equator : the length of a similar arc in *that latitude.*

PRACTICAL EXAMPLES.

1. Suppose a ship in latitude 40° north, sail due east until her difference of longitude be 60°; required the distance sailed in that parallel.

rad : cos 40° : : diff. long = 3600 miles : the distance 2757·7 miles.

EXAMPLE II.

2. Suppose a ship, in latitude 40° north, sail due east 2757·7 miles, what is her difference of longitude?

Answer. 3600 miles.

(581) By the first of these examples, geographers have calculated tables showing how many miles answer to a degree of longitude for each degree of latitude; thus in latitude 45°, a degree of longitude is 42·43 miles; in the latitude of London 51° 32′, the length of a degree is 37·32 miles, &c.

3. If a ship sail 2757·7 miles directly eastward or westward, so as to alter her longitude 60°, what latitude is she in?

Answer. 40° north or south.

4. Suppose a ship to sail from latitude 40° north to latitude 10° north, till her difference of longitude be 60°, and it is required to find her departure, course, and distance sailed.

Here *l*L the parallel of 40° is the latitude left, A*a* the parallel of 10° is the latitude arrived at, and MN the parallel of 25° is the middle latitude between them; hence MP is the complement of the middle latitude, and MN is the meridional distance, or the true departure nearly *; PW is a quadrant or 90°, and WE is the difference of longitude.

Hence the following CONSTRUCTION (*Plate III. Fig.* 3.):

1. With the chord of 60°, which is equal to the sine of 90°, and centre P, describe the arc WE, on which set off the difference of longitude (60° = 3600 miles); draw PW and PE.

2. With the sine of (65°) the complement of the middle latitude, taken from the line of sines on Gunter's scale, describe the arc MN, and draw the straight line MN.

3. Set off the difference of latitude (30° = 1800 miles) from P to *m*; draw *mn* perpendicular to P*m*, and equal to MN; lastly, draw P*n*, and the construction is finished. The several lines are named as in the figure.

OR THUS.

Make the angle MPN (Plate III. Fig. 4.) = the complement of the middle latitude, and PN the difference of longitude, then MN will be the departure or meridional distance; for, in the plane triangle MPN, *per* Plane Trigonometry, rad : PN : : sin ∠ MPN : MN, the same proportion as is used in the first example. The triangle *m*P*n* is the same as before.

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* See the Theory of Navigation, Book IV. Chapter II.

OR THUS. (*Plate III. Fig.* 5.)

Draw the meridian AP, in which assume any point P; make the angle at P equal to the complement of the middle latitude, in the opposite quadrant to which the ship sails; set off PN equal to the difference of longitude, and draw MN perpendicular to AP; lastly, set off the difference of latitude from M to A, and draw AN; then the names of the several lines are as expressed in the figure.

BY CALCULATION. (*Plate III. Fig.* 3, 4, *or* 5.)

rad : diff. long : : sin comp. midd. lat. : merid. dist. or departure $=3262{\cdot}7$.

diff. lat. : rad : : departure : tan course $=61^\circ\ 7'$.

sin course : dep. : : rad : distance $=3726{\cdot}2$.

CHAP. XII.

THE APPLICATION OF OBLIQUE-ANGLED SPHERICAL TRIANGLES TO ASTRONOMICAL PROBLEMS.

PROBLEM VII. (*Plate III. Fig.* 1.)

(582) *Given the sun's declination, and the latitude of the place, to find the apparent time of day-break in the morning, and the end of twilight in the evening.*

1. In the oblique-angled spherical triangle ⊙ *Zenith* N.

1. ⊙ N = the sun's distance from the north pole.
2. ⊙ *Zenith* = the sun's distance from the zenith.
3. *Zenith* N = the complement of the latitude.
4. ⊙N *Zenith*, measured by the arc *a*Æ = the time from noon.

OR, 2. In the oblique-angled spherical triangle ⊙ *Nadir* S.

1. ⊙ S = the sun's distance from the south pole.
2. ⊙ *Nadir* = the sun's distance from the nadir.
3. S *Nadir* = the complement of the latitude.
4. ⊙ S *Nadir*, measured by the arc *a*Q, = the time from midnight.

EXAMPLE.

Given the latitude of the place 51° 32′ N., and the sun's declination 10° N., to find the time of day-break in the morning, and the end of twilight in the evening.

In the triangle ⊙ *zenith* N.

⊙ N $=80^\circ$ the sun's distance from the north pole $=90^\circ-10^\circ$.

⊙ *Zenith* $=108^\circ$ the sun's distance from the zenith $=18^\circ+90^\circ$.

Zenith N $=38^\circ\ 28'$ the complement of the latitude.

To find the angle *Zenith* N⊙, measured by the arc *a*Æ, = the time from noon.

⊙ N - =	80° 0′	cosec ZN = 38°28′	·20617 } reject
⊙ *Zenith* =	108 0	cosec ⊙N = 80 0	·00665 } indices.
Zenith N =	38 28	sin - - 113 14	9·96327
		sin - - 5 14	8·96005
2 |	226 28		
		2 |	19·13614
Half sum -	113 14		
⊙ *Zenith* =	108 0	cos 68° 17′ 28″ -	9·56807
		2	
Remainder -	5 14		

136 34 56 = ∠ ⊙NZ = $9^h 6^m 20^s$, time from noon* when the sun is 18° below the horizon. Consequently the day breaks at 2^h 53^m 40^s in the morning, and twilight ends at 9^h 6^m 20^s in the evening, supposing the sun's declination to undergo no change between the beginning of twilight in the morning, and the ending thereof at night, being about 18 hours.

The same might have been found from the triangle ⊙S*Nadir*, for S⊙ = 90° + 10° = 100°, *Nadir* ⊙ = 90° − 18° = 72°, and *Nadir* S = comp. lat. = 38° 28′. Then, by the method above, find the angle ⊙ S *Nadir* (measured by the arc *a*Q) = 43° 25′ = 2^h 53^m 40^s as before, the time from midnight, when the sun is 18° below the horizon.

(583) When the declination of the sun, the latitude and declination being of the same name, is greater than the difference between the complement of latitude and 18°, the parallel of declination (♋ SSS ♋) will not cut the parallel of 18° (T⊙W) below the horizon: consequently there will be no real night at these times, but constant day or twilight; as is the case at London from the 22d of May to the 21st of July.

(584) Since the sun sets more obliquely at some times of the year than at others, it necessarily follows that he will be longer in descending 18° below the horizon at one season than at another.

When the sun is on the same side of the equinoctial as the visible pole, the duration of twilight will constantly increase as he approaches that pole, till he enters the tropic, at which time the duration of twilight will be the longest. It will then decrease till some time after the sun passes the equinox, but will increase again before he arrives at the other tropic; therefore there must be a point between the tropics, where the duration of twilight is the shortest, which point may be found by the following proportion; viz.

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* The ∠ ⊙ NZ represents the time from *apparent noon*, that is, from the time the ⊙ comes to the meridian, and not the time from 12 o'clock, as shown by a well-regulated clock or time-piece. If the equation of time, given in page I. of each month in the *Nautical Almanac*, be applied to this *apparent time* by addition or subtraction, as there directed, you will obtain the mean time, or that shown by a well-regulated clock or chronometer.

rad : tan 9° : : sine of the latitude : sine of the sun's declination.[*]

The declination must always be of a contrary name with the latitude.

PRACTICAL EXAMPLES.

1. Given the sun's declination 10° south, and the latitude of the place 51° 32′ N. to find the time of day-break in the morning, and the end of twilight in the evening.

Answer. The $\angle \odot$s *Nadir* $= 73° 35' 34'' = 4^h\ 54^m\ 22^s$ the time from midnight. Hence day breaks at $4^h\ 54^m\ 22^s$ in the morning, and twilight ends at $7^h\ 5^m\ 38^s$ in the evening; admitting the sun's declination to be constant for one day.

2. Given the sun's declination 23° 28′ S. and the latitude of the place 51° 32′ N. to find the time of day-break in the morning, and the end of twilight in the evening.

Answer. The $\angle \odot$ s *Nadir* $= 90° 16' = 6^h\ 1^m\ 4^s$ the time from midnight when the $\odot$ is 18° below the horizon. Consequently day breaks at $6^h\ 1^m\ 4^s$, and twilight ends at $5^h\ 58^m\ 56^s$, on the shortest day at London.

3. In the latitude of London, 51° 32′ N., what time does the day break when the sun has 11° 30′ N. declination?

Answer. $2^h\ 41^m$, rejecting seconds.

4. Required the beginning and ending of twilight at London, lat. 51° 32′ N., when the sun's declination is 15° 12′ N.

Answer. $2^h\ 4^m\ 11^s$ and $9^h\ 55^m\ 49^s$.

5. Required the time the twilight begins and ends at London, latitude 51° 32′, when the sun has 11° 30′ south declination.

Answer. $5^h\ 2^m$ and $6^h\ 58^m$, rejecting seconds.

6. At London, latitude 51° 32′ N., required the sun's declination, day of the month, and duration of twilight when it is the shortest.

Answer. Sun's declination $= 7° 7' 25''$ south, answering to March 2nd and October 11th. The duration of twilight is

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[*] Vide Dr. Gregory's Astron. page 328. Vince's Astronomy, vol. i. page 18.&c. Also, Leybourn's edition of the Ladies' Diary, vol. iv., from page 314 to 339, where the subject is amply discussed.

$1^h\ 56^m\ 32^s$; this is found by taking the difference between the time of sun-rise and day-break.

PROBLEM VIII. (*Plate III. Fig.* 6.)

(585) *Given the day of the month, the latitude of the place, the horizontal refraction, and the sun's horizontal parallax, to find the apparent time of his centre appearing in the Eastern and Western part of the horizon.*

Example. Given the sun's declination at noon on the 21st of June 1840 = 23° 28′, the latitude of the place = 51° 32′ N., the horizontal refraction = 33′, the sun's horizontal parallax = 8″; to find the apparent time of his rising and setting.

This example being given for the longest day, the sun's declination may be considered the same at his rising as at noon; for the declination does not vary above 5″ in 24 hours at this time, though near the equinoxes it varies about 1′ in an hour.

Let s be the true point of the sun's rising, and *b* that of his apparent* rising, then *b*⊙ = 33′ − 8″ = 32′ 52″ the distance of the sun's centre below the horizon; hence the true zenith distance ZS will be increased, by refraction to Z⊙, but the declination will remain the same, for NS = N⊙.

In the oblique-angled triangle ZN⊙.

Z⊙ = 90° 32′ 52″ app. dist. of ⊙'s centre from the zenith.
N⊙ = 66 32 0 app. dist. of ⊙'s centre from the north pole.
ZN = 38 28 0 the complement of the latitude.

2)195 32 52	ZN = 38° 28′ 0″ cosec ·20617
	N⊙ = 66 32 0 cosec ·03749
½ Sum 97 46 26	Half sum = 97 46 26 sin 9·99599
Z⊙ = 90 32 52	Remainder = 7 13 34 sin 9·09963
Rem. 7 13 34	2)19·33928
	62° 8′ 15″ cos 9·66964
	2
	∠ZN⊙ = 124 16 30 = $8^h\ 17^m\ 6^s$ the

apparent time from noon when the sun's centre appears in the western part of the horizon. Hence the apparent time of

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* The apparent time of rising and setting of the heavenly bodies always differs from the true time, for they are elevated by refraction (165), and depressed by parallax (184).

All the heavenly bodies, when in the horizon, appear 33′ above their true places, by the effect of refraction (see Table IV.); but they are depressed by the [*] sun's apparent rising is $3^h\ 42^m\ 54^s$, and the time of apparent setting is $8^h\ 17^m\ 6^s$. The time of the sun's true rising and setting on this day, has been determined (page 287) to be $3^h\ 47^m\ 32^s$ and $8^h\ 12^m\ 28^s$; hence the apparent day is $9^m\ 16^s$ longer than the true day.

(586) As the refraction causes an error in the rising and setting of the celestial objects, so it will cause an error in the amplitudes, as may be seen by comparing the triangle ABS with the triangle ADb. If the sun's amplitude be taken when his centre is in the visible horizon, an allowance depending on the horizontal refraction, parallax, height of the eye, and latitude of the place, should be applied to the observed amplitude, in order to obtain the true amplitude. This inconvenience may be avoided, by observing the amplitude when the altitude of the sun's lower limb is equal to 16′ + the dip of the horizon.

PRACTICAL EXAMPLES.

1. Required the apparent time of the sun's rising and setting at Glasgow, latitude 55° 32′ N., longitude 4° 15′ W., on the 12th of October 1840.

The sun's declination at apparent noon, for the meridian of Greenwich (*Naut. Alm.*), being 7° 31′ 50″ S., and on the 13th 7° 54′ 19″ S.

Answer. The sun's declination, when it is noon at Glasgow, is 7° 32′ 6″ S. (558). The ⊙'s declination being south, the triangle ZN⊙ will fall on the left hand of the prime vertical ZA. Then

$$\left.\begin{array}{l} Z\odot = 90^\circ\ 32'\ 52'' \\ N\odot = 97\ \ 32\ \ 6 \\ ZN\ = 34\ \ 28\ \ 0 \end{array}\right\}$$

With these three sides, find the hour $\angle ZN\odot = 79^\circ\ 53'\ 6'' = 5^h\ 19^m\ 32^s$, the time from noon when the sun sets; hence the sun rises at $6^h\ 40^m\ 28^s$.

Call these the *approximate* times of rising and setting, calculate the sun's declination for these times of rising and setting, which you will find to be 7° 27′ 7″ S., and 7° 37′ 5″ S.;

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parallax. Hence, by these causes combined, a body becomes visible when its distance from the zenith is equal to 90° + 33′ − horizontal parallax, and as the sun's horizontal parallax is about 8″ (see Table VI.) his centre will appear in the horizon when it is 32′ 52″ below it, or 90° 32′ 52″ from the zenith. If to this zenith distance, the sun's semi-diameter 15′ 45″ be added, the zenith distance of the centre for the setting of the upper limb will be had; if subtracted, that for the lower limb will be obtained; the former being 90° 48′ 37″, and the latter 90° 17′ 7″, and the respective times $8^h\ 19^m\ 20^s$ and $8^h\ 14^m\ 52^s$.

A star has no sensible parallax, and, therefore, will appear in the horizon when it is 33′ below it, or 90° 33′ from the zenith.

[*] The equation of time on this day at noon (see page I. of the *Nautical Almanac*) is $1^m\ 24^s$ at Greenwich; this, applied as directed in the *Nautical Almanac*, will give the mean time of the sun's rising and setting, as shown by a well-regulated clock or time-piece.

then, with the polar distance $N\odot = 97° 27' 7''$ repeat the operation, and the *apparent* time of apparent rising will be $6^h\ 39^m\ 58^s$; with the polar distance $N\odot = 97° 37' 5''$ repeat the operation a second time, and the *apparent* time of the sun's apparent setting will be $5^h\ 20^m\ 2^s$. To obtain the *mean* time of rising and setting, as shown by a well regulated clock, or time-piece, the equation of time must be applied as directed in the *Nautical Almanac.*

2. Required the apparent time of the rising and setting of the sun, in latitude 51° 32′ N., supposing his declination to be 23° 29′ S.

Answer. $8^h\ 8^m\ 2^s$ and $3^h\ 51^m\ 58^s$.

3. Required the apparent time of the rising and setting of the sun, in latitude 51° 32′ N. when his declination is 17° 32′ N., supposing it to undergo no change from sun-rise to sun-set.

Answer. $4^h\ 22^m\ 14^s$ and $7^h\ 37^m\ 46^s$.

4. Required the apparent time of the rising and setting of the sun in latitude 56° N., supposing the sun's declination at his rising to be 22° 34′ 35″ N., and at his setting 22° 30′ 7″ N.

Answer. $3^h\ 22^m\ 19^s$ and $8^h\ 36^m\ 59^s$.

PROBLEM IX. (*Plate III. Fig.* 7.)

(587) *Given the latitude of the place, the day of the month, the moon's horizontal parallax and refraction, to find the mean and apparent time of her rising.*

Required the time of the moon's rising at Greenwich, latitude 51° 28′ 39″ N., on the 4th of August 1840.

The horizontal parallax of the moon varying from 53′ to 62′, always exceeds the horizontal refraction; therefore, when the moon's centre appears in the horizon it is really above it, by a quantity equal to the horizontal parallax minus the refraction.

The moon's declination must be found as near to the time of her rising as possible, because it is subject to a considerable variation in the course of a few hours.

Let n be the place of the moon, when in the horizon, m the point where she becomes visible, and e her place on the meridian; then de will represent her change of declination from the time of her rising to the time of her transit.

The time of the moon's passage over the meridian at Greenwich is given in page IV. of the *Naut. Alm.*, and may be reduced to any other meridian by art. (562); reduce the declination to this time (558). With this declination, the latitude, &c. calculate the hour angle, which subtract from the time of the moon's passage over the meridian, and call the difference the *estimated* time of the moon's rising. Correct the moon's declination, and her horizontal parallax, to the estimated time of rising, repeat the operation, and correct the result for the daily retardation of the moon's transit, which will be the *mean* time of the moon's rising, to which apply the *equation of time* for the nearest noon, as directed in Page II. of the *Naut. Alm.*, for the *apparent* time.

SOLUTION.

1. *Find the time of the moon's passage over the meridian.*

The moon passes over the Greenwich meridian at $5^h\ 17^m{\cdot}4$ mean time. (*Naut. Alm.*, Page IV.)

2. *To find the ☽'s declination when she is on the meridian.*

☽'s dec. Aug. 4th, at 5^h (*Naut. Alm.*) - -	$18°\ 22'\ 2''$ S.
——————— 6^h ——————— - -	$18\ \ 33\ \ 9$ S.
Increase of dec. in 1 hour - - - - -	$0\ \ 11\ \ 7$

Hence $60^m : 11'\ 7'' :: 17^m{\cdot}4 : 3'\ 13''$; this added to the dec. at 5^h gives $18°\ 25'\ 15''$ S., the moon's dec. when on the meridian.

3. *In the triangle* ZN*m*, *find the* ∠ ZN*m*.

The horizontal refraction $=33'$ (Table IV.) The ☽'s horizontal parallax at noon (*Naut. Alm.*) $=55'\ 4''$, at midnight $=54'\ 49''$, and by calculating the proportional part for $5^h\ 17^m{\cdot}4$ (559), the time past noon, when the ☽ is on the meridian, the horizontal parallax at that time will be $54'\ 57''$. Hence we have

$$\text{Given}\left\{\begin{array}{l} zm = 90° + 33' - 54'\ 57'' = \ 89°\ 38'\ \ 3'' \\ \text{N}m = 90\ +\ 18°\ 25'\ 15'' = 108\ \ 25\ \ 15 \\ zn = 90\ -\ 51\ \ 28\ \ 39\ = \ 38\ \ 31\ \ 21 \end{array}\right\}\ \begin{array}{c}\text{To find the}\\ \angle\,\text{ZN}m.\end{array}$$

The ∠ ZN*m* will be found $= 64°\ 35' = 4^h\ 18^m{\cdot}3 =$ nearly; whence $5^h\ 17^m{\cdot}4 - 4^h\ 18^m{\cdot}3 = 0^h\ 59^m{\cdot}1$ P. M. the *estimated* time of the moon's rising; and the time of setting will be nearly $9^h\ 35^m{\cdot}7$ P. M. on the 4th of August.

4. *To find the ☽'s declination at the estimated time of rising.*

☽'s dec. Aug. 4th, at 0^h - - -	$17°\ 25'\ 10''$ S.
——————— 1^h - - -	$17\ \ 36\ \ 42$ S.
Increase of dec. in 1 hour - -	$0\ \ 11\ \ 32$

Hence $60^m : 11'\ 32'' :: 59^m{\cdot}1 : 11'\ 22''$; this added to the dec. at 0^h, gives $17°\ 36'\ 32''$ S., the moon's declination at the estimated time of her rising. The moon's horizontal parallax, at $0^h\ 59^m{\cdot}1$ is found to be $55'\ 3''$ from the parallaxes at noon and midnight given above.

5. *Again, in the triangle* ZNm, *we have*

$$\text{Given}\left\{\begin{array}{l} zm = 90^\circ + 33' - 55'\ 3'' = \ 89^\circ\ 37'\ 57'' \\ \text{N}m = 90 \ + \ 17^\circ\ 36'\ 32'' = 107 \ \ 36\ 32 \\ \text{ZN} = 90 \ - \ 51 \ \ 28\ 39 \ = \ 38 \ \ 31\ 21 \end{array}\right\}\begin{array}{c}\text{To find the} \\ \angle\,\text{ZN}m.\end{array}$$

The ∠ ZNm will be found $= 65^\circ\ 50' = 4^h\ 23^{m}\cdot 3$, the *interval* between rising and the meridian passage; whence $5^h 17^{m}\cdot 4 - 4^h\ 23^{m}\cdot 3 = 0^h\ 54^{m}\cdot 1$ P. M., the time of the moon's rising, nearly.

To find the correct time. The ☽ passes the meridian on the 3d of August (*Naut. Alm.*) at $4^h\ 33^{m}\cdot 1$, then $5^h\ 17^{m}\cdot 4 - 4^h\ 33^{m}\cdot 1 = 44^{m}\cdot 3$ daily variation of transit; hence $24^h : 44^{m}\cdot 3 ::$ (*interval*) $4^h\ 23^{m}\cdot 3 : 8^{m}\cdot 6$, and $0^h\ 54^{m}\cdot 1 - 8^{m}\cdot 6 = 0^h\ 45^{m}\cdot 5$ P. M. mean time of the moon's rising. The equation of time at mean noon, on Aug. 4th, is $5^{m}\cdot 8$ to be subtracted; hence the *apparent* time of the moon's rising is $0^h\ 39^{m}\cdot 7$ P. M.

Should more examples be thought necessary, they may be formed and solved in a similar manner; likewise the rising of the different planets may be found by the same method. The times of the risings and settings of the sun, moon, and planets are given in *White's Ephemeris*, and not in the Nautical Almanac.

PROBLEM X. (*Plate III. Fig.* 1.)

(588) *The latitude and longitude of a fixed star, or planet, being given, to find its right ascension and declination, et contra.*[*]

In the oblique triangle NnC.

Nn = obliquity of the ecliptic, or distance between the poles of the equator and ecliptic.

nC = the complement of the star's latitude VC.

NC = the complement of the star's declination RC.

The angle NnC, measured by the arc V♋, is the complement of the star's longitude from aries.

The supplement of the angle nNC, measured by the arc RQ, is the complement of the star's right ascension from aries.

Or, *In the oblique triangle* SmX.

Sm = the obliquity of the ecliptic.

mX = the complement of the star's latitude VX.

SX = the complement of the star's declination RX.

The supplement of the angle XmS, measured by the arc VV♑, is the complement of the star's longitude from aries.

The angle XSm, measured by the arc RQ, is the complement of the right ascension from aries.

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[*] A neat method of solving this problem may be seen, if necessary, in Dr. Maskelyne's introduction to Taylor's logarithms. The problem may be varied so as to admit of several cases, but the two above specified are the most useful ones

EXAMPLE.

The latitude of Aldebaran in Taurus being=5° 28′ 51″ S., its longitude 67° 33′ 11″, Jan. 1. 1840; required its right ascension and declination, the obliquity of the ecliptic being 23° 27′ 45″.

In the triangle s*m*x.

s*m* = 23° 27′ 45″=the obliquity of the ecliptic.
x*m*s=157 33 11 =star's longitude+90°.
*m*x = 84 31 9 =the complement of the star's lat.

1. To find the co-declination sx.

s*m* =23° 27′ 45″ log sin - - = 9·60005
*m*x =84 31 9 log sin - - = 9·99801
½ x*m*s=78 46 36 log sin - - = { 9·99161
9·99161

{ Sum of logs - —1·58128
nat no - ·38131
2

·76262
*m*x—s*m*= 61° 3′ 24″ nat cos - - ·48395

sx=106 10 51 nat cos - - ·27867

Hence 106° 10′ 51″—90°=16° 10′ 51″=RX the dec. N.

2. *To find the co-right ascension* xs*m*.

sin sx=106° 10′ 51″ : sin x*m*s=157° 33′ 11″ : :
sin *m*x= 84° 31′ 9″ : sin xs*m*= 23° 18′ 48″.

Hence AR=66° 41′ 12″, the right ascension.

OR THUS:

Draw the perpendicular XL.

rad : cos L*m*x : : tan *m*x : tan L*m*, and L*m*+*m*s=LS.
cos *m*L : cos *m*x : : cos SL : cos sx.
sin *m*L : sin SL : : cot L*m*x : cot xs*m*.

PRACTICAL EXAMPLES.

1. Required the right ascension and declination of Regulus in the Lion's heart, latitude being 0° 27′ 30″ N., longitude 147° 36′ 22″, and obliquity of the ecliptic 23° 27′ 45″.

Answer. Right ascension=149° 57′ 42″, and declination= 12° 44′ 46″ N.

2. Required the right ascension and declination of Spica Virginis, in the sheaf of Virgo, latitude 2° 2′ 29″ South, longitude 201° 36′ 32″, and obliquity of the ecliptic 23° 27′ 45″.

Answer. Right ascension = 199° 11′ 36″, and declination = 10° 19′ 26″ South.

3. Required the right ascension and declination of Betilguese in the shoulder of Orion, latitude being 16° 3′ 1″ South, longitude 86° 31′ 12″, and obliquity of the ecliptic 23° 27′ 45″.

Answer. Right ascension = 86° 37′ 41″, declination = 7° 22′ 17″ North.

4. Required the right ascension and declination of Antares in Scorpio, latitude being 4° 32′ 47″ S., longitude 247° 31′ 45″, and obliquity of the ecliptic 23° 27′ 45″.

Answer. Right ascen. = 244° 54′ 8″, declination = 26° 4′ 13″ South.

5. Required the right ascension and declination of the moon, on Nov. 22. 1840, at noon, when her latitude is 4° 55′ 37″ S., longitude 222° 36′ 40″, and the obliquity of the ecliptic 23° 27′ 43″.

Answer. Right ascension = 218° 33′ 45″, declination = 20° 19′ 28″ South.

6. Given the right ascension of Aldebaran = $4^h\ 26^m\ 44^s{\cdot}8$, and its declination 16° 10′ 54″ N., Jan. 1, 1840; required its latitude and longitude, the obliquity of the ecliptic being 23° 27′ 45″.

*In the triangle n*NC.

*n*NC = 156° 41′ 12″ = AR + 90°, AR being = $4^h\ 26^m\ 44^s{\cdot}8$.
NC = 73 49 6 = complement of RC.
N*n* = 23 27 45 = obliquity of the ecliptic.

1. *To find the co-latitude n*C.

N*n*	=23° 27′ 45″	log sin - - =	9·60005
NC	=73 49 6	log sin - - =	9·98244
½ *n*NC	=78 20 36	log sin - - =	9·99095 9·99095
		Sum of logs - -	1·56439
		nat no - -	·36677
			2
			·73354
NC—N*n*	=50° 21′ 21″	nat cos - - -	·63802
Co-lat	=95 28 52	nat cos - - -	·09552

Hence the latitude = 5° 28′ 52″ S.*

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* The latitude of this star, in the year 1796, was 5° 28′ 27″ S., making a difference of only a few seconds in 44 years.

OR THUS,

Draw the perpendicular CL, then, in the right-angled triangle CLN,

rad. sin of 90°	- -	10·00000
: cos CNL = RQ	= 23° 18′ 48″	9·96301
: : tan NC	= 73 49 6	10·53734
: tan NL	= 72 27 53	0·50035

Then NL + N*n* = *n*L = 95° 55′ 38″

In the oblique-angled triangle *n*NC (404)

cos NL	= 72° 27′ 53″	-	9·47899
: cos *n*L	= 95 55 38	-	9·01395
: : cos NC	= 73 49 6	-	9·44512
: cos *n*C	= 95 28 52	-	8·98008

Hence the lat = 5° 28′ 52″ South.

2. *To find the co-longitude* C*n*N.

sin *n*C = 95° 28′ 52″ : sin *n*NC = 156° 41′ 12″ : : sin NC = 73° 49′ 6″ : sin C*n*N = 22° 26′ 48″. Hence the longitude = 67° 33′ 12″.*

7. Required the latitude and longitude of Sirius in the Great Dog, right ascension being 99° 31′ 29″, and declination 16° 30′ 7″ South; the obliquity of the ecliptic being 23° 27′ 45″.

Answer. Latitude 39° 34′ 18″ S., longitude 101° 52′ 41″.

8. Required the latitude and longitude of Procyon in the Little Dog, right ascension being 112° 43′ 50″, declination 5° 37′ 47″ N., and obliquity of the ecliptic 23° 27′ 45″.

Answer. Latitude 15° 59′ 23″ S., longitude 113° 34′ 43″.

9. Required the latitude and longitude of Arcturus in Bootis, right ascension being 212° 5′ 30″, declination 20° 1′ 5″ North, and obliquity of the ecliptic 23° 27′ 45″.

Answer. Latitude 30° 50′ 56″ N., longitude 202° 0′ 0″.

10. Required the latitude and longitude of Fomalhaut, in the Southern Fish, right ascension being 342° 11′ 54″, declination 30° 28′ 5″ S., and obliquity of the ecliptic 23° 27′ 45″.

Answer. Latitude 21° 6′ 47″ S., longitude 331° 36′ 24″.

11. Required the latitude and longitude of the moon, on the 25th of June 1840, when her right ascension is 32° 38′ 1″, declination 18° 11′ 0″ North, and the obliquity of the ecliptic 23° 27′ 44″.

Answer. Latitude 4° 44′ 10″ N., longitude 36° 36′ 5″.

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* The longitude of this star in the year 1796 was found, by the same process, to be 66° 56′ 24″, making an increase of 36′ 48″ in 44 years.

By comparing together the places of the fixed stars deduced from observation, astronomers have found that their longitudes increase about 50¼″ annually. This increase of longitude must necessarily cause an irregular motion in the same star with regard to the equinoctial. Hence the right ascensions and declinations of stars are continually varying, so that stars which formerly had north declination have now south declination, *et contra*. Their latitudes are also subject to a small variation. See the Tables, page 206, &c. *Naut. Alm.* 1833.

PROBLEM XI. (*Plate III. Fig.* 8.)

(589) *The right ascensions and declinations of two stars, or the latitudes and longitudes of two stars given, to find their distance.*

Required the distance between Sirius in the Great Dog, right ascension = 99° 31′ 29″, and declination = 16° 30′ 7″ S., and Procyon in the Little Dog, right ascension = 112° 43′ 50″, and declination 5° 37′ 47″ N.

In the triangle XST let the star at T represent Procyon, because its right ascension is the greatest, and the star at X represent Sirius.

The ∠ XST = 13° 12′ 21″ diff. of their right ascensions = RQ.

SX = 73° 29′ 53″, the complement of Sirius's declination RX.

ST = 95° 37′ 47″, Procyon's declination + 90° = SQ + QT.

Hence, here are two sides and the contained angle given, to find the third side XT, by any of the methods in the preceding problems.

Answer. XT = 25° 42′ 0″.

EXAMPLE II.

The latitude and longitude of Sirius being = 39° 34′ 18″ S., and 101° 52′ 41″; and the latitude and longitude of Procyon = 15° 59′ 23″ South, and 113° 34′ 43″. Required their distance.

In the triangle X*m*T as before, we have

X*m* = 50° 25′ 42″ the complement of Sirius's lat. VX.

T*m* = 74° 0′ 37″ the complement of Procyon's lat. ♋ T.

The ∠ X*m*T = 11° 42′ 2″ the difference of their longitudes V♋.

Hence, as before, here are two sides and the contained angle given, to find the third side, XT = 25° 42′ *Answer.*

EXAMPLE III.

Required the distance between Capella in the Goat, right ascension = 76° 13′ 12″, declination = 45° 49′ 39″ North, and Procyon in the Little Dog, right ascension = 112° 43′ 50″, and declination = 5° 37′ 47″ North.

In the triangle TNC, let the star at T represent Procyon, and C Capella.

NC = 44° 10′ 21″ the co-declination of Capella.

TN = 84° 22′ 13″ the co-declination of Procyon.

∠ TNC = 36° 30′ 38″ the diff. of their right ascensions = RQ.

Here we have two sides and the contained angle given, to find the third side, as in the preceding examples.

Answer. CT = 51° 7′ 5″.

EXAMPLE IV.

Required the distance between Capella, whose latitude is 22° 51′ 37″ North, longitude 79° 37′ 23″, and Procyon, latitude 15° 59′ 23″ South, longitude 113° 34′ 43″.

In the triangle T*n*C, let the star at T represent Procyon, that at C, Capella.

*n*C = 67° 8′ 23″ = comp. Capella's lat. OC.
*n*T = 105 59 23 = 90° + Procyon's lat. ♋T.
∠T*n*C = 33 57 20 = diff. of their longitudes = O♋.

Here we have two sides and the contained angle given, to find the third side CT = 51° 7′ 3″ *Answer.*

(590) In the same manner we may find the distance between any two places on the earth; suppose X and T to be two places whose latitudes and longitudes are known, then in the triangle XNT, XN is the distance of X from the pole, TN the distance of T from the same pole, and the angle XNT, measured by the arc RQ, the difference of longitude between the two places, hence the distance XT is easily found. In a similar manner the distance between T and C, or C and X may be obtained.

EXAMPLE V.

What is the extent of Africa, from Cape Verde, latitude 14° 46′ N., longitude 17° 47′ W., to Cape Guardafui, latitude 11° 47′ N., longitude 51° 35′ E.?

Answer. 67° 19′, the arc of a great circle contained between the places, or 4679 *English* miles, of 69½ to a degree, or 4039 geographical miles, of 60 to a degree.

PROBLEM XII. (*Plate III. Fig.* 8.)

(591) *The places of two stars being given, and their distances from a third star, to find the place of this third star.*

EXAMPLE I.

Suppose the distance of a comet or new star C, was measured by a sextant to be 65° 47′ 42″ from Sirius the dog-star, lat. 39° 34′ 18″ South, longitude 101° 52′ 41″; and 51° 6′ 56″, from Procyon the Little Dog, lat 15° 59′ 23″ South, longitude 113° 34′ 43″; it is required to find the latitude and longitude of this comet or star.

Let X represent the place of Sirius, or one of the known stars, T that of Procyon the other known star, having the greater longitude, and C the place of the comet or unknown star.

First. In the triangle X*n*T, we have given

X*n*=129° 34′ 18″ Sirius's distance from the pole of the ecliptic nearest the comet or star.
T*n*=105 59 23 Procyon's distance from the same pole.
∠X*n*T= 11 42 2 the difference of longitude between the given stars.

To find XT the distance between the two known stars= 25° 42′ 0″, and the angle *n*XT=26° 42′ 53″: To find the former of these quantities, there are two sides and the included angle given, and to find the latter the three sides and one angle are given.

Secondly. In the triangle XTC, we have giver

XT=25° 42′ 0″
TC=51 6 56
and XC=65 47 42

To find the ∠CXT at the star X. By any of the methods in arts. (484) and (502), you will find CXT=49° 13′ 42″.

This angle being greater than *n*XT, shews the longitude of the star at C to be less than at X. Hence CXT−*n*XT=22° 30′ 49″ =*n*XC.

Thirdly. In the triangle X*n*C, we have given

X*n*=129° 34′ 18″
XC= 65 47 42
and *n*XC= 22 30 49

To find the side *n*C the comet or star's co-lat., and X*n*C the diff. between the star's long. at C and Sirius's long.

*n*C=67° 9′ 7″ the star's co-latitude, hence its latitude OC= 22° 50′ 53″ N., and X*n*C=22° 16′ 16″; and since the star's longitude has been shown to be greater than the comet's, from Sirius's longitude=101° 52′ 41″ take 22° 16′ 16″, the remainder 79° 36′ 25″ is the unknown star's longitude.

EXAMPLE II.

Suppose the distance of an unknown star at X was measured to be 65° 47′ 42″, from a known star (Capella) at C, having 22° 51′ 37″ North latitude, and 79° 37′ 23″ of longitude; and its distance from another known star at T (Procyon) = 25° 42′ 10″, this star having 15° 59′ 23″ South latitude, and 113° 34′ 43″ of longitude; required the latitude and longitude of the unknown star.

First. In the triangle C*m*T, we shall have

C*m*=*m*O+OC=112° 51′ 37″ = Capella's distance from the pole of the ecliptic nearest the comet, or unknown star.

C*m*T=33° 57′ 20″=the diff. long. between Capella and Procyon.

T*m* =74 0 37 =Procyon's distance from the pole of the ecliptic.

To find TC = 51° 7′ 3″, the distance between the known stars, and the angle *m*CT = 43° 36′ 42″.

Secondly. In the triangle XCT, we have given

XC = 65° 47′ 42″
XT = 25 42 10 } To find the angle XCT at the star C.
TC = 51 7 3

By any of the methods in arts. (484) and (502) XCT will be found = 24° 57′ 34″; this angle being less than *m*CT shows the longitude of the star at X to be greater than that at C.

Hence *m*CT − XCT = 18° 39′ 8″ = XC*m*.

Thirdly. In the triangle XC*m*, we have

C*m* = 112° 51′ 37″
CX = 65 47 42
∠ XC*m* = 18 39 8
{ To find X*m* the unknown star's co-latitude, and X*m*C the difference between the star's longitude at X, and Capella's longitude at C.

X*m* = 50° 25′ 46″ the star's co-lat., hence its latitude VX = 39° 34′ 14″ S., and X*m*C = 22° 14′ 6″; and since the given star's longitude has been shown to be less than the required star's longitude, this difference must be added to Capella's longitude, viz. to 79° 37′ 23″ add 22° 14′ 6″, the sum is 101° 51′ 29″, the longitude required.

EXAMPLE III.

Suppose the distance from an unknown star at T was measured to be 25° 42′ 10″, from a star at X, whose latitude is 39° 33′ 30″ South, longitude 101° 13′ 44″; and 51° 6′ 56″ from another known star at C, whose latitude is 22° 51′ 57″ North, and 78° 57′ 57″ longitude; required the latitude and longitude of the unknown star.

First. In the triangle X*m*C, you will have given

X*m* = the complement of X's latitude.

C*m* = the distance of C from the same pole of the ecliptic, and the angle X*m*C = the difference between their longitudes = VO.

Hence XC will be found = 65° 47′ 51″, and the angle *m*XC = 157° 29′ 50″.

Secondly. In the triangle XTC, are given

XC
XT } To find the angle CXT = 49° 13′ 30″.
and TC } Now *m*XC − CXT = 108° 16′ 20″ = *m*XT.

Thirdly. In the triangle *m*XT, are given.

X*m*
XT
∠ *m*XT
{ To find T*m* = 62° 1′ 50″, the distance of the star T from the pole of the ecliptic, and the angle X*m*T = 27° 47′ 38″, the difference between the longitude of X and T.

Hence the lat. = 27° 58′ 10″ S., and the longitude = long. of X + angle X*m*T = 101° 13′ 44″ + 27° 47′ 38″ = 129° 1′ 22″.

(592) By the assistance of the foregoing problem, and a knowledge of any one fixed star's true situation, the places of all the rest may be determined. Indeed it is by this problem principally that astronomers have rectified the places of all the fixed stars; and thence, by a similar mode of proceeding, found the true places of the planets. *Leadbetter* tells us in his Astronomy, page 230., that he made use of this method for constructing his astronomical tables. The preceding examples contain every variation that can happen with respect to the stars' latitudes and longitudes, but the same things may be determined from their right ascensions and declinations, as in the following example: —

EXAMPLE IV. (*Plate III. Fig.* 9.)

Suppose the distance of a star or planet at Y was observed to be 65° 47′ 42″, from Sirius the Dog-star at X, whose right ascension is 99° 31′ 29″, and declination 16° 30′ 7″ South; and to be 51° 6′ 56″ distant from Procyon at T in the Little Dog, whose right ascension is 112° 43′ 50″, and declination 5° 37′ 47″ North; it is required to find the right ascension and declination of this unknown star or planet.

First, In the triangle XNT, we have

The angle XNT = 13° 12′ 21″ = the difference between the right ascensions of X and T, measured by the arc RQ.

XN = 106° 30′ 7″ = X's distance from the pole N.
TN = 84 22 13 = T's distance from the same pole.

Hence two sides and the included angle are given, to find the third side XT = 25° 42′, and the angle NXT = 31° 37′ 6″.

Secondly. In the triangle YXT, we have given

YX = 65° 47′ 42″
YT = 51 6 56
XT = 25 42 0

To find the angle YXT = 49° 13′ 42″
Then YXT − NXT = 17 36 36
= YXN.

The ∠ YXT being greater than the ∠ NXT shows the right ascension of the unknown star at Y to be less than the right ascension of that at X.

Thirdly. In the triangle XYN, are given

XY = 65° 47′ 42″
XN = 106 30 7
∠ YXN = 17 36 36

To find YN the complement of the declination BY, and the angle YNX the difference between the right ascensions of Y and X = BR.

You will find
YN = 44° 11′ 12″, consequently BY = 45° 48′ 48″ N.
YNX = 23° 19′ 19″, hence the right ascension of Y = AB = 76° 12′ 10″.

PROBLEM XIII. (*Plate III. Fig.* 1.)

(593) *Given the latitude of the place, the sun's declination and altitude, to find the azimuth.*

EXAMPLE I.

In latitude 51° 32′ North the sun's altitude was observed to be 50°, and his declination 23° 28′ North; required his azimuth from the North.

In the triangle SZN, we have given

SN = 66° 32′ the co-declination, or sun's polar distance; for DS is the sun's declination North.

SZ = 40 0 the co-altitude of the sun; for CS is his altitude above the horizon HO.

ZN = 38 28 the co-latitude of the place, or what ON the elevation of the pole wants of 90°.

To find the angle SZN, measured by the arc of the horizon CO, the sun's distance from the North.

Call the co-latitude ZN the base, bisect ZN in *t*, and draw the perpendicular S*f*. Then (484)

$\tan \frac{1}{2}$ ZN = zt = 19° 14′ - - - 9·54269

: $\tan \frac{1}{2}$ (SN + SZ) = 53 16 - - - 10·12710

: : $\tan \frac{1}{2}$ (SN − SZ) = 13 16 - - - - 9·37250

: $\tan ft$ = 42 10 - - - 9·95691

Take the difference between *ft* and half the complement of latitude. Then

Radius, sin of 90° - - - - - - 10·00000

: $\tan (ft - zt)$ = 22° 56′ - - - - 9·62645

: : tan of the sun's altitude = CS = 50° - - 10·07619

: sin of azimuth from the east or west = 30° 17′ 9·70264

Hence 90° + 30° 17′ = 120° 17′ the sun's azimuth from the North.

(594) A slight review of *Figure I. Plate III.* will naturally point out to us the following observations:

If the latitude of the place and the declination of the sun be of the same name, the sun is less than 90° from the elevated pole; therefore the co-declination is the polar distance.

If the sun or star be in the equinoctial, it is 90° from the pole, and forms a quadrantal triangle. If the declination be of a contrary name with the latitude, the distance of the object from the elevated pole will be equal to the declination increased by 90°.

(595) The preceding solution is on the supposition that the sun's declination, as taken from the Nautical Almanac, remains the same in all places, and at all times of the day. But this cannot be strictly true, for the declination is subject to a daily variation, and is adapted to the meridian of Green-

wich. An azimuth will not be materially affected by this change, but it will make a considerable difference when we want to find the time of the day.

EXAMPLE II.

Suppose the sun's altitude at London, latitude 51° 32′ North, to be 1° 14′ when the declination is 20° 11′ South, what is the azimuth from the North?

Answer. 125° 34′.

EXAMPLE III.

Given the latitude of the place 51° 32′ North, the sun's altitude 25°, his declination 4° 47′ South; required the azimuth from the North.

Answer. 137° 18′.

EXAMPLE IV.

Suppose in latitude 51° 32′ North, the altitude of the star Arcturus in Bootis was 44° 30′, when its declination was 20° 1′ N.; required its azimuth from the North at that time.

Answer. 117° 44′.

EXAMPLE V.

In latitude 48° 51′ North, when the sun's declination is 18° 30′ North, and his altitude 52° 35′; what is his azimuth from the North?

Answer. 134° 36′.

(596) This problem is not only useful for finding the variation of the compass, but it will be found very convenient for determining the true meridian, and the four cardinal points. For, if with a line representing the vertical on which the sun is, an angle be made equal to the supplement of the azimuth from the North, the meridian will be obtained. provided care has been taken to correct the sun's altitude for refraction, &c.

(597) An azimuth may be constructed by a scale of chords, sufficiently accurate for Nautical purposes: Thus,

With the chord of 60° describe a circle (*Plate IV. Fig.* 2.), set off the complement of the latitude from Z to P, and the complement of the altitude from Z to *a*, draw ZON through the centre C and *aoa* at right angles to it. Set off the polar distance from P to *p*, draw *pp* through the centre C, and from the point I where it intersects *aoa* draw IR parallel to ZON, and through C the centre draw CS parallel to *aoa*.

With C as a centre, and the distance O*a*, cross the line IR in *m;* through *m* draw C*n*, then S*n*, measured on a scale of chords, will be the azimuth from the south (in North latitude) if the point I fall on the left of ZN, and from the north if it fall on the right.

See *Leybourn*'s Mathematical Repository, old series, vol. i.

page 283., second edition, 1799, and vol. ii. pages 26 and 27. See also Dr. Mackay's Navigation, first edition, 1804, page 200.

PROBLEM XIV. (*Plate III. Fig.* 1.)

(598) *Given the latitude of the place, the sun's declination and altitude, to find the hour of the day.*

EXAMPLE I.

At London, latitude 51° 32′ North, the altitude of the sun's centre was 38° 19′, and his declination 19° 39′ North; required the hour of the day.

In the triangle SZN, we have given

SN = 70° 21′ the sun's distance from the elevated pole.
SZ = 51 41 the sun's co-altitude.
ZN = 38 28 the co-latitude.

To find the angle ZNS, measured by the arc ÆD, the distance between the meridian which the sun is on at the time of observation, and the meridian of the place of observation.

Since ÆZ is equal to the latitude ON; it follows, that if the latitude and declination be both north or both south, the difference between them is equal to the meridional zenith distance; if the one be north and the other south, their sum is the meridional zenith distance.

Nat cos of merid zenith dist
(51° 32′ − 19° 39′) - - - 31° 53′ = 84913
Nat sine of the sun's altitude - - 38 19 = 62001

Remainder - - 22912

Common logarithm of - - 22912 = 4·36007
Secant of the latitude - - 51° 32′ = ·20617
Secant of the declination - 19 39 = ·02606

Num. in col. ris., Requisite Tables, $3^h\ 30^m$ 4·59230
Or, Increase the index by 5 +

Log versed sine - - - 52° 29′ 24″ - 9·59230
Or add the logarithm of 5000000000 = ½ radius. - - + 9·69897

Divide by 2 | 19·29127

Log sine of half the hour angle 26° 14′ 42″ = 9·64563
2

The hour angle - - 52 29 24 = $3^h\ 29^m\ 57^s\ 36^t$.

This solution is analogous to Problem 8th in the *Requisite Tables* published by order of the Commissioners of Longitude; or to Problem VII. Chap. II. Book III. of Dr. Mackay's Navigation. If you have the *Requisite Tables* * by you, the sum of the three logarithms above may be looked for in the column of rising, and the hours and minutes answering thereto will be the time from noon. Or, increase the index by 5, and the sum is the log. versed sine of the same angle. Or, the log. versed sine of an arc increased by the logarithm of 5 (with 9 for its index), will give double the logarithmic sine of half the required angle (281).

EXAMPLE II.

In latitude 39° 54′ North, longitude 80° 39′ 45″ West of Greenwich, suppose the altitude of the sun's lower limb to be 15° 40′ 57″ on the 9th of May, 1840, at 5ʰ 34ᵐ 21ˢ P. M. per watch; required the correct apparent and mean time at that place, and the error of the watch.

1. *To find the time by acct. at Greenwich, and equation of time.*

May 9ᵈ 5ʰ 34ᵐ 21ˢ time by watch.
5 22 39 long in time W.

10 57 0 Gr. time by acct.

Eq. of time May 9, 1840 = −3ᵐ 47ˢ
——— May 10 ——— = −3 49

Increase in 24ʰ - - = 2

24ʰ : 2ˢ : : 11ʰ : 1ˢ
Hence the cor. eq. is −3ᵐ 48ˢ

2. *To correct the ⊙'s declin.*

⊙'s decl. May 9th 840 = 17° 27′ 12″ N.
Ditto May 10th - = 17 42 57

Increase in 24 hours = 15 45

24ʰ : 15′ 45″ : : 10ʰ 57ᵐ : 7′ 11″
Hence the ⊙'s declin. at the time and place of observation = 17° 34′ 23″ N.

3. *To correct the ⊙'s altitude.*

⊙'s observed alt. - = 15° 40′ 57″
Refraction† - - = − 3 21

15 37 36
⊙'s semidiameter ‡ - = + 15 51
⊙'s parallax ‖ - - = + 8

True alt. ⊙'s centre - = 15 53 35

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* The XVIth of the *Requisite Tables*, or the XXXVth, XXXVIth, and XXXVIIth Tables in Dr. Mackay's Navigation.

The XVIth of the Requisite Tables is divided into three columns. The FIRST column, called *half elapsed time*, is only a table of secants with 10 rejected from the index; the time being reduced to degrees.

The SECOND, called *middle time*, is the logarithmic sine of half the elapsed time, increased by the logarithm of 2, and the index of the sum diminished by 5.

The THIRD, called *column of rising*, is a table of logarithmic versed sines with their indices diminished by 5.

Thus for 1 hour of time, or 15 degrees.

cosec 15° = 10·58700	sin 15° = 9·41300	vers 15° = 8·53243
−10	log 2 = 0·30103	−5
log ½ elap. time 0·58700	sum 9·71403	log rising 3·53243
	−5	
	log mid. time 4·71403	

Proceed in the same manner for any other time.

† Table IV. ‡ Nautical Almanac. ‖ Table VI

4. *To find the correct time. In the triangle* SZN,

reject indices.
Complement of latitude NZ = 50° 6′ 0″ cosec - ·11511
Complement of altitude SZ = 74 6 25
Polar distance SN = 72 25 37 cosec - ·02076

2 | 196 38 2

½ sum - - 98 19 1 sin- - 9·99541
SZ = 74 6 25

Remainder - 24 12 36 sin - 9·61287

2 | 19·74415

Cosine of half ZNS = 41° 51′ 5″ - - = 9·87208
2

ZNS = 83 42 10 = $5^h\ 34^m\ 49^s$ App. time
− 3 48 Equation

5 31 1 Mean time
(*Subtract* from App. time) 5 34 21 Time per watch

28 Watch too slow.

(599) In the practical application of this problem, it will be proper to take several altitudes of the sun within a minute or two of each other, and to note the corresponding times per watch. The mean of these altitudes, and of the times, will be more correct than a single altitude and time, and will in a great measure counteract the errors arising from the imperfection of the instrument.

EXAMPLE III.

On the 4th of March, 1840, in latitude 45° 37′ North, and longitude 169° 59′ 30″ West, suppose the following altitudes of the sun's lower limb to be observed: —

App. time P. M. per watch.		Alt. of the ⊙'s lower limb.
$2^h\ 53^m\ 32^s$	- - -	24° 55′
2 54 30	- - -	24 48
2 55 36	- - -	24 40
2 56 46	- - -	24 31
4 | 11 40 24		4 | 98 54
Mean 2 55 6 P. M.		Mean 24 43 30.

Required the correct apparent and mean time of observation, and the error of the watch.

The sun's declination at Greenwich, app. noon, on March 4th is 6° 17′ 7″ S. (*Naut. Alm.*), and on the 5th 5° 53′ 57″ S., the equation of time at noon on March 4th is $+11^{m}\ 55^{s}$, and on the 5th $+11^{m}\ 42^{s}$, and the ⊙'s semi-diameter 16′ 8″.

Answer. The time at Greenwich is $14^{h}\ 15^{m}\ 4^{s}$, the ⊙'s declination at that time = 6° 3′ 21″ S., the equation of time $11^{m}\ 47^{s}$, and true altitude of the ⊙'s centre = 24° 57′ 43″; hence the apparent time of observation = $2^{h}\ 57^{m}\ 22^{s}$, mean time = $3^{h}\ 9^{m}\ 9^{s}$, and the watch is $2^{m}\ 16^{s}$ too slow.

EXAMPLE IV.

On the 16th of September, 1840, in latitude 33° 56′ South, and longitude 84° 6′ 15″ West, suppose the app. time per watch of several altitudes of the sun's lower limb to be $8^{h}\ 5^{m}\ 43^{s}$ A.M., and the mean of the altitudes themselves 24° 43′; required the correct apparent and mean time at that place, and the error of the watch.

The ⊙'s declination at app. noon (*Naut. Alm.*) on the 16th is 2° 31′ 39″ N., and on the 17th 2° 8′ 25″, the equation of time at noon on the 16th is $-5^{m}\ 20^{s}$, and on the 17th $-5^{m}\ 41^{s}$, and the ⊙'s semi-diameter 15′ 57″.

Answer. The time at Greenwich on the 16th is $1^{h}\ 42^{m}\ 8^{s}$ P.M., the ⊙'s declination at that time = 2° 30′ 0″ N., the equation of time = $-5^{m}\ 21^{s}$, and true altitude of the ⊙'s centre = 24° 57′ 2″; hence the apparent time is $8^{h}\ 10^{m}\ 16^{s}$ A.M., mean time $8^{h}\ 4^{m}\ 55^{s}$ A.M., and the watch is $4^{m} 33^{s}$ too slow.

PROBLEM XV. (*Plate III. Fig.* 1.)

(600) *Given the latitude of the place, the declination and altitude of a known fixed star, to find the hour of the night when the observation was made.*

EXAMPLE I.

In latitude 7° 45′ South, and longitude 30° 18′ East of Greenwich, on the 7th of September, 1840, suppose the altitude of the star Procyon, when east of the meridian, to be 28° 12′; required the apparent and mean time of observation, the declination of Procyon being 5° 37′ 40″ N.

Procyon's alt = (28° 12′ 0″ − refraction 1′ 46″ =) 28° 10′ 14″

Co-lat NZ	= 82° 15′ 0″	cosec	·00399
Co-alt SZ	= 61 49 46		
Polar dist SN	= 95 37 40	cosec	·00210
2 |	239 42 26		
Half sum	= 119 51 13	sin	9·93817
Co-alt SZ	= 61 49 46		
Remainder	= 58 1 27	sin	9·92854
		2 |	19·87280
Cos of half the hour ∠	30° 15′ 26″		9·93640
	2		
The hour ∠ ZNS	= 60 30 52 = $4^h\ 2^m\ 3^s$		

(601) The hour ∠ found by calculation, as above, is the distance of the star from the meridian of the place of observation; consequently when the star is eastward of the meridian at the time of observation, this hour angle must be subtracted from the time of the star's culminating, and when westward it must be added.

*'s Rt. Asc. Jan. 1. 1840 (Tab. VIII.) + 24^h = $31^h\ 30^m 55^s$ (+3·14 an. var.)

3·14 × 0·7 =	+ 2	
*'s Rt. Asc. at given time	31 30 57	
☉'s Rt. Asc. at noon, Sept. 7. 1840	11 4 19	(+9^s·00 hourly diff.)
Required time nearly	20 26 38	
Longitude 30° 18′ E. in time	2 1 12	
Greenwich time	18 25 26	
9·00 × 18·4 =	−2 46	
Time of *'s culminating	20 23 52	
Hour angle	4 2 3	
Apparent time of observation	16 21 49	
Equation of time, Sept. 7. 1840	−2 12	
— ——— Sept. 8. —	−2 32	
Variation in 24^h	20	
24^h : 20^s : : 16^h·4 : 14^s		
Hence the corr. equation of time is	−2 26	
Mean time of observation	16 19 23	

(602) In order to attain the greatest accuracy several altitudes of a star should be observed in succession, and if two or more separate stars be observed, and the error of the watch be deduced from each star, the medium result will be more correct than that derived from the observation of a single star.

If an equal number of stars can be observed on each side of the meridian, and nearly equidistant therefrom, the errors arising from the imperfection of the instruments, &c. will be rendered almost insensible.

EXAMPLE II.

On the 29th of January, 1840, in latitude 53° 24′ North, and longitude 25° 18′ West; suppose the following altitudes of Procyon to the west, and Alphacca to the east of the meridian, were observed by two separate persons at the same instants of time; required the error of the watch.

The right ascension and declination of Alphacca being $15^h\,27^m\,55^s$ and 27° 15′ 24″ N., and of Procyon $7^h\,30^m\,56^s$ and 5° 37′ 46″ N. The sun's right ascension at apparent noon on the 29th being $20^h\,44^m\,40^s$ and its hourly diff. $10{\cdot}3^s$; also the equation of time on Jan. 29. is $+13^m\,23^s$, and on Jan. 30 it is $+13^m\,33^s$.

	Apparent Time per watch.		Procyon's altitude.		Alphacca's altitude.
	$14^h\,53^m\,0^s$		20° 41′ 40″		41° 17′ 40″
	14 57 0		20 7 40		41 57 40
	15 0 32		19 37 40		42 24 40
	15 4 0		19 8 40		42 55 40
4	59 54 32	4	79 35 40	4	168 35 40
Mean	14 58 38	Mean	19 53 55	Mean	42 8 55

Answer. The hour ∠ by Alphacca's corrected altitude is $3^h\,40^m\,28^s$, the ✱ culminates at $18^h\,39^m\,45^s$; hence the apparent time $= 14^h\,59^m\,17^s$.

The hour ∠ by Procyon's corrected altitude is $4^h\,15^m\,39^s$, the ✱ culminates at $10^h\,44^m\,7^s$; hence the apparent time $= 14^h\,59^m\,46^s$. The mean between these two times is $14^h\,59^m\,32^s$, consequently the watch is 54^s too slow.

EXAMPLE III.

In latitude 48° 56′ North, longitude 66° West, suppose the altitude of Aldebaran, when west of the meridian, to be 22° 20′ 15″ on the 15th of April, 1840; required the apparent time of observation.

The right ascension of Aldebaran being $4^h\ 26^m\ 46^s$, and declination 16° 10′ 56″ N. Also the ⊙'s right ascension at apparent noon on the 15th (*Naut. Alm.*) $= 1^h\ 34^m\ 48^s$, and its hourly difference $9{\cdot}24^s$.

Answer. The hour ∠ by the ✳'s corrected altitude $= 4^h\ 57^m\ 44^s$, and the ✳ culminates at $2^h\ 50^m\ 51^s$, hence the apparent time $= 7^h\ 48^m\ 35^s$.

SCHOLIUM.

(603) The three preceding problems (XIII., XIV., and XV.) are all included in one triangle (*Zenith* NS *Plate III. Fig.* 1.), but they have been considered separately, on account of their importance, and to prevent confusion.

Any three of the five quantities, viz. the sun's or star's declination, the latitude of the place, the altitude of the object, the azimuth, and the hour of the day being given, the rest may be found.

A sixth quantity might be taken into consideration, were it materially useful, viz. the ∠ opposite to the complement of the latitude, being formed by the intersection of the azimuth circle ZSC, and hour circle NSD.

PRACTICAL EXAMPLES.

1. Given the sun's declination 19° 39′ North, in north latitude, his correct altitude 38° 19′, and his azimuth = S. 72° 13′ E.; required the hour of the day and the latitude of the place.

Answer. The hour angle is 52° 30′ $= 3^h\ 30^m$, the hour from noon, hence the time is $8^h\ 30^m$ A.M., and the latitude 51° 32′ North.

When the altitude of the sun and his declination are equal, the triangle is isosceles, and the hour angle is equal to the azimuth. But when the altitude of the sun is less than the declination, the case is ambiguous.

2. Given the sun's declination = 19° 39′ North, his altitude corrected = 38° 19′ at $3^h\ 30^m$ in the afternoon; required the latitude of the place, supposed to be north, and the sun's azimuth.

Answer. This problem is ambiguous, the azimuth may be N. 72° 13′ 30″ W. and the latitude 9° 15′ 14″ N., or the azimuth may be N. 107° 46′ 30″ W. and the latitude 51° 30′ 54″ N. But when the sun's altitude is less than the declination, the problem will be limited.

3. Given the latitude of the place = 51° 32′ N. the sun's declination = 19° 39′ North, the sun's azimuth = S. 72° 13′ W.; required the sun's altitude and the hour of the day.

Answer. The sun's altitude is 38° 18′ 26″, the hour angle 52° 30′ 16″, and the time from noon $3^h\ 30^m$. If the latitude of the place be less than the sun's declination, when both have the same name, the case will be ambiguous; see the following example.

4. Given the latitude of the place = 13° 30′ North, the sun's declination = 23° 28′ N., the sun's azimuth = N. 67° 30′ East; required the sun's altitude and the hour of the day.

Answer. The angle opposite to the complement of latitude may be either acute = 78° 20′, or obtuse = 101° 40′. In the former case the sun's altitude is 7° 8′ nearly, and the time from noon $5^h\ 52^m\ 3^s$; and in the latter case the sun's altitude is 57° 4′ 33″, and the time from noon $2^h\ 12^m\ 46^s$. Hence it follows that in this latitude on the longest day, the sun appears upon the same point of the compass twice in the forenoon and twice in the afternoon.

5. Given the latitude 13° 30′ North, the sun's corrected altitude = 60° 30′ at two o'clock in the afternoon; required the sun's azimuth and its declination.

Answer. The angle opposite to the complement of latitude may be either acute = 80° 52′, or obtuse = 99° 8′; in the former case the declination is 10° 21′ 40″, and the azimuth 87° 12′ 34″, and in the latter case the declination is 20° 37′ 36″, and azimuth 71° 51′ 29″.

6. In latitude 51° 32′ North, at $3^h\ 30^m$ in the afternoon, the sun's azimuth was S. 72° 13′ West; required the sun's altitude, his declination, and the angle formed by the intersection of the azimuth circle, and the hour circle passing through the sun.

Answer. The sun's altitude = 38° 18′ 53″, the declination = 19° 39′ 19″, and the angle *Zenith* SN formed by the vertical and azimuth circle = 38° 58′ 30″.

7. The sun's corrected altitude at $8^h\ 30^m$ in the forenoon was 38° 19′, the angle of position, viz. formed by the azimuth and hour circles, was 38° 58′; required the sun's azimuth, declination, and the latitude of the place.

Answer. The latitude is 51° 32′ 34″; but it cannot be determined by the *data* whether it be north or south, the sun's declination = 19° 39′ 46″, and the azimuth = 72° 13′ from the north or south.

PROBLEM XVI. (*Plate III. Fig.* 10. *and* 11.)

(604) *Given two altitudes of the sun and the time between the observations, to find the latitude of the place.*

EXAMPLE I. (*Plate III. Fig.* 10.)

Given the sun's declination 19° 39′ North, his altitude in the forenoon 38° 19′, and at the end of one hour and a half the same morning the altitude was 50° 25′; required the latitude of the place, supposing it north.

On a supposition that the sun's declination is the same at both observations, (which will not materially affect the result of the operation, when the observations are taken at no great distance from each other, and the time is not near the equinoxes,) the triangle AND is isosceles, A being the place of the sun at the first observation, D its place at the second, and AN=ND. Draw NG perpendicular to AD, then will AG=GD and GND=½AND half the elapsed time (316).

1. In the right-angled triangle NGD,

there is given ND=70° 21′ the sun's distance from the pole at each observation, and the angle GND=11° 15′=½ elapsed time, to find GD.

*
rad × sin GD=sin GND × sin ND.
∴ log sin GND+log sin ND−10=log sin GD.
or rad : sin ND = 70° 21′ : : sin GND = 11° 15′ : sin GD = 10° 35′ 13″, the double of which is 21° 10′ 26″=AD.

2. *To find the* ∠GDN. *

rad × cos ND=cot GND × cot GDN.
∴ 10+log cos ND−log cot GND=log cot GDN.
or cot GND = 11° 15′ : rad : : cos ND = 70° 21′ : cot GDN = 86° 10′ 24″.

3. *To find the* ∠ADZ.

In the oblique-angled triangle ADZ, there is

Given { 1st co-alt. AZ=51° 41′ 0″
2d co-alt. DZ=39 35 0
and AD=21 10 26 }
Hence ½ADZ=57° 41′ 30″
and ∠ADZ=115° 23′ 0″

The ∠ADZ being greater than the ∠GDN, this example belongs to Figure 10.

And ∠ADZ − ∠GDN=115° 23′ 0″−86° 10′ 24″=29° 12′ 36″ = ∠NDZ.

. *To find* NZ *the co-latitude.*

In the oblique triangle DZN, there is

Given { the co-dec. ND=70° 21′ 0″
2nd co-alt. DZ=39 35 0
and the ∠NDZ=29 12 36 }
To find NZ the co-lat. =38° 28′ 16″.

This may be done by the Rules Case III. (500), or by Case VIII. page 204, viz.

$$\tan \varphi = \frac{\cos \angle \text{NDZ} \cdot \tan \text{ND}}{\text{rad}}; \text{ and } \cos \text{NZ} = \frac{\cos \text{ND}}{\cos \varphi} \cdot \cos (\text{DZ} \sim \varphi).$$

cos ∠ NDZ	=29° 12′ 36″ -	9·94093	cos ND	=70° 21′ 0″ -	9·52669
tan ND	=70 21 0 -	10·44725	cos (DZ∼φ)	=28 10 3 -	9·94526
	Sum	20·38818		Sum	19·47195
	Subtract rad =	10·	cos φ	=67 45 3 sub.	9·57823
tan φ	=67° 45′ 3″ -	10·38818	cos NZ	=38 28 16 -	9·89372

Hence the latitude is 51° 31′ 44″ N.

(605) The principal use to which this problem is applied, is in questions of a similar nature with the preceding; but several other things may be determined from the same *data;* such as the hour from noon when each altitude was taken, the azimuth at each observation, the difference of azimuths, &c. It will be sufficient in this place to point out the method of solution of these less useful cases.

The hour from noon when the second altitude was taken, is measured by the angle DNZ, for ⊙ is the place of the sun when on the meridian. Now in the triangle DNZ, we have DZ = 39° 35′, ZN = 38° 28′ 16″, and the angle NDZ = 29° 12′ 36″, to find the angle DNZ the hour from noon = 30° or 2 hours; consequently the second altitude was taken at 10 o'clock, and the first at 8^h 30^m in the forenoon. Again, in the triangle ANZ we have AN = 70° 21′, the sun's polar distance at the first observation, AZ = 51° 41′ his zenith distance, and the angle ANZ = 3^h 30^m or 52° 30′ the hour from noon when the first altitude was taken, to find the angle AZN = 107° 47′ the azimuth from the North at the first observation. Also, in the triangle DNZ we have DN = 70° 21′ the sun's polar distance at the second observation, DZ = 39° 35′ the zenith's distance, and the angle DNZ = 30° or 2 hours, the hour from noon when the second altitude was taken, to find the angle NZD = 132° 21′, the azimuth from the north at the second observation. The difference of the azimuth is therefore = 24° 34′. The azimuths might have been found independent of the time, for in each of the above triangles, the co-declination, the co-altitude, and the co-latitude are given.

EXAMPLE II. (*Plate III. Fig.* 11.)

When the sun's declination was 22° 40′ North, his altitude at 10^h 54^m in the forenoon was 53° 29′, and at 1^h 17^m in the afternoon it was 52° 48′; required the latitude of the place of observation, supposing it to be north.

$12^{h}\ 0^{m}$
Time of the first alt. 10 54

The time from noon 1 6
The time from noon 1 17 when the second alt. was taken.

Elapsed time - - 2 23=35° 45′ the angle AND.

Half elapsed time - $1^{h}\ 11^{m}\ 30^{s}$=17° 52′ 30″ the angle GND.

Supposing the sun's declination constant during the elapsed time, AND is an isosceles triangle, and the method of solution is precisely the same as in the former example.

I. For ND=AN=67° 20′, and the angle GND=17° 52′ 30″, whence the angle GDN is found = 82° 54′ 55″, and AD = 32° 54′ 22″.

II. In the triangle ADZ are given AZ the first co-alt.=36° 31′, DZ the second co-alt.=37° 12′, and the side AD=32° 54′ 22″, to find the angle ADZ=65° 44′ 30″; hence the difference between GDN and ADZ gives NDZ=17° 10′ 26″.

The ∠ GDN being greater than the ∠ ADZ, this example belongs to Figure 11.

III. In the triangle DZN there are given ND the polar distance=67° 20′, DZ the second co-altitude=37° 12′, and the angle NDZ=17° 10′ 26″, to find the co-lat. NZ=32° 51′ 37″; hence the latitude is 57° 8′ 23″.

The azimuth at the time of each observation may be found as in the preceding example.

(606) If the times as given in Example II., and such like, were exactly true, one altitude *only* would at all times determine the latitude. But we must not depend upon these times; they only serve to point out the elapsed time between the observations. If the watch go true during the elapsed time, or measure that time truly, it is of no consequence to the solution of the problem, whether it be too fast or too slow.

The methods of solving this important problem directly by trigonometry, as is done in the foregoing examples, have been considered by mathematicians as too troublesome for general practice. They have, therefore, contrived various other rules dependent upon an assumed latitude, in such a manner that when the latitude assumed is true, the result of the operation will exactly agree with it. But when the latitude obtained by calculation differs materially from the assumed latitude, the solution must be repeated.

The rule now chiefly used, and which is printed in almost every Treatise on Navigation, the Requisite Tables, &c., was first given by Mr. Dowes, examiner of the marine cadets at Amsterdam; this rule, and tables adapted to it, were first published in England, by Mr. R. Harrison, in the year 1759, but without any account of the construction of the tables. In the year 1760 the learned Dr. Pemberton gave a very circumstantial account of the construction of the tables, and the reason of the process; showed the nature and limits of the approximation, and estimated the errors which may arise from the several circumstances that occur in it. Vide Phil. Trans. 1760.

The following rule depending only on the elapsed time, the two altitudes and declination, is deduced from examples I. and II. It is in all cases accurate, and in general less troublesome in the process than the common method by assumed latitudes.*

GENERAL RULE.

I. Radius is to the sine of the polar distance, as the sine of half the elapsed time is to the sine of half the first arc, the double of which gives *arc the first.*

II. The cotangent of half the elapsed time is to radius, as the cosine of the polar distance is to the cotangent of *arc the second.* This arc is acute or obtuse like the polar distance.

III. Add the two zenith distances and arc the first into one sum. From half this sum subtract the greatest zenith distance. Add together the cosecant of arc the first, the cosecant of the least zenith distance (rejecting the indices), the sine of the half sum, and the sine of the remainder; half this sum is the logarithmic cosine of half the third arc, double of which gives *arc the third.* The *difference* † between arc the third and arc the second gives *arc the fourth.*

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* Dr. Brinkley, Professor of Astronomy in the University of Dublin, has published a set of rules for solving this problem, which are annexed to the Nautical Almanac for the year 1822.

† 1. When the supposed latitude, or latitude by account, is less than, or nearly equal to, the declination, and both the same way, the magnetic azimuths of the sun ought to be attended to at the time of taking the altitudes. For when the magnetic azimuth at taking the least altitude is greater than the magnetic azimuth at taking the greatest altitude, counting from the elevated pole, the *supplement* of arc the second must be *added* to the *supplement* of arc the third to obtain *arc the fourth.*

2. Should the double natural number used near the conclusion of the problem, at any time exceed the natural cosine of the difference between the least zenith distance and polar distance, the latitude will be of a different name to the supposed latitude.

3. Hence it appears that in places near the equator where the azimuths in-

IV. Add double the sine of half the fourth arc, the sine of the polar distance, and the sine of the least zenith distance into one sum, and reject 30 from the index. Consider the remainder as a common logarithm, and find the natural number answering to it, to as many places of figures as the tables extend to, if the index be 9; if 8, to one less; if 7, to two less; and if 6, to three less (for this index can never exceed 9, nor fall short of 6). Subtract double this natural number from the natural cosine of the difference between the polar distance and the least zenith distance, the remainder will be the natural sine of the true latitude.

EXAMPLE III.

In a supposed latitude of 50° 41′ N. at $10^h\ 17^m\ 30^s$ *per* watch, the true altitude of the sun's centre was 17° 13′, and at $11^h\ 17^m\ 30^s$ it was 19° 41′, the sun's declination on that day was 20° South; required the true latitude independent of the supposed one.

Here $11^h\ 17^m\ 30^s - 10^h\ 17^m\ 30^s = 1^h$ the elapsed time, hence ½ the elapsed time $= 30^m = 7°\ 30'$.

I.				
rad = sine of	90°	0′	0″	10·00000
: sin pol. dist.	110	0	0	9·97299
: : sin ½ elap. time	7	30	0	9·11570
: sin ½ arc 1st	7	2	44	9·08869
Arc 1st =	14	5	28	

II.				
cot ½ elap. time	7°	30′	0″	10·88057
: rad = sine of	90	0	0	10·00000
: : cos pol. dist.	110	0	0	9·53405
: cot arc 2nd	92	34	41	8·65348

III.					
Arc first - - -	14°	5′	28″	- cosec	·61357
Greatest zen. dist. -	72	47	0		
Least zen. dist. - -	70	19	0	- cosec	·02615
2 |	157	11	28		
Half sum - - - -	78	35	44	- sin -	9·99134
Greatest zen. dist. -	72	47	0		
Remainder - - -	5	48	44	- sin -	9·00548
				2 |	19·63654
½ third arc - - -	48°	50′	51″	- cos -	9·81827

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crease and decrease slowly, the above method of solution requires a little more attention. However, should any doubt remain whether the latitude found by the general rule be true or not: find the fourth arc as directed in the first note, and use it instead of that found by the rule: one of the two results will most certainly be the true latitude, and some circumstances will generally occur that will seldom fail to point out which is the proper one.

½ third arc - - -		48° 50′ 51″		
		2		
Third arc - -	=	97 41 42		
Second arc - -	=	92 34 41		
Fourth arc - -	=	5 7 1		
IV. ½ fourth arc - -	=	2 33 31	- sin -	8·64973
				2
				17·29946
Polar distance - -		110° 0′ 0″	- sin -	9·97299
Least zen. dist. - -		70 19 0	- - -	9·97385
		176	- log -	7·24630
		2		
		352		
Polar dist. – zen. dist. =39° 41′	nat cos	76959		
True lat. 50° 0′ -	nat sin	76607		

Note. This example, by the approximating rules, requires a repetition of the process, vide page 18, Requisite Tables, Example 2.

EXAMPLE IV.

In a supposed latitude of 51° 0′ North, when the sun's declination was 22° 23′ South, some time past noon the sun's corrected altitude was found by observation to be 14° 46′, and 1^h 22^m afterwards his altitude was 8° 27′; required the true latitude independent of the supposed one.

Answer. The true latitude is 50° 33′ 44″ North.

EXAMPLE V.

In a supposed latitude of 47° 19′ North, when the sun's declination was 12° 16′ North, some time in the afternoon the true altitude of the sun's centre was observed to be 51° 59′, and 2^h 50^m afterwards it was 49° 9′; required the true latitude independent of the supposed one.

Answer. The true latitude is 49° 19′ 30″ North.

EXAMPLE VI.

In a supposed latitude of 30° North, when the sun's declination was 10° 24′ North, at 8^h 54^m in the forenoon the sun's true altitude was 41° 30′, and at 1^h 24^m in the afternoon his altitude was 61° 47′; required the true latitude independent of the supposed one.

Answer. The latitude is 31° 33′ 26″ North.

EXAMPLE VII.

In a supposed latitude of 40° North, when the sun's declination was 2° 46′ South, at 9^h 20^m in the forenoon the sun's altitude corrected was 33° 11′, and at 1^h 20^m in the afternoon his altitude was 42° 44′; required the true latitude independent of the supposed one.

Answer. The latitude is 40° 50′ 10″ North.

EXAMPLE VIII.

In a supposed latitude of 50° 40′ North, when the sun's declination was 20° South, the true altitude of the sun's centre was 19° 41′, and one hour afterward his altitude was 17° 13′; required the true latitude independent of the supposed one.

Answer. The true latitude is 50° North.

EXAMPLE IX.

In a supposed latitude of 60° North, when the sun was on the equinoctial, or had no declination, the true altitude of his centre was 28° 53′, and two hours afterwards his corrected altitude was 20° 42′; required the true latitude independent of the supposed one.

Answer. The true latitude is 59° 59′ 32″ North.

In this example the polar distance is equal to 90°; and in all such examples, the elapsed time will be arc the first, and arc the second will be 90°.

EXAMPLE X.

In North latitude, when the sun's declination was 23° 29′ North, at 8^h 54^m in the forenoon the sun's corrected altitude was 48° 42′, and at 9^h 46^m the altitude was 55° 48′; required the true latitude. *British Palladium*, 1773, *page* 72.

Answer. The true latitude is 49° 49′ 28″ North.

EXAMPLE XI.

In a supposed latitude of 6° 50′ North, at 7^h 30^m in the forenoon the true altitude of the sun's centre was 22° 30′, and at 10^h 36^m 40^s his altitude was 63° 40′, the sun's declination being 22° 48′ North; required the true latitude, without using the supposed one.

Here $10^h\ 36^m\ 40^s - 7^h\ 30^m = 3^h\ 6^m\ 40^s$ the elapsed time, hence half the elapsed time $= 1^h\ 33^m\ 20^s = 23°\ 20'$.

I.			
rad sine of -	90°	0′	10·00000
: sin pol. dist. -	67	12	9·96467
: : sin ½ elap time	23	20	9·59778
: sin ½ arc 1st -	21	25	9·56245
Arc 1st =	42	50	

II.				
cot ½ elap. time	23°	20′	0″	10·36516
: rad sin of -	90	0	0	10·00000
: : cos pol. dist.	67	12	0	9·58829
: cot arc 2nd	80	30	37	9·22313

III.	Arc first - - -	42° 50′	cosec		·16758
	Greatest zenith dist.	67 30			
	Least zenith dist.	26 20	cosec		·35302
	2	136 40			
	Half sum - -	68 20	sin		9·96818
	Greatest zenith dist.	67 30			
	Remainder -	0 50	sin		8·16268
				2	18·65146
	½ third arc -	77° 46′ 40″	cos		9·32573
		2			
	Third arc -	155 33 20			
	Second arc -	80 30 37			
	Fourth arc -	75 2 43			

IV.	½ fourth arc	37° 31′ 21″	sin		9·78467
					2
					19·56934
	Polar distance	67 12 0	sin		9·96467
	Least zenith dist.	26 20 0	sin		9·64698
			log	15170	9·18099
				2	
				30340	
	Polar dist. — zen. dist. =40° 52′		nat cos	75623	
	Latitude -	26° 56′	nat sin	45283	

This latitude being so very different from the latitude by account, it will be necessary to examine it by the notes to the rule.

There can be no error, except in the fourth arc, and this is examined with very little additional trouble, there being only three numbers to take out of the tables:

Arc the second	= 80° 30′ 37″	its supplement	= 99° 29′ 23″
Arc the third	= 155 33 20	its supplement	= 24 26 40
		Sum = arc the fourth	123 56 3

½ arc fourth	61° 58′ - sin -		9·94580
			2
			19·89160
sin polar distance	- - -		9·96467
sin least zenith dist.	- - -		9·64698
		31860 log	9·50325
		2	
		63720	
Natural cosine of the diff. polar dist. and the least zenith dist.		75623	
Latitude - 6° 50′ N. nat sin		11903	

One of the two latitudes found above is certainly the true one; the latter, being the nearest to the latitude by account, may be taken as the proper one.

EXAMPLE XII.

In latitude 1° 50′ North by account, at $10^h\ 24^m$ in the forenoon the true altitude of the sun's centre was 64° 59′, and at $11^h\ 20^m$ it was 78° 57′, the sun's declination being 0° 30′ North; required the true latitude, without making use of the supposed one.

Answer. The latitude by the rule is 1° 36′ 24″ North, but the place being so near to the equator, it will be proper to examine arc the fourth as directed in the notes to the rule. By this method the latitude is 0° 39′ South. The former is nearest the latitude by account.

(607) In all the preceding examples the two observations of the sun's altitude are supposed to have been made at the same place, and the latitude determined by the solution agrees

to that place. Although the two altitudes are generally taken at the same place at land, yet at sea that is seldom the case. An allowance, therefore, ought to be made for the run of the ship during the elapsed time; thus, find the angle contained between the ship's course and the sun, if it be eight points no correction is necessary, but if less or more than eight points the correction must be applied to the first altitude, by addition or subtraction. Consider the angle contained between the ship's course and the bearing of the sun as a course, the distance made good during the elapsed time as a distance; with these find a difference of latitude and apply it as above. The result will reduce the first altitude to what it would have been if taken at the same place where the second was taken. The latitude must be found with these altitudes, thus corrected, in the same manner as before: this will be the latitude of the place where the second altitude was taken.

The difference of longitude during the elapsed time may likewise be taken into consideration, though in general it is of little or no consequence. The change of the sun's declination during the elapsed time might likewise be considered, but this, like the longitude, will cause no sensible error: particularly if the declination answering to the middle time between the observations be used.

PROBLEM XVII. (*Plate III. Fig.* 12.)

(608) *Given the apparent distance of the moon from the sun, or a star, and their apparent zenith distances, to find their true distance, as seen from the earth's centre.*

Let ZM be the observed zenith distance of the moon, *zm* the true zenith distance; M*m* being the difference between the moon's refraction and her parallax in altitude. And let Z⊙ represent the observed zenith distance of the sun or of a star: ZS the true zenith distance, ⊙S being the difference between the sun's refraction and his parallax, or the refraction of a star.

GENERAL PRINCIPLES.

I. Find the segments of the base ⊙V and VM, (by art. 484). With ⊙Z and ⊙V find the angle Z⊙V, which will be equal to the angle T⊙S (295). With ⊙S, and the angle T⊙S, find ⊙T, which will be equal to SP. The triangle T⊙S, being indefinitely small, may be considered as a plane triangle.

Again, with VM and ZM find the angle ZMV, with M*m* and the angle ZMV find the base RM, considering the right-angled triangle *m*RM as a plane triangle.

Lastly, $M\odot + SP - RM = sm$ the true distance in all cases except where the angle at the zenith is acute, and the angle at the moon obtuse, then $M\odot + SP + RM = sm$.

II. Or, with $Z\odot$, ZM, and $\odot M$ find the vertical angle $\odot ZM$ (502); and with ZS, zm, and the angle $\odot ZM$, find the true distance sm (500).

The various methods which have hitherto been made use of for determining the distance between the moon and the sun, or a star, are derived from one or other of the above principles. Those methods which are derived from the latter, are generally preferable to those derived from the former, as being more correct and simple. When the observed distance is small, or the moon's parallax great, and the star's refraction considerable, two other corrections are necessary in order to render the first of these principles generally correct. We have considered the small triangles as right-angled, but the fact is $NP\odot$ and NRM are each of them isosceles triangles, and therefore $\odot P$ and Rm are not strictly perpendicular to sm and $\odot M$; these corrections, therefore, consist in determining how much SP and RM deviate from the bases of right-angled triangles. A true method of determining these *four* corrections may be seen in the Edinburgh Transactions, Article VII., *Physical Class.*

INVESTIGATION OF A GENERAL RULE FOR DETERMINING THE TRUE DISTANCE OF THE MOON FROM THE SUN, OR FROM A FIXED STAR. (*Plate III. Fig.* 13.)

(609) Let $\odot M$ be the observed distance, and sm the true distance. Also let ZM be the observed zenith distance of the moon and zm the true zenith distance; $Z\odot$ the observed zenith distance of the sun or of a star, and zs the true zenith distance, as above. Then in the triangle $\odot ZM$, we have by art. (422),

$$\frac{2\sin^2 \frac{1}{2} Z}{\text{rad}} = \frac{\cos(Z\odot - ZM) - \cos\odot M}{\sin Z\odot \,.\, \sin ZM};$$

and likewise in the triangle szm,

$$\frac{2\sin^2 \frac{1}{2} Z}{\text{rad}} = \frac{\cos(zs - zm) - \cos sm}{\sin zs \,.\, \sin zm}.$$

$$\therefore \frac{\cos(Z\odot - ZM) - \cos\odot M}{\sin Z\odot \,.\, \sin ZM} = \frac{\cos(zs - zm) - \cos sm}{\sin zs \,.\, \sin zm};$$

hence $\cos sm - \cos(zs - zm) = [\cos\odot M - \cos(Z\odot - ZM)] \times \frac{\sin zs \,.\, \sin zm}{\sin Z\odot \,.\, \sin ZM}.$

$$\text{or}\ \cos sm = [\cos \odot\mathrm{M} - \cos (z\odot - z\mathrm{M})] \times \frac{\sin zs\,.\,\sin zm}{\sin z\odot\,.\,\sin z\mathrm{M}} + \cos (zs - zm).$$

Hence the following

GENERAL RULE.*

Take the *sum* of the natural cosine of the difference between the apparent altitudes†, and the natural cosine of the apparent distance, if *greater* than 90°, or their *difference* if *less* than 90°, and find the logarithm of the remainder, to which add the logarithmic secants of the apparent altitudes, the logarithmic cosines of the true altitudes, and reject 40 from the index. The difference between the natural number answering to this sum in the table of logarithms, and the natural cosine of the difference between the true altitudes, will give the natural cosine of the true distance, which will be *less* or *greater* than 90°, according as the natural cosine is *greater* or *less* than the natural number.

Note. The moon's correction, which is the difference between the refraction and the parallax in altitude, must always be added to the apparent altitude, to obtain the true altitude; but the sun or star's correction must always be subtracted. The sun's correction is the difference between the refraction (Table IV.) and the parallax in altitude (Table VI.), and a star's correction is the refraction (Table IV.).

EXAMPLE I.

The apparent distance of the moon's centre from the star Regulus was 63° 35′ 13″, when the apparent altitude of the moon's centre was 24° 29′ 44″, the apparent altitude of the star 45° 9′ 12″; and the moon's horizontal parallax 55′ 2″; required the true distance.

☽'s horizontal parallax=55′ 2″, the parallax in altitude= 50′ 5″ (185), refraction=2′ 5″ (Table IV.); hence the ☽'s correction=(50′ 5″−2′ 5″=)48′ 0″; and the *'s correction =57″ (Table IV.).

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* For other methods, see Professor Young's Trigonometry, page 171, and Riddle's Navigation, page 263.

† The apparent altitudes are the observed altitudes corrected for dip and semidiameter, and the apparent distance is the observed distance corrected for semidiameter

Diff. app. alts.	20° 39′ 28″	nat cos	·93570
Apparent dist.	63 35 13	nat cos	·44484
Difference	·49086	log—	1·69095
Moon's app. alt.* - -	24° 29′ 44″	log sec	10·04096
Star's app. alt. - -	45 9 12	log sec	10·15168
Moon's corr. alt. - -	25 17 44	log cos	9·95623
Star's corr. alt. - -	45 8 15	log cos	9·84844
nat no.	·48782	log—	1·68826
Diff. true alts.	19° 50′ 31″	nat cos	·94063
True distance	63 4 32	nat cos	·45281

EXAMPLE II.

The apparent distance of the moon's centre from the sun's was 106° 46′ 44″, when the apparent altitude of the sun's centre was 45° 32′ 30″, and the moon's 19° 43′ 22″; the moon's correction 50′ 3″, and the sun's 50″; required the true distance of their centres.

Diff. app. alts.	25° 49′ 8″	nat cos	·90017
Apparent dist.	106 46 44	nat cos	·28868
Sum	1·18885	log	0·07513
Moon's app. alt. - -	19° 43′ 22″	log sec	10·02625
Sun's app. alt. - -	45 32 30	log sec	10·15466
Moon's corr. alt. - -	20 33 25	log cos	9·97143
Sun's corr. alt. - -	45 31 40	log cos	9·84544
nat no.	1·18281	log	0·07291
Diff. true alts.	24° 58′ 15″	nat cos	·90652
True distance	106 2 19	nat cos	·27629

EXAMPLE III.

The apparent distance of the sun and moon's centres was 68° 42′ 11″, when the apparent altitude of the sun's centre was

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* Instead of the logarithmic secants of the apparent altitudes, we may take the logarithmic cosines, add them together and take the sum from 20, the remainder will be the same as the sum of the logarithmic secants without the indices.

32° 0′ 1″, and that of the moon's 24° 0′ 10″; the sun's correction was 1′ 23″, and the moon's 51′ 1″; required the true distance of their centres.

Answer. 68° 19′ 46″.

EXAMPLE IV.

The apparent distance of the moon's centre and a star was 2° 20′, when the apparent altitude of the star's centre was 11° 14′, and that of the moon's 9° 39′; the moon's correction was 51′ 30″, and the star's 4′ 40″; required the true distance of their centres.

Answer. 1° 49′.

EXAMPLE V.

The apparent distance of the moon from the star Spica was observed to be 31° 17′ 53″, when the apparent altitude of the moon's centre was 18° 56′ 45″, and the apparent altitude of the star 20° 10′ 56″; the moon's correction was 50′ 46″, and the star's 2′ 35″; required the true distance of their centres.

Answer. The true distance is 31° 11′ 44″.

EXAMPLE VI.

The apparent distance of the moon's centre from that of the sun was 70° 8′ 23″ at a time when the apparent altitude of the sun's centre was 22° 14′ 55″, and the apparent altitude of the moon's centre 80° 52′ 22″; the sun's correction was 2′ 11″, and the moon's correction 8′ 47″; required the true distance of their centres.

Answer. 70° 7′ 56″.

EXAMPLE VII.

The apparent distance of the sun and moon's centres was 63° 5′ 46″, the apparent altitude of the sun 45° 9′ 12″, that of the moon 24° 29′ 40″; the sun's correction 57″, and the moon's 47′ 47″; required the true distance of their centres.

Answer. 62° 44′.

EXAMPLE VIII.

The apparent distance of the sun and moon's centres was 72° 21′ 40″, the apparent altitude of the moon 19° 19′, that of the sun 25° 16′; the moon's correction being 50′ 40″, and the sun's 1′ 53″; required the true distance of their centres.

Answer. 72° 3′ 51″.

PROBLEM XVIII.

(610) *The latitude of a place and its longitude by account, the distance between the sun and the moon, or the moon and a star* in the* NAUTICAL ALMANAC, *being given, to find the correct longitude.*

I. Turn the longitude by account into time (556), and add it to the time by the watch when the observations are taken, if west longitude, but subtract it from that time if east, and you will have the apparent *Greenwich time* nearly.

II. In page III. of the Nautical Almanac, for the given month and day, take out the moon's semidiameter and horizontal parallax, for the noon and midnight between which the Greenwich time falls, subtract the less semidiameter from the greater, and the less parallax from the greater.

Then, as 12 hours are to the first difference, so is the reduced time to a fourth number, which must be added to the moon's semidiameter if the tables are increasing, but subtracted if they are decreasing; to the sum or difference add the augmentation (Table VII.), and you will have the moon's *true semidiameter* at the *Greenwich time.*

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* The principal stars used for finding the longitude at sea are the following:

1. *α Arietis* in the head of Aries. This is a small star, without the zodiac, and cannot be readily found and applied by the generality of persons; it appears about 22° to the right hand of the Pleiades.

2. *Aldebaran* in the Bull's eye is easily distinguished by its largeness, colour, and position to the other stars, being half way between the Pleiades and the star which forms the western shoulder of Orion.

3. *α Pegasi* or *Markab*, about 44° to the right hand of Arietis, a line drawn in the imagination from the Pleiades through Arietis will pass through *α* Pegasi.— The constellation Pegasus is very remarkable, the three principal stars in it, with the head of Andromeda, form a large square, of which the four corner stars are all of the second magnitude.

4. *Pollux*, a little northward of Aldebaran, and about 45° towards the left hand, there are two stars nearly together: the right hand one is Castor, the left hand one Pollux.

5. *Regulus*, about 38° S. E. of Pollux, is easily distinguished by being the southernmost of four bright stars resembling the letter Z inverted, it lies to the north-east of Aldebaran.

6. *Spica*, or *α Virginis*, a white sparkling star, about 54° south-east of Regulus.

7. *Antares*. The arc of a great circle passing through Regulus and *Spica Virginis* east-south-east, or to the left hand of Regulus southward, will pass through Antares, which is about 45° from *Spica Virginis*.

8. *Fomalhaut*, about 45° south of *α* Pegasi.

9. *α Aquila*, 47° westward, or to the right hand of *α* Pegasi.

The distance of the Moon from the Sun, Mars, Venus, Jupiter, Saturn, and the nine stars above described (being near her path), are given in pages XIII. to XVIII. of the Nautical Almanac, for every three hours of mean time for the meridian of Greenwich. The observer who uses the Nautical Almanac, — and certainly no observer ought to be without it, — is under the necessity of taking the distance of one or other of the above stars from the moon. The distances calculated in the Nautical Almanac afford, perhaps, the readiest method of knowing the star from which the moon's distance ought to be observed. For the sextant being fixed to that distance, and the moon found upon the horizon glass, there is nothing more to do than to look to the east or west of the moon, according as the distance corresponds to that given in the Nautical Almanac, guiding the sextant in a line with the moon's shortest axis.

Again, as 12 hours are to the second difference, so is the reduced time to a fourth number, which must be added to or subtracted from the horizontal parallax at the nearest noon, or midnight, preceding the Greenwich time, according as the tables are increasing or decreasing, and it will give the *horizontal parallax* at the *Greenwich time.* (*See Example, page* 272.)

III. Clear the observed altitude of the moon of dip (Table V.) and semidiameter *, and you have the *apparent altitude of her centre:* to the cosine of the moon's apparent altitude, add the logarithm of the horizontal parallax in seconds at the *Greenwich time;* the sum, rejecting 10 from the index, will be the logarithm of the moon's parallax in altitude in seconds (185), from which take the refraction of the moon in altitude (Table IV.), the remainder will be the *moon's correction.*

The moon's correction must always be added to the *apparent altitude* to obtain the *true altitude.*

IV. *Additional Preparation for the Sun and Moon.*

Clear the observed altitude of the sun of dip and semidiameter †, and you have the *apparent altitude of his centre.*

From the refraction of the sun's altitude (Table IV.) take his parallax in altitude (Table VI.), and you have the *correction of the sun's altitude,* which must always be subtracted from the apparent altitude to obtain the *true altitude.*

V. To the observed distance of the sun and moon's nearest limbs, add their semidiameters at the *Greenwich time,* and the sum will be the *apparent distance* of their centres.

IV. *Additional Preparation for the Moon and a Star.*

From the star's observed altitude take the dip of the horizon (Table V.), the remainder will be its *apparent altitude.* The refraction of a star (Table IV.) is the *correction of its altitude,* and must always be subtracted from the apparent altitude to obtain the *true altitude.*

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* *Viz.* take their difference, and add it to the observed altitude of the moon's lower limb; or take their sum, and subtract it from the observed altitude of the moon's upper limb, according as the lower or upper limb has been observed.

† *Viz.* take their difference, and add it to the observd altitude of the ⊙'s lower limb; or take their sum, and subtract it from the observed altitude of the ⊙'s upper limb, according as the lower or upper limb has been observed.

V. To the observed distance of the moon from a star, add the moon's semidiameter at the *Greenwich time*, the sum will be the *apparent distance;* if the remote limb is observed, subtract the semidiameter.

VI. *To find the true Distance.*

With the apparent altitudes, their corrections, and the apparent distance, find the true distance by the general rule (609).

Note. If the watch has not been previously regulated, the apparent time must now be found with the mean altitude of the sun or star, and the latitude of the ship, as in Prob. XIV. and XV., taking care to reduce the sun's declination to the *Greenwich time.*

VII. *To find the Longitude.*

In pages XIII. to XVIII. of the Nautical Almanac for the given month and day, look for the computed distance between the moon and the other object; if you find it there exactly, the time at Greenwich stands at the top of the column; but if you do not find it exactly, take the nearest distances to it, both less and greater; take their difference, and likewise the difference between the computed distance and the *earliest* Ephemeris distance.

Then, as the first difference is to 3 hours, so is the second difference to a fourth number, which being added to the time standing over the *earliest* Ephemeris distance, will give the *mean time* at Greenwich, to which apply the equation of time as directed in Page II. of the Nautical Almanac, and we have the *apparent time* at Greenwich.

The difference between the ship's *apparent time*, and the *apparent time* at Greenwich (turned into degrees) will be the longitude of the place, where the altitude of the object was taken for determining the apparent time. If the ship's time be greater, the longitude is east; if less, west.

EXAMPLE I.

Suppose on the 11th of March 1840, in longitude 7° west by account, at $5^h\ 15^m\ 45^s$ A. M. apparent time per watch, well regulated, the distance of the moon's farthest limb from the star Regulus to be 64° 2′, the altitude of the moon's lower limb 24° 18′ 30″, the star's altitude 45° 13′ 15″, and the eye 18 feet above the surface of the sea; required the true longitude.

I.

Apparent time (*astron.*) at ship, Mar. 10th	17^h	15^m	45^s
Long. 7° W. in time - - -	0	28	0
Apparent *Greenwich time* -	17	43	45

II.

Mar. 10, ☽'s semidiam. at midnight	=16′	5″	Hor. paral. - -	=59′	2″
Mar. 11, ☽'s semidiam. at noon	=16	3	Hor. paral. - -	=58	54
First diff.	0	2	Second diff.	0	8

$12^h : 2'' :: 5^h\ 44^m : 0'\ 1''$

☽'s semidiam. at midnight	=16	5
☽'s semidiam. at Gr. time	=16	4
☽'s augmentation (tab. VII.)	= +	7
☽'s true semidia. at Gr. time	=16	11

$12^h : 8'' :: 5^h\ 44^m : 0'\ 4''$

Hor. parallax at midnight	=59	2
Hor. parallax at Gr. time	=58	58
		60
In seconds	3538	

III.

☽'s observed altitude	=24°	18′	30″
☽'s semidia. =16′ 11″ } Dip. (Tab. V.)= 4 3 }	= +12		8
App. altitude ☽'s centre	=24	30	38
☽'s correction -	=	51	35
☽'s corrected altitude	=25	22	13

Cos ☽'s apparent altitude	=9·95899
Horizontal paral. 3538″ log	=3·54876
Paral. in altitude 3219″ log	=3·50775

3219″ = 53′ 39″

☽'s refr. (Tab. IV.) = 2 4

☽'s correction - =51 35

IV.

✱'s observed altitude	=45°	13′	15″
Dip. (Table V.) -	= −	4	3
✱'s apparent altitude -	=45	9	12
✱'s refraction (Tab. IV.)	=	−	57
✱'s corrected altitude	=45	8	15

V.

Observed distance -	=64°	2′	0″
☽'s semidiameter -	= −16		11
Apparent distance -	=63	45	49

VI. *To find the true Distance.*

✱'s app. alt.	45°	9′	12″	log sec	-	10·15168
☽'s app. alt.	24	30	38	log sec	-	10·04101
Diff. app. alts.	20	38	34	nat cos	·93579	
App. dist.	63	45	49	nat cos	·44208	
				Common log of	·49371	−1·69347
✱'s cor. alt.	45°	8′	15″	log cos	-	9·84844
☽'s cor. alt.	25	22	13	log cos	-	9·95596
				Natural number	·49041	− 1·69056
Diff. of true alts.	19°	46′	2″	nat cos	·94108	
True dist. -	63	12	48	nat cos	·45067	

VII. *To find the Longitude.*

Distance at 15^h - -	$=64°\ 54'\ 37''$	Earliest ephem. distance	$=64°\ 54'\ 37''$
Distance at 18 - -	$=63\ \ 9\ \ 11$	Computed distance -	$=63\ \ 12\ \ 48$
First difference -	$=\ 1\ \ 45\ \ 26$	Second difference -	$=\ 1\ \ 41\ \ 49$

Then $1°\ 45'\ 26'' : 3^h :: 1°\ 41'\ 49'' :$ $2^h\ 53^m\ 49^s$
Time above the earliest distance $=15$

Mean time at Greenwich - - $=17\ \ 53\ \ 49$
Mar. 10. Equat. of time $=-10^m\ 27^s\!\cdot\!0$
Mar. 11. Equat. of time $=-10\ \ 10\cdot9$

$16\cdot1$

As $24^h : 16^s\!\cdot\!1 :: 17^h\ 54^m :$ $12\cdot0$

Equat. of time at $17^h\ 54^m$ - $=\ -10^m\ 15^s$

Apparent time at Greenwich - - 17 43 34
Apparent time at ship - - - 17 15 45

Difference - - 0 27 49 $=6°\ 57'\ 15''$
the longitude west of Greenwich, because Greenwich time exceeds the ship's time.

EXAMPLE II.

Suppose on the 7th of May 1840, in longitude 105° E. of Greenwich by account, at $5^h\ 1^m\ 37^s$ A.M. apparent time, the distance between the sun and moon's nearest limbs to be $68°\ 9'\ 30''$, the altitude of the sun's lower limb $31°\ 48'\ 16''$, the altitude of the moon's lower limb $23°\ 44'\ 16''$, and the eye 18 feet above the level of the sea; required the true longitude.

☽'s semi-diameter at noon, May 6th (*Naut. Alm.*) $=16'\ 11''$, and at midnight $=16'\ 4''$; horizontal parallax at noon $=59'\ 22''$, and at midnight $=58'\ 58''$; also the ⊙'s semidiameter $=15'\ 52''$. Equation of time (*added* to mean time) May 6th is $3^m\ 35^s\!\cdot\!0$, and May 7th it is $3^m\ 39^s\!\cdot\!4$.

Answer. The apparent time at Greenwich is May 6th $10^h\ 1^m\ 37^s$; ☽'s true semidiameter $=16'\ 12''$; horizontal parallax $=59'\ 2''$; apparent altitude of the ☽'s centre $23°\ 56'\ 25''$; and the correction of her altitude $=51'\ 50''$.

The apparent altitude of the ⊙'s centre = 32° 0′ 5″, correction 1′ 23″; apparent distance of the ⊙ and ☽'s centres 68° 41′ 34″, true distance 68° 18′ 49″, and the true longitude = 104° 56′ 30″ East.

(*Naut. Alm.* distance at IX^h, being = 67° 47′ 7″, and at midnight = 69° 25′ 6″.)

EXAMPLE III.

Suppose on the 6th of August 1840, in longitude 102° W., by account, the following observations to be taken: —

Apparent Time A. M.		Alt. ⊙'s lower limb.		Alt. ☽'s upper limb.		Distance of nearest limbs.
$9^h 24^m 28^s$	-	44° 48′ 30″	-	20° 32′ 0″	-	106° 19′ 15″
9 25 28	-	45 4 0	-	20 18 30	-	106 18 45
9 26 18	-	45 22 45	-	20 10 0	-	106 18 30
9 26 38	-	45 45 0	-	19 50 0	-	106 17 45
9 27 13	-	45 50 0	-	19 30 0	-	106 16 45
5 | 47 10 . 5		5 | 226 50 15		5 | 100 20 30		5 | 531 31 0
Mean 9 26 1	-	45 22 3	-	20 4 6	-	106 18 12
Errors of the quadrant		−58	-	−1 0	-	−2 37
True mean		45 21 5	-	20 3 6	-	106 15 35

Required the true longitude, the eye being 21 feet above the level of the sea.

☽'s semidiameter at noon, Aug. 6th (*Naut. Alm.*) = 14′ 48″, and at midnight 14′ 47″; horizontal parallax at noon = 54′ 19″, at midnight 54′ 14″; also the ⊙'s semi-diameter = 15′ 48″. Equation of time (*subtracted* from mean time) Aug. 6th is $5^m 33^s \cdot 6$, and on Aug. 7th it is $5^m 26^s \cdot 7$.

Answer. The reduced time is Aug. 6th $4^h 14^m 1^s$; ☽'s true semi-diameter 14′ 54″; horizontal parallax 54′ 17″; apparent altitude of her centre 19° 43′ 50″; and correction of her altitude 48′ 29″.

The apparent altitude of the sun's centre is 45° 32′ 31″; correction 50″; apparent distance of centres 106° 46′ 17″; true distance 106° 3′ 11″, and the true longitude 102° 10′ 45″ W.

(*Naut. Alm.* distance at III^h = 105° 26′ 44″, and at VI^h = 106° 48′ 28″).

EXAMPLE IV.

Suppose on the 22d of March, 1840, in longitude 127° 40′ E. by account, the following observations to be taken: —

	Apparent Time P. M.	Alt. ☽'s lower limb.	Alt. ✱ Spica ♍.	Dist. of ✱ from ☽'s far limb.
	$12^h\ 30^m\ 16^s$	19° 10′ 20″	20° 39′ 40″	31° 35′ 45″
	12 32 26	18 50 10	20 29 20	31 34 10
	12 34 36	18 40 30	20 15 50	31 32 50
	12 36 26	18 20 10	20 4 10	31 31 35
	12 38 24	18 9 20	19 47 30	31 30 40
	5 | 62 52 8	5 | 93 10 30	5 | 101 16 30	5 | 157 45 0
Mean	12 34 26	18 38 6	20 15 18	31 33 0
Error of the quad.		+ 7 39	Error − 19	
		18 45 45	20 14 59	

Required the true longitude, the eye being 18 feet above the level of the sea.

☽'s semidiameter at noon, March 22d (*Naut. Alm.*) = 14′ 48″, and at midnight 14′ 47″; horizontal parallax at noon 54′ 18″, and at midnight 54′ 13″. Equation of time (*subtracted* from mean time) on March 22d is 6^m 57^s·6, and on March 23d it is 6^m 39^s·1.

Answer. The apparent time at Greenwich is 4^h 3^m 46^s, ☽'s true semidiameter = 14′ 54″, horizontal parallax = 54′ 16″, apparent altitude of the ☽'s centre = 18° 56′ 36″, and her correction = 48′ 35″.
App. altitude of the star 20° 10′ 56″, correction 2′ 34″.
App. distance of centres 31° 18′ 6″; true distance 31° 12′ 12″.
The true longitude 127° 45′ East.

(*Naut. Alm.* distance at III^h = 30° 37′ 24″, and at VI^h = 32° 6′ 27″.)

(611) In all the preceding examples the watch is supposed to have been previously regulated; when that is not the case, the error of the watch must be found from observations of the altitudes of the sun or of a star, taken either before or after that of the distance, as directed in Problems XIV. or XV., pages 316 and 319. Or if the sun or star be sufficiently distant from the meridian, the mean of the sun's or star's altitudes, taken at the same time as the distance is taken, together with the latitude of the place and the sun's declination, &c. may be used to correct the watch; with this corrected time proceed as before.

EXAMPLE V.

At sea, April 7th, 1840, in latitude 47° 39′ N. and longitude 184° 45′ West from Greenwich by account at 3^h 56^m P. M. apparent time, per watch not previously regulated; suppose the observed altitude of the sun's lower limb to be 25° 57′ 10″, and that of the moon's lower limb 46° 22′ 32″, and at the same time the distance of the sun's and moon's nearest limbs to be 75° 33′ 37″; required the longitude, the eye being 18 feet above the level of the sea.

☽'s semidiameter at midnight, April 7th (*Naut. Alm.*) = 16′ 9′, and at noon April 8th = 16′ 4″; horizontal parallax April 7th at midnight = 59′ 15″, and at noon April 8th = 58′ 58 ; also the ⊙'s semidiameter = 15′ 59″. Equation of time (*subtracted* from mean time) April 7th is $2^{m}\ 7^{s}{\cdot}0$, and April 8th it is $1^{m}\ 49^{s}{\cdot}9$.

Answer. The apparent time at Greenwich is April $7^{d}\,16^{h}\,15^{m}$; ☽'s true semidiameter 16′ 18″; horizontal parallax 59′ 9″; apparent altitude of her centre 46° 34′ 47″, and correction of her altitude 39′ 45″.

The apparent altitude of the sun's centre is 26° 9′ 6″; correction 1′ 47″; apparent distance of centres 76° 5′ 54″; true distance 75° 51′ 24″. Apparent time at Greenwich $16^{h}\ 12^{m}\ 19^{s}$.

The sun's declination at apparent noon (*Naut. Alm.*) on the 7th, is 6° 58′ 3″ N., and hourly difference 56″·1, which reduced to the apparent time at Greenwich is 7° 13′ 15″ N.; with this declination, the ⊙'s true altitude 26° 7′ 19″; and the co-latitude 42° 21′, find the correct apparent time at ship = $3^{h}\ 54^{m}\ 43^{s}$; and hence the true longitude is 184° 24′ West.

(*Naut. Alm.* distance at XV^{h} = 75° 10′ 44″, and at $XVIII^{h}$ = 76° 49′ 21″.)

CHAP. XIII.

ON THE DIFFERENTIAL ANALOGIES OF SPHERICAL TRIANGLES.*

PROPOSITION I. (*Plate IV. Fig.* 3.)

A preparatory Proposition.

(612) CONSTRUCT a general figure as at Prop. XIX. Book III. Chap. I. Thus, let A be the pole of the circle HGFE; F the pole of ABH; E the pole of CGI; and C the pole of EDI.

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[*] See Simpson's Fluxions, vol. ii. page 278. Traité de Trigonométrie, par M Cagnoli, chap. xix. page 310. Hind's Trigonometry, 2nd edition, page 286. Young's Trigonometry, page 203.

Suppose these circles to be invariable whilst another great circle DFCB revolves about the pole F, and let cn be at right angles to the great circle mnoFsd; then the three triangles ABC, CGF, and EDF will be variable, viz.

I. In the right-angled triangle ABC, the ∠ A will be a fixed quantity, and the other parts will be variable; viz. Bm will be the increment of AB; no the increment of BC; co the increment of AC; and Ds the increment of the arc ID which measures the ∠ C.

II. In the right-angled triangle CGF, the side FG will be a fixed quantity, and the other parts will be variable; viz. co will be the decrement of CG; no the decrement of FC; Bm the decrement of the ∠ CFG; and Ds the increment of the ∠ C.

III. In the right-angled triangle FDE, the hypothenuse EF will be a fixed quantity, and the other parts will be variable; viz. $sd = no$, will be the increment of FD = BC; sD the decrement of ED; and Bm the decrement of the ∠ EFD = ∠ CFG.

PROPOSITION. II. (*Plate IV. Fig.* 3.)

(613) *In any right-angled spherical triangle* ABC, *right-angled at* B, *suppose one of the angles as* A *to remain constant, it is required to find the ratio of the differentials of the other parts.*

1. In the triangles FBm, FCn, having the same acute ∠ at F,

$$\sin \text{FB} : \sin \text{FC} :: \tan \text{B}m : \tan cn \quad (382).$$

But FB = 90°, FC is the complement of BC, and Bm and cn, being very small arcs, have the same ratio to each other as their tangents.

$$\therefore \text{rad} : \cos \text{BC} :: \text{B}m : cn = \frac{\cos \text{BC}}{\text{rad}} \cdot \text{B}m.$$

2. In the triangles DCI, con, where the ∠ DCI may be supposed equal to the ∠ con,

$$\tan \text{DI} : \sin \text{CI} :: \tan cn : \sin no \quad (382).$$

But DI is the measure of the ∠ C, CI = 90°, and the tangent of cn and the sine of no, have the same ratio to each other as the arcs,

$$\therefore \tan \text{C} : \text{rad} :: cn \left(= \frac{\cos \text{BC}}{\text{rad}} \cdot \text{B}m\right) : no = \frac{\cos \text{BC}}{\tan \text{C}} \cdot \text{B}m.$$

Again, sin DI : sin DC :: sin cn : sin co (382).

But DI = ∠ C, DC = 90°, and the very small arcs cn and co, have the same ratio to each other as their sines,

$$\therefore \sin \text{C} : \text{rad} :: cn \left(= \frac{\cos \text{BC}}{\text{rad}} \cdot \text{B}m\right) : co = \frac{\cos \text{BC}}{\sin \text{C}} \cdot \text{B}m.$$

3. In the triangles BFm, DFs, where $Fs = FD$ ultimately,

$$\sin FB : \sin Bm :: \sin Fs : \sin Ds \quad (382).$$

But $FB = 90°$, $Fs = FD = BC$, and the small arcs are as their sines,

$$\therefore \text{rad} : Bm :: \sin BC : Ds = \frac{\sin BC}{\text{rad}} \,.\, Bm.$$

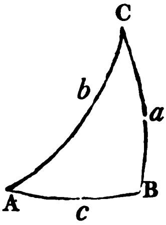
$$\text{Hence } no = \frac{\cos BC}{\tan C} \,.\, Bm; \quad co = \frac{\cos BC}{\sin C} \,.\, Bm;$$

$$\text{and } Ds = \frac{\sin BC}{\text{rad}} \,.\, Bm.$$

But no represents the differential of BC, co of AC, Ds of DI $= \angle C$, and Bm of AB (269). Therefore,

$$d\text{BC} = \frac{\cos BC}{\tan C} \,.\, d\text{AB}; \quad d\text{AC} = \frac{\cos BC}{\sin C} \,.\, d\text{AB}; \quad d\text{C} = \frac{\sin BC}{\text{rad}} \,.\, d\text{AB}.$$

(614) Hence, in any right-angled spherical triangle ABC, right-angled at B; by denoting the sides opposite to the angles A, B, C, by a, b, c, respectively, we derive by substitution and reduction, the following general equations, *which include all the varieties that can possibly happen wherein the* $\angle$ B *is* 90° *and one of the other angles* (*viz.* A) *constant.*



*I. $da = \dfrac{\cos a}{\tan C} \,.\, dc = \dfrac{\cos C}{\text{rad}} \,.\, db = \dfrac{\cot a}{\tan C} \,.\, dc.$

II. $db = \dfrac{\cos a}{\sin C} \,.\, dc = \dfrac{\text{rad}}{\cos C} \,.\, da = \dfrac{\cot a}{\sin C} \,.\, dc.$

III. $dC = \dfrac{\sin a}{\text{rad}} \,.\, dc = \dfrac{\tan C}{\cot a} \,.\, da = \dfrac{\sin C}{\cot a} \,.\, db.$

IV. $dC = \dfrac{\tan C}{\cos a} \,.\, da = \dfrac{\sin C}{\cos a} \,.\, db = \dfrac{\text{rad}}{\sin a} \,.\, dc.$

(615) Any of the foregoing equations may be turned into proportions, or varied in the expression, by reference to pages 104, 105, &c.

Thus, from the fourth set of equations $dC \,.\, \cos a = db \,.\, \sin C$.

$$\text{that is, } db : dC :: \cos a : \sin C.$$

$$:: \frac{\cos b}{\cos c} : \frac{\sin c}{\sin b} \quad (384).$$

$$:: \cos b \,.\, \sin b : \cos c \,.\, \sin c.$$

$$:: \sin 2b : \sin 2c\dagger \quad (228).$$

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* Simpson's Fluxions, page 280.

† Vince's Trigonometry, 2d edition, page 139.; Traité de Trigonométrie, par M. Cagnoli, art. 677. page 329

PROPOSITION III. (*Plate IV. Fig.* 3.)

(615) *In any right-angled spherical triangle* CGF, *right-angled at* G, *suppose one of the sides as* FG *to remain constant, it is required to find the ratio of the differentials of the other parts.*

By the foregoing proposition $no = \frac{\cos \mathrm{BC}}{\tan \mathrm{C}} \,.\, \mathrm{B}m$; $co = \frac{\cos \mathrm{BC}}{\sin \mathrm{C}} \,.\, \mathrm{B}m$; and $\mathrm{D}s = \frac{\sin \mathrm{BC}}{\mathrm{ad}} \,.\, \mathrm{B}m$. But no is the decrement of FC; Bm the decrement of the ∠ F; co the decrement of CG; and Ds the increment of the ∠ C, also FC is the complement of BC. Therefore,

$-d\mathrm{CF} = \frac{\sin \mathrm{CF}}{\tan \mathrm{C}} \times -d\mathrm{F}$; $-d\mathrm{CG} = -\frac{\sin \mathrm{CF}}{\sin \mathrm{C}} \times -d\mathrm{F}$; and $d\mathrm{C} = \frac{\cos \mathrm{CF}}{\mathrm{rad}} \times -d\mathrm{F}$; hence $d\mathrm{F} = \frac{\tan \mathrm{C}}{\sin \mathrm{CF}} \,.\, d\mathrm{CF} = \frac{\sin \mathrm{C}}{\sin \mathrm{CF}} \,.\, d\mathrm{CG} = -\frac{\mathrm{rad}}{\cos \mathrm{CF}} \,.\, d\mathrm{C}$.

(616) Hence, if the fixed side FG be represented by c, the hypothenuse FC by b, the side CG by a, and the angles opposite to these sides by C, B, and A respectively, as at (614), we derive the following general equations or formulæ, *which comprehend all the different cases that can possibly happen wherein the angle* B *is* 90°, *and one of its adjacent sides* (viz c) *constant.*

* I. $d\mathrm{A} = \frac{\tan \mathrm{C}}{\sin b} \,.\, db = \frac{\sin \mathrm{C}}{\sin b} \,.\, da = -\frac{\mathrm{rad}}{\cos b} \,.\, d\mathrm{C}$.

II. $d\mathrm{C} = -\frac{\cos b}{\mathrm{rad}} \,.\, d\mathrm{A} = -\frac{\tan \mathrm{C}}{\tan b} \,.\, db = -\frac{\sin \mathrm{C}}{\tan b} \,.\, da$.

III. $da = \frac{\sin b}{\sin \mathrm{C}} \,.\, d\mathrm{A} = \frac{\mathrm{rad}}{\cos \mathrm{C}} \,.\, db = -\frac{\tan b}{\sin \mathrm{C}} \,.\, d\mathrm{C}$.

IV. $db = \frac{\sin b}{\tan \mathrm{C}} \,.\, d\mathrm{A} = \frac{\cos \mathrm{C}}{\mathrm{rad}} \,.\, da = -\frac{\tan b}{\tan \mathrm{C}} \,.\, d\mathrm{C}$.

(617) The preceding formulæ may be varied in the expression by reference to pages 104, 105, &c. or they may be turned into proportions thus, from the first set of equations $d\mathrm{A} \,.\, \sin b = da \,.\, \sin \mathrm{C}$, that is,

$$da : d\mathrm{A} :: \sin b : \sin \mathrm{C}.\dagger$$

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* Simpson's Fluxions, page 280.

† Vince's Trigonometry, 2d edition, page 140.

PROPOSITION IV. (*Plate IV. Fig.* 3.)

(618) *In any right-angled spherical triangle* FDE, *right-angled at* D, *suppose the hypothenuse* EF *to remain constant, it is required to find the ratio of the differentials of the other parts.*

By Prop. II., $no = \frac{\cos \mathrm{BC}}{\tan \mathrm{C}} \cdot \mathrm{B}m$; $co = \frac{\cos \mathrm{BC}}{\sin \mathrm{C}} \cdot \mathrm{B}m$; and $\mathrm{D}s = \frac{\sin \mathrm{BC}}{\mathrm{rad}} \cdot \mathrm{B}m$.

But *no* is the increment of BC, or of its = FD; B*m* is the measure of the ∠ BF*m*, or of its equal *s*FD, and consequently it is the decrement of the ∠ EFD; also BC = FD, and the ∠ C is measured by the arc DI, which is the complement of ED.

$$\text{Hence } d\mathrm{FD} = \frac{\cos \mathrm{FD}}{\cot \mathrm{ED}} \times -d\mathrm{F}.$$

Again, *co* is the decrement of CG, and CG is the complement of GI, the measure of the ∠ E, therefore *co* is the increment of the ∠ E.

$$\text{Hence } d\mathrm{E} = \frac{\cos \mathrm{FD}}{\cos \mathrm{ED}} \times -d\mathrm{F}.$$

Also D*s* is the decrement of ED, consequently we have

$$-d\mathrm{ED} = \frac{\sin \mathrm{FD}}{\mathrm{rad}} \times -d\mathrm{F}.$$

$$\therefore\ d\mathrm{F} = -\frac{\cot \mathrm{ED}}{\cos \mathrm{FD}} \cdot d\mathrm{FD} = -\frac{\cos \mathrm{ED}}{\cos \mathrm{FD}} \cdot d\mathrm{E} = \frac{\mathrm{rad}}{\sin \mathrm{FD}} \cdot d\mathrm{ED}.$$

(619) Hence, if the hypothenuse EF be represented by *b*, the side FD by *c*, ED by *a*, and their opposite angles by B, C, and A respectively, as at (614), we derive the following general formulæ, *which include all the varieties that can possibly happen, wherein the angle* B *is* 90°, *and the hypothenuse* (*viz. b*) *constant.*

$$\text{I. } d\mathrm{A} = -\frac{\cot a}{\cos c} \cdot dc = -\frac{\cos a}{\cos c} \cdot d\mathrm{C} = \frac{\mathrm{rad}}{\sin c} \cdot da.$$

$$\text{II. } d\mathrm{C} = -\frac{\cos c}{\cos a} \cdot d\mathrm{A} = -\frac{\cot c}{\cos a} \cdot da = \frac{\mathrm{rad}}{\sin a} \cdot dc.$$

$$\text{III. } dc = -\frac{\cos c}{\cot a} \cdot d\mathrm{A} = -\frac{\cot c}{\cot a} \cdot da = \frac{\sin a}{\mathrm{rad}} \cdot d\mathrm{C}.$$

$$\text{IV. } da = \frac{\sin c}{\mathrm{rad}} \cdot d\mathrm{A} = -\frac{\cos a}{\cot c} \cdot d\mathrm{C} = -\frac{\cot a}{\cot c} \cdot dc.$$

(620) These formulæ may be varied in the expression or turned into proportions, &c. as before observed. Thus from the third set of equations.

$$dc = -\frac{\cot c}{\cot a} \cdot da\text{; but } \cot c = \frac{\text{rad}^2}{\tan c}\text{, and } \cot a = \frac{\text{rad}^2}{\tan a}.$$

$$\therefore\ dc = -\frac{\tan a}{\tan c} \cdot da. \quad \text{Hence } da : dc :: -\tan c : \tan a.^{*}$$

PROPOSITION V. (*Plate IV. Fig.* 4.)

(621) *In any oblique-angled spherical triangle* ABC, *suppose the angle* A *and its adjacent side* AB *to remain constant*[†], *it is required to find the ratio of the differentials of the other parts.*

Let BC, by its revolution about B, be changed to Bm, then AC will become Am, and the ∠ ABC will be reduced to the ∠ ABm. Now mC will be the decrement of AC, the ∠ CBm will be the decrement of the ∠ B, and if Cn be drawn perpendicular to Bn, mn will be the decrement of BC; produce Bn and BC so that Bp and Bo may be quadrants, then op will be the measure of the ∠ CBm. Hence,

I. Cn : op : : sin BC : sin Bo,

and in the small rectilineal triangle Cnm right-angled at n

mn : Cn : : sin ∠ mCn : cos ∠ mCn (109).

∴ mn : op : : sin BC . sin ∠ mCn : sin Bo . cos ∠ mCn.

But ∠ mCn will be the complement of the angle C, because BCn is a right angle, and Bo = 90° by the construction, hence

dBC : dB : : sin BC . cos C : rad . sin C.

$$: :\ \sin \text{BC} : \frac{\text{rad} . \sin \text{C}}{\cos \text{C}}.$$

: : sin BC : tan C.

II. Again, rad : mC : : sin mCn : mn,

that is, rad : dAC : : cos C : dBC.

III. Also, mC : nC : : rad : cos ∠ mCn,

and nC : op : : sin BC : sin Bo = rad;

∴ mC : op : : sin BC : sin C,

that is, dAC : dB : : sin BC : sin C.

IV. By taking the supplemental[‡] triangle to ABC, viz. DFE (302), the ∠ E and the side FE will be constant quantities, and the rest will be variable; now it is shown in the first case of this proposition, that the differential of the side opposite a constant ∠ : the differential of the ∠ adjacent to the constant

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[*] Vince's Trigonometry, Prop. 49. page 140. 2d edition.

[†] Traité de Trigonométrie, par Cagnoli, page 311. La Lande's Astron. vol. iii. art. 3998 to 4003.

[‡] Called by the French writers the polar triangle, from the manner in which it is described.

side : : sine of the side opposite to the constant ∠ : tan ∠ opposite to the constant side.*

Hence *d*DF : *d*F : : sin DF : tan ∠ D, that is (302),

*d*C : *d*AC : : sin C : —tan BC,

and *d*B : *d*AC : : sin C : sin BC, (III. case above.)

∴ *d*B : *d*C : : —tan BC : sin BC,

: : rad : —cos BC.

OTHERWISE. (*Plate IV. Fig.* 4.)

(622) Let a perpendicular BD be drawn from the end of the fixed side AB and opposite to the fixed ∠ A, then the several parts of the right-angled triangle BDA will be invariable; and in the right-angled triangle BDC, the perpendicular BD will be a fixed quantity, and all the other parts will be variable; hence by the assistance of the formulæ already given (616), all the variations of which this proposition is susceptible may easily be derived; some of the principal will here be inserted, denoting the sides of the triangle by *a*, *b*, *c*, and their opposite angles by A, B, C, as in the figure, where A is the constant angle, and *c* the constant side.

db : *d*B : : sin *a* : sin C

db : *da* : : rad : cos C

db : *d*C : : tan *a* : —sin C

da : *d*B : : sin *a* : tan C

*d*B : *d*C : : rad : —cos *a*

*d*C : *da* : : tan C : —tan *a*

PROPOSITION VI. (*Plate IV. Fig.* 5.)

(623) *In any oblique-angled spherical triangle* ABC, *suppose the angle* A *and its opposite side* BC *to remain constant* †, *it is required to find the ratio of the differentials of the other parts.*

Let BC change its position to *mo*, and let these circles intersect each other in an indefinitely small angle at *r*; make *rn* = *r*B, and *r*C = *rp*; then *nm* will be the decrement of *r*B, and *po* will be the increment of *r*C; and because *np* = BC, by construction, and *mo* = BC, by hypothesis; if the common part *mp* be taken from each, there will remain *nm* = *op*; then,

I. Considering the indefinitely small triangles B*nm*, C*po*, as rectilineal, and right-angled at *n* and *p*, we have

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[*] Cagnoli, page 314. art. *555.*

[†] La Lande's Astronomy, vol. iii. art. 4003 to 4010. Cagnoli, page 316, &c.

In the right-angled triangle cpo, rad : co :: cos $\angle poc$: op, and in the right-angled triangle Bnm, rad : Bm :: sin $\angle nBm$: $nm = op$;

$$\therefore co : Bm :: \sin \angle nBm : \cos \angle poc.$$

that is, $d\text{AC} : d\text{AB} :: \cos \text{B} : -\cos \text{C}$.

For, co is the increment of AC, Bm is the decrement of AB; and the three angles pco, pcr, and ACr are together equal to two right angles (294), of which pcr is a right angle, therefore $\angle pco + \angle \text{AC}r$ = one right angle $= \angle \text{P}co + \angle poc$, hence $\angle poc = \angle \text{ACB}$.

II. Take the supplemental triangle to ABC, viz. DFE (302), then, since $\angle$ A and BC are constant quantities in the triangle ABC, EF and the $\angle$ D will be constant quantities in the triangle DFE; whence by the first part of this proposition, we have $d\text{DE} : d\text{DF} :: \cos \text{F} : -\cos \text{E}$; but DE is the measure of B, and DF is the measure of C, also $\cos \text{F} = -\cos \text{AC}$, and $\cos \text{E} = -\cos \text{AB}$; $\therefore d\text{B} : d\text{C} :: \cos \text{AC} : -\cos \text{AB}$.

III. Again, $\sin \text{A} : \sin \text{BC} :: \sin \text{C} : \sin \text{AB} :: \sin \text{B} : \sin \text{AC}$. That is, the sines of the variable angles have a constant ratio to the sines of their opposite sides, consequently their differentials will be in the same ratio; but the rectangle of radius and the differential of the sine of any arc = the rectangle of the differential of that arc and its cosine (1st equation, page 127.) Hence,

$$d \sin \text{C} = \frac{\cos \text{C}}{\text{rad}} \,.\, d\text{C}, \quad d \sin \text{B} = \frac{\cos \text{B}}{\text{rad}} \,.\, d\text{B},$$

$$d \sin \text{AB} = \frac{\cos \text{AB}}{\text{rad}} \,.\, d\text{AB}, \quad d \sin \text{AC} = \frac{\cos \text{AC}}{\text{rad}} \,.\, d\text{AC};$$

$$\therefore \sin \text{A} : \sin \text{BC} :: \cos \text{C} \,.\, d\text{C} : \cos \text{AB} \,.\, d\text{AB},$$
$$:: \cos \text{B} \,.\, d\text{B} : \cos \text{AC} \,.\, d\text{AC},$$

that is, $d\text{C} : d\text{AB} :: \sin \text{A} \,.\, \cos \text{AB} : \sin \text{BC} \,.\, \cos \text{C}$,

and $d\text{B} \,.\, d\text{AC} :: \sin \text{A} \,.\, \cos \text{AC} : \sin \text{BC} \,.\, \cos \angle \text{B}$.

But it has been shown in the first part of this proposition, that

$$d\text{AB} : d\text{AC} :: \cos \text{C} : -\cos \text{B},$$

$$\therefore d\text{C} : d\text{AC} :: \sin \text{A} \,.\, \cos \text{AB} : -\sin \text{BC} \,.\, \cos \text{B}.$$

(624) Hence, as in the former propositions, if we denote the sides of the triangle ABC, by a, b, c, and their opposite angles by A, B, C, we derive the following proportions, wherein A and a are invariable quantities.

1. $db : dc :: \cos \text{B} : -\cos \text{C}$
2. $d\text{B} : d\text{C} :: \cos b : -\cos c$
3. $d\text{C} : dc :: \sin \text{A} \,.\, \cos c : \sin a \,.\, \cos \text{C}$
4. $d\text{B} : db :: \sin \text{A} \,.\, \cos b : \sin a \,.\, \cos \text{B}$
5. $d\text{C} : db :: \sin \text{A} \,.\, \cos c : -\sin a \,.\, \cos \text{B}$

These proportions may be varied in their form, thus from the third.

$$d\mathrm{C} : dc :: \frac{\sin \mathrm{A}}{\cos \mathrm{C}} : \frac{\sin a}{\cos c}.$$

$$\text{But } \sin \mathrm{A} : \sin a :: \sin \mathrm{C} : \sin c,$$

$$\therefore\ d\mathrm{C} : dc :: \frac{\sin \mathrm{C}}{\cos \mathrm{C}} : \frac{\sin c}{\cos c},$$

$$\text{viz. } d\mathrm{C} : dc :: \tan \mathrm{C} : \tan c.$$

And exactly in the same manner, from the fourth proportion, we derive $d\mathrm{B} : db :: \tan \mathrm{B} : \tan b$.

PROPOSITION VII. (*Plate IV. Fig.* 6.)

(625) *In any oblique-angled spherical triangle* ABC, *suppose the two sides* AB *and* AC *to remain constant**, *it is required to find the ratio of the differentials of the other parts.*

Let BC and AC change their positions to $\mathrm{B}n$ and $\mathrm{A}n$, $\mathrm{A}n$ being equal to AC; produce AC and $\mathrm{A}n$ to the points o and p, so that $\mathrm{A}o$ and $\mathrm{A}p$ may be quadrants, and join $\mathrm{C}n$.

Also, produce BC and $\mathrm{B}n$ to the points q and r, making $\mathrm{B}q$ and $\mathrm{B}r$ quadrants, and through C draw $\mathrm{C}m$ parallel to qr.

Then op will be the measure of the increment of the $\angle$ A, qr the measure of the decrement of the $\angle$ B, and mn will be the increment of the side BC.

I. $\sin \mathrm{A}p : \sin \mathrm{A}n :: \sin op : \sin \mathrm{C}n,$

$$\text{Viz. rad} : \sin \mathrm{AC} :: d\mathrm{A} : \mathrm{C}n = \frac{\sin \mathrm{AC}}{\text{rad}} d\mathrm{A};$$

$$\text{also, } \sin \mathrm{B}q : \sin \mathrm{BC} :: \sin qr : \sin \mathrm{C}m,$$

$$\text{Viz. rad} : \sin \mathrm{BC} :: -d\mathrm{B} : \mathrm{C}m = -\frac{\sin \mathrm{BC}}{\text{rad}} d\mathrm{B}.$$

II. In the small rectilineal triangle $\mathrm{C}mn$,

$$1.\ \text{rad} : \mathrm{C}n :: \sin \angle m\mathrm{C}n : mn,$$

$$\text{Viz. rad} : \frac{\sin \mathrm{AC}}{\text{rad}} d\mathrm{A} :: \sin \mathrm{C} : d\mathrm{BC} = \frac{\sin \mathrm{AC} \,.\, \sin \mathrm{C}}{\text{rad}^2} d\mathrm{A}.$$

$$\text{But } \sin \mathrm{C} : \sin \mathrm{AB} :: \sin \mathrm{B} : \sin \mathrm{AC},$$

$$\text{or } \sin \mathrm{C} \,.\, \sin \mathrm{AC} = \sin \mathrm{B} \,.\, \sin \mathrm{AB},$$

$$\therefore\ d\mathrm{BC} = \frac{\sin \mathrm{B} \,.\, \sin \mathrm{AB}}{\text{rad}^2} d\mathrm{A}.$$

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* La Lande's Astronomy, vol. iii. art. 4015 to 4034; Cagnoli, page 321, &c.

2. $\text{rad} : \cos\angle mcn :: cn : cm,$

$$\text{Viz. rad} : \cos \text{C} :: \frac{\sin \text{AC}}{\text{rad}} d\text{A} : -\frac{\sin \text{BC}}{\text{rad}} d\text{B}.$$

$$\text{Hence } d\text{B} = -\frac{\cos \text{C} \,.\, \sin \text{AC}}{\text{rad} \,.\, \sin \text{BC}} d\text{A}.$$

3. $\cot\angle mcn : \text{rad} :: cm : mn,$

$$\text{Viz. } \cot \text{C} : \text{rad} :: -\frac{\sin \text{BC}}{\text{rad}} d\text{B} : d\text{BC} = -\frac{\sin \text{BC}}{\cot \text{C}} d\text{B}.$$

Hence, by substituting AB for AC, and C for B, we obtain

$$d\text{C} = -\frac{\cos \text{B} \,.\, \sin \text{AB}}{\text{rad} \,.\, \sin \text{BC}} d\text{A}, \text{ and } d\text{BC} = -\frac{\sin \text{BC}}{\cot \text{B}} d\text{C}.$$

(626) By denoting the three sides of the triangle by a, b, c, and their opposite angles by A, B, C, the following proportions are deduced, where c and b are constant quantities.

1. $d\text{A} : da :: \text{rad}^2 : \sin b \,.\, \sin \text{C}.$

2. $d\text{A} : da :: \text{rad}^2 : \sin c \,.\, \sin \text{B}.$

$$\text{But } \frac{\text{rad}^2}{\sin b} = \text{cosec } b, \text{ and } \frac{\text{rad}^2}{\sin c} = \text{cosec } c;$$

$\therefore d\text{A} : da :: \text{cosec } b : \sin \text{C} :: \text{cosec C} : \sin b,$

and $d\text{A} : da :: \text{cosec } c : \sin \text{B} :: \text{cosec B} : \sin c.$

3. $d\text{A} : d\text{B} :: \text{rad} \,.\, \sin a : -\cos \text{C} \,.\, \sin b.$

4. $d\text{A} : d\text{C} :: \text{rad} \,.\, \sin a : -\cos \text{B} \,.\, \sin c.$

5. $d\text{C} : da :: -\cot \text{B} : \sin a.$

$d\text{B} : da :: -\cot \text{C} : \sin a.$

6. $\therefore d\text{B} : d\text{C} :: \cot \text{C} : \cot \text{B},$

and $d\text{B} : d\text{C} :: \tan \text{B} : \tan \text{C}.$

PROPOSITION VIII. (*Plate IV. Fig.* 6.)

(627) *In any oblique-angled spherical triangle* ABC, *suppose the two angles* B *and* C *to remain constant*[*], *it is required to find the ratio of the differentials of the other parts.*

Take the supplemental triangle DEF (302), then DE and DF will be constant, therefore by Proposition VII.

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[*] La Lande's Astronomy, art. 4034 to 4045; Cagnoli, page 325, &c.

1. $dD : dEF :: \text{cosec}\ DF : \sin F :: \text{cosec}\ F : \sin DF$,

Viz. $dBC : dA :: \text{cosec}\ C : \sin AC :: \text{cosec}\ AC : \sin C$.

2. $dD : dEF :: \text{cosec}\ DE : \sin E :: \text{cosec}\ E : \sin DE$,

Viz. $dBC : dA :: \text{cosec}\ B : \sin AB :: \text{cosec}\ AB : \sin B$.

3. $dD : dE :: \text{rad} \cdot \sin EF : \cos F \cdot \sin FD$,

Viz. $dBC : dAB :: \text{rad} \cdot \sin A : \cos AC \cdot \sin C$.

4. $dD : dF :: \text{rad} \cdot \sin EF : \cos E \cdot \sin DE$,

Viz. $dBC : dAC :: \text{rad} \cdot \sin \angle A : \cos AB \cdot \sin B$.

5. $dF : dEF :: \cot E : \sin EF$,

and $dE : dEF :: \cot F : \sin EF$,

Viz. $dAC : dA :: \cot AB : \sin \angle A$,

and $dAB : dA :: \cot AC : \sin \angle A$.

6. $\therefore dAC : dAB :: \cot AB : \cot AC$,

and $dAC : dAB :: \tan AC : \tan AB$.

(628) By denoting the three sides of the triangle by a, b, c, and their opposite angles by A, B, C, we derive the following proportions, wherein B and C are constant quantities.

1. $da : dA :: \text{cosec}\ C : \sin b :: \text{cosec}\ b : \sin C$.

2. $da : dA :: \text{cosec}\ B : \sin c :: \text{cosec}\ c : \sin B$.

3. $da : dc :: \text{rad} \cdot \sin A : \cos b \cdot \sin C$.

4. $da : db :: \text{rad} \cdot \sin A : \cos c \cdot \sin B$.

5. $db : dA :: \cot c : \sin A$.

6. $dc : dA :: \cot b : \sin A$.

7. $db : dc :: \cot c : \cot b$.

8. $db : dc :: \tan b : \tan c$.

SCHOLIUM.

(629) In all the preceding propositions if the sides of the triangle be diminished without limit, the triangle may be considered as plane, and instead of the sines and tangents of the sides, we may substitute the sides themselves (440). Hence

the variations of plane triangles are readily deduced from those of spherical triangles, in every case where the differentials are proportional to the sines or tangents of the sides. Thus, by the 4th proportion (626) we have shown that $d\text{A} : dc :: \text{rad} . \sin a : \cos \text{B} . \sin c$, that is (supposing the triangle plane), $d\text{A} : dc :: a . \text{rad} : c . \cos \text{B}$.

Again, by the 8th proportion (628) $db : dc :: \tan b : \tan c$, that is, when the triangle is rectilineal, $db : dc :: b : c$, and in the same manner the rest may be deduced.

The variations of rectilineal triangles may be deduced from the triangles themselves, without reference to spherical triangles, in a manner exactly similar to those deduced from the spherical triangles. *Vide* Traité de Trigonométrie, par M. Cagnoli, Chapitre X.

THE APPLICATION OF THE DIFFERENTIAL ANALOGIES.*

PROPOSITION IX.

(630) *To find when that part of the equation of time dependent on the obliquity of the ecliptic is the greatest possible.*†

Here the sun's longitude will form the hypothenuse of a right-angled spherical triangle, his right ascension will be the base, and the obliquity of the ecliptic will be a constant angle.

Let the hypothenuse be denoted by b, and the base by c,

$$\text{then } db : dc :: \sin 2b : \sin 2c \text{ (346).}$$

Hence, when $b-c$ is a maximum, $db-dc=0$, or $db=dc$, and therefore $\sin 2b=\sin 2c$; consequently $2b$ must be the supplement of $2c$, or $b+c=90°$.

The equation of time dependent on the obliquity of the ecliptic is therefore the greatest possible when the sun's longitude and his right ascension together are equal to 90° from the equinoctial points. This will happen, in 1840, on May 6 and November 8.

PROPOSITION X.

(631) *Given the parallax in altitude of a planet, to find its parallax in latitude and longitude.*

Let B represent the pole of the ecliptic, A the zenith, and C the place of the planet.

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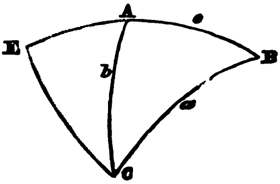
* A variety of examples will be met with in the perusal of La Lande's Astronomy, vol. iii.

† Simpson's Fluxions, page 550.

Then db will represent the parallax in altitude, $d\mathrm{B}$ the parallax in longitude, and da the parallax in latitude.



Now $d\mathrm{B} = \frac{\sin \mathrm{C}}{\sin a} db$, and $da = \frac{\cos \mathrm{C}}{\mathrm{rad}} db$ (350).

If EC be supposed to represent a part of the ecliptic, then BE and BC will be quadrants (290) and CEB and ECB will be right angles (291).

In the right-angled triangle CEA, making EA the middle part, we have $\frac{\sin \mathrm{ECA}}{\mathrm{rad}} = \left(\frac{\cos \mathrm{C}}{\mathrm{rad}} =\right) \frac{\sin \mathrm{EA}}{\sin \mathrm{AC}}$, hence by substitution, $da = \frac{\sin \mathrm{EA}}{\sin b} db$, and because the altitude of the nonagesimal degree of the ecliptic is an arc of a great circle comprehended between the zenith of any place of the pole of the ecliptic (550), we obtain the following proportion :

sine of the zenith distance is to cosine of the altitude of the nonagesimal degree, as the parallax in altitude is to the parallax in latitude, viz. $\sin b : \sin \mathrm{EA} = \cos c :: db : da$.

(632) Again, in the right-angled triangle CEA, making the $\angle \mathrm{ECA}$ the middle part, we have $\mathrm{rad} . \cos \mathrm{ECA} = \tan \mathrm{EC} . \cot b$; but $\cot b = \frac{\mathrm{rad}^2}{\tan b}$, hence $\frac{\cos \mathrm{ECA}}{\mathrm{rad}} = \left(\frac{\sin \mathrm{C}}{\mathrm{rad}} =\right) \frac{\tan \mathrm{EC}}{\tan b}$; therefore by substitution $d\mathrm{B} = \frac{\tan \mathrm{EC}}{\tan b} db$ (because the planet is supposed to be in or very near to the ecliptic, $\sin a = \mathrm{rad}$) hence,

tangent of the planet's zenith distance is to the tangent of its longitude from the nonagesimal degree, as the parallax in altitude is to the parallax in longitude, viz.

$$\tan b : \tan \mathrm{EC} :: db : d\mathrm{B}.$$

PROPOSITION XI.

(633) *Given the altitude of the nonagesimal degree of the ecliptic, the longitude of a planet from the nonagesimal degree, and its horizontal parallax, to find its parallax in latitude and longitude.*

It is shown in the preceding proposition, that $da = \frac{\cos c}{\sin b} db$ and $\mathrm{B}d = \frac{\tan \mathrm{EC}}{\tan b} db$. Now, if H represent the horizontal parallax of any planet, its parallax in altitude $db = \frac{\mathrm{H} \sin b}{\mathrm{rad}}$ nearly (185). By substituting this value of db in the above equations $da = \frac{\mathrm{H} \cos c}{\mathrm{rad}}$, and $d\mathrm{B} = \frac{\mathrm{B} \tan \mathrm{EC} \,.\, \sin b}{\mathrm{rad} \,.\, \tan b}$.

In the right-angled triangle CEA, making b the middle part, $\mathrm{rad} \,.\, \cos b = \cos \mathrm{EA} \,.\, \cos \mathrm{EC}$; but $\cos b = \frac{\mathrm{rad} \,.\, \sin b}{\tan b}$, and $\cos \mathrm{EC} = \frac{\mathrm{rad} \,.\, \sin \mathrm{EC}}{\tan \mathrm{EC}}$, hence $\frac{\sin b}{\tan b} = \frac{\cos \mathrm{EA} \,.\, \sin \mathrm{EC}}{\mathrm{rad} \,.\, \tan \mathrm{EC}}$; and by substitution $d\mathrm{B} = \frac{\tan \mathrm{EC}}{\mathrm{rad}} \,.\, \frac{\mathrm{H} \cos \mathrm{EA} \,.\, \sin \mathrm{EC}}{\tan \mathrm{EC} \,.\, \mathrm{rad}} = \frac{\mathrm{H} \cos \mathrm{EA} \,.\, \sin \mathrm{EC}}{\mathrm{rad}^2}$, but EC is the measure of the $\angle \mathrm{B}$ (474), and $\cos \mathrm{EA} = \sin c$, therefore $d\mathrm{B} = \frac{\mathrm{H} \sin c \,.\, \sin \mathrm{B}}{\mathrm{rad}^2}$: hence are derived the following general rules.

1. *Radius is to the cosine of the altitude of the nonagesimal degree of the ecliptic, as the horizontal parallax is to the parallax in latitude.*

2. *The square of the radius is to the rectangle of the sines of the altitude of the nonagesimal degree and the planet's longitude from thence, as the horizontal parallax is to the parallax in longitude.*[*]

PROPOSITION XII.

(634) *To determine the correction for finding the time of apparent noon, from equal altitudes of the sun.*

It is obvious that if the sun's declination was invariable, half the interval of time between equal altitudes would show the instant of noon; but by the variation in the sun's declin-

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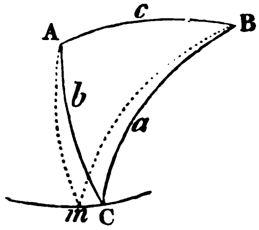
[*] Simpson's Fluxions, vol. ii. page 286.



ation he will have the same altitude at different distances from the meridian: this variation will, in general, be very small; to correct for which suppose B to represent the pole of the equinoctial, A the zenith, and C the place of the sun; AB and AC will be constant quantities.

By art. (626) $d\text{B}=-\frac{\cot \text{C}}{\sin a}da$,

and by (430) $\cot \text{C}=\frac{\sin a\,.\cot c-\cos a\,.\cos \text{B}}{\sin \text{B}}$

$$=\frac{\sin a\,.\cot c}{\sin \text{B}}-\frac{\sin a\,.\cot a}{\tan \text{B}};$$

$$\therefore\ d\text{B}=-\left(\frac{\cot c}{\sin \text{B}}-\frac{\cot a}{\tan \text{B}}\right)da.$$

Let l be the latitude of the place; h, h' the hour angles, and δ, δ' the declinations of the sun corresponding to the common zenith distance b. Then $\cot c=\tan l$, $\cot a=\tan \delta$, $-da=d\delta$, and therefore $d\text{B}$ and $d\delta$ cotemporary increments of $h-h'$ and $\delta-\delta'$; hence the last formula becomes

$$h-h'=\left(\frac{\tan l}{\sin h}-\frac{\tan \delta}{\tan h}\right)(\delta-\delta')$$

or if $\delta-\delta'$ be seconds of arc,

$$\frac{h-h'}{2}\ (\textit{in seconds of time})=\left(\frac{\tan l}{\sin h}-\frac{\tan \delta}{\tan h}\right).\frac{\delta-\delta}{30},$$

and subtracting this quantity from the middle time between the instants shown by a chronometer, at which, on a given day, the sun has equal altitudes, gives the time by the chronometer when the sun is on the meridian.*

PROPOSITION XIII.

(635) *The error in taking the altitude of a star being given, to find the corresponding error in the hour angle.*

As in the preceding proposition, let B represent the pole of the equinoctial, A the zenith, and C the observed place of the star. Then c will be the co-latitude, a the star's co-declination, and b its co-altitude; the sides c and a will be constant quantities. It is shown (626) that $d\text{B} : db :: \text{rad}^2 : \sin \text{A}\,.\sin c$, hence

$$d\text{B}=\frac{\text{rad}^2}{\sin \text{A}\,.\sin c}db.$$

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* Riddle's Navigation, page 252.

Now when the errors are very small, dB and db are the corresponding errors of the hour angle and zenith distance; therefore, radius being *unity*,

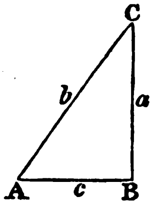
$$\text{the error } \textit{in time} = \frac{\text{error in altitude}}{15 \sin c \,.\, \sin A}$$

A being the star's azimuth.

COR. Hence, the error in time, from a given error in altitude, is less the greater sin A is, or the nearer the body is to the prime vertical, at the time of observation.

PROPOSITION XIV.

(636) *The error in the altitude of any tower, or other object, is to the error committed in taking the angle of elevation, as double the height of the observed object is to the sine of double the angle of elevation.*



This proposition is designed to illustrate the scholium (629). From the first set of equations (616) we have $dA = \frac{\sin C}{\sin b} \,.\, da$, that is, $da : dA :: \sin b : \sin C$, and considering the triangle as plane (629), $da : dA :: b : \sin C = \cos A$, $\therefore \frac{da}{dA} = b \times \frac{1}{\cos A}$.

But, $\sin 2A = 2 \sin A \,.\, \cos A$, rad $= 1$ (228), hence

$$\frac{1}{\cos A} = \frac{2 \sin A}{\sin 2A} \text{ and } \frac{da}{dA} = \frac{2\, b \sin A}{\sin 2A}.$$

Again, $\sin A : a ::$ rad $= 1 : b$ (110), consequently $b \sin A = a$, and by substitution $\frac{da}{dA} = \frac{2a}{\sin 2A}$, that is $da : dA :: 2a : \sin 2A$ which was to be shown.

COR. Because $da = \frac{2a\, dA}{\sin 2A}$ it is obvious that da the error in altitude will be a minimum, when $\sin 2A$ is a maximum, that is sine of 90°. Hence if dA the error in the observed *angle* be $1'$, the sine or arc of which is $\cdot 0002909$, and the observed angle be 45°, we have $da = \frac{2a\, dA}{\sin 2A} = 2a \times \cdot 0002909 = \cdot 0005818\, a = \frac{a}{1718 \cdot 8}$. *See the note page* 77.

CHAP. XIV.

MISCELLANEOUS PROPOSITIONS, ETC.

Of the French Division of the Circle.

(637) The modern French writers on Trigonometry divide the circumference of the circle into 400 * equal parts or degrees, each degree into 100 equal parts or minutes, each minute into 100 equal parts or seconds, &c. which degrees, minutes, &c. they write in the usual manner, thus 126° 80′ 64″, &c.

A French degree is therefore less than an English degree, in the ratio of 90 to 100, or of 9 to 10; a French minute is less than an English minute in the ratio of 90 × 60 to 100 × 100, or of 27 to 50; and a French second is less than an English second in the ratio of 90 × 60 × 60 to 100 × 100 × 100 or of 81 to 250.

Hence, if n = any number of degrees, to turn English degrees into French, we have $9 : 10 :: n : \frac{10n}{9} = n + \frac{n}{9}$, and to turn French degrees into English, $10 : 9 :: n : \frac{9n}{10} = n - \frac{n}{10}$.

PROPOSITION I.

(638) *To turn French degrees, minutes, &c. into English.*

Rule. Consider the degrees as a whole number, after which place the minutes and seconds † as decimals; $\frac{1}{10}$ of this mixed decimal deducted from itself will give the English degrees corresponding to the French.

Example I. The latitude of Paris is 54° 26′ 36″ in the

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* Eléments de Géométrie, par A. M. Legendre, 6th ed. page 328. Preface to Borda's Trigonometrical Tables (Paris, An. IX.), page 18, *et seq.*

† The minutes and seconds, if under 10, must have a cipher prefixed; thus 27° 7′ 35″ must be written 27° 07′ 35″ or 27°·0735; 45° 18′ 4″ = 45° 18′ 04″ or 45°·1804, &c.

French division of the circle, what is the corresponding latitude in the English division?

$$\begin{array}{rl}
54^\circ\,26'\,36''=54^\circ{\cdot}2636 & \text{French degrees.}\\
\tfrac{1}{10}\quad=\ 5\ {\cdot}42636 & \\
\hline
48\ {\cdot}83724 & \text{English degrees.}\\
60 & \\
\hline
50\ {\cdot}23440 & \\
60 & \\
\hline
14\ {\cdot}06400 & \\
\hline
\end{array}$$

Answer. 48° 50′ 14″ English.

2. What number of degrees, &c. in the English division of the circle will correspond to 74° 4′ 8″ in the French division?

$$\begin{array}{r}
74^\circ\,4'\,8''=74^\circ\,04'\,08''=74^\circ{\cdot}0408\\
\tfrac{1}{10}\quad=\ 7\ {\cdot}40408\\
\hline
66\ {\cdot}63672\\
60\\
\hline
38\ {\cdot}20320\\
60\\
\hline
12\ {\cdot}19200\\
\hline
\end{array}$$

Answer. 66° 38′ 12″ English.

PROPOSITION II.

(639) *To turn English degrees, minutes, &c. into French.*

RULE. Reduce the minutes and seconds, &c. to the decimal of a degree, and annex it to the given number of degrees; this mixed decimal increased by $\frac{1}{9}$ of itself will give the French degrees corresponding to the English.

EXAMPLE. The latitude of Greenwich Observatory is 51° 28′ 39″ N., according to the English division of the circle, what is the corresponding latitude by the French division?

$$\begin{array}{rl}
51^\circ\,28'\,39''=\ 51^\circ{\cdot}4775 & \\
\tfrac{1}{9}\quad=\quad 5\ {\cdot}7197 & \\
\hline
57\ {\cdot}1972 & =57^\circ\,19'\,72''.\\
\hline
\end{array}$$

Answer. 57° 19′ 72″ North, by the French division of the circle.

On the 12th March, 1840, the moon's distance from the sun, at midnight, was 116° 20′ 38″ by the *Nautical Almanac*, what was the distance according to the French division of the circle?

$$\begin{array}{rl} 116^\circ\, 20'\, 38'' = & 116^\circ{\cdot}343889 \\ \frac{1}{9} = & 12\ {\cdot}927099 \\ \hline & 129\ {\cdot}270988 = 129^\circ\, 27'\, 10''. \\ \hline \end{array}$$

Answer. 129° 27′ 10″.

(640) A table of arcs differing by 10 degrees according to the French division of the circle, or by 9 degrees by the English division, with the corresponding natural and logarithmic sines.

French Division.	*English Division.*	*Nat. sines.*	*Log. sines.*
sin 10° = cos 90°	sin 9° = cos 81°	·1564345	9·1943324
sin 20 = cos 80	sin 18 = cos 72	·3090170	9·4899824
sin 30 = cos 70	sin 27 = cos 63	·4539905	9·6570468
sin 40 = cos 60	sin 36 = cos 54	·5877853	9·7692187
sin 50 = cos 50	sin 45 = cos 45	·7071068	9·8494850
sin 60 = cos 40	sin 54 = cos 36	·8090170	9·9079576
sin 70 = cos 30	sin 63 = cos 27	·8910065	9·9498809
sin 80 = cos 20	sin 72 = cos 18	·9510565	9·9782063
sin 90 = cos 10	sin 81 = cos 9	·9876883	9·9946199
sin 100 = cos 0	sin 90 = cos 0	1·0000000	10·0000000

EXAMPLE.

The latitude of the Observatories of Paris and Pekin are 54° 26′ 36″ N., and 44° 33′ 73″ N., and their difference of longitude 126° 80′ 56″, according to the French division of the circle, what is their distance?[*]

SOLUTION BY THE FRENCH DIVISION OF THE CIRCLE.[†]

Here we have two sides a and b of a spherical triangle given, and the included angle c, to find the third side c.

By the formulæ (434) $\tan \varphi = \frac{\cos \mathrm{c}\ .\ \tan b}{\text{rad}}$, and $\cos c = \frac{\cos b}{\cos \varphi} . \cos (a - \varphi)$, where $a = 100^\circ - 54^\circ\, 26'\, 36'' = 45^\circ\, 73'\, 64''$, $b = 100^\circ - 44^\circ\, 33'\, 73'' = 55^\circ\, 66'\, 27''$, and $\mathrm{c} = 126^\circ\, 80'\, 56''$.

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[*] Legendre's Geometry, page 403.

[†] By Borda's Tables, Paris, An. IX.

$$\begin{array}{rl}
\log\cos c = \log\cos 126^\circ\ 80'\ 56'' = & 9{\cdot}6114358 \\
\log\tan b = \log\tan\ \ 55\ \ 66\ \ 27\ \ = & 10{\cdot}0776713 \\
\hline
\log\tan\varphi\ \ -\ -\ -\ = & 9{\cdot}6891071
\end{array}$$

the arc corresponding in the tables is 28° 94′ 23″·7. But because the cosine of c is negative (200), it is plain, by the formula above, that tan φ is also negative; hence $\varphi = -28^\circ\ 94'\ 23''{\cdot}7$, and $a - \varphi = 45^\circ\ 73'\ 64'' + 28^\circ\ 94'\ 23''{\cdot}7 = 74^\circ\ 67'\ 87''{\cdot}7$.

$$\begin{array}{rl}
\log\cos(a-\varphi) = \log\cos 74^\circ\ 67'\ 87''{\cdot}7 = & 9{\cdot}5880927 \\
\log\cos b\ \ = \log\cos 55\ \ 66\ \ 27\ \ = & 9{\cdot}8071949 \\
\hline
 & 19{\cdot}3952876 \\
\log\cos\varphi = \log\cos 28\ \ 94\ \ 23{\cdot}7\ = & 9{\cdot}9534821 \\
\hline
\log\cos c\ \ - = & 9{\cdot}4418055
\end{array}$$

the corresponding arc is 82° 16′ 04″ the distance between Paris and Pekin = 821604 seconds, or 821·604 myriametres; a myriametre being an arc of 10 minutes, and a metre an arc of $\frac{1}{10}$ of a second.*

SOLUTION BY THE ENGLISH DIVISION OF THE CIRCLE.†

Lat. of Paris = 54° 26′ 36″ French = 48° 50′ 14″·06 English. (638).

Lat. of Pekin = 44° 33′ 73″ French = 39° 54′ 12″·85 English.

Diff. Long. = 126° 80′ 56″ French = 114° 7′ 30″·14 English.

Hence $a = (90^\circ - 48^\circ\ 50'\ 14''{\cdot}06 =)\ 41^\circ\ 9'\ 45''{\cdot}94$; $b = (90^\circ - 39^\circ\ 54'\ 12''{\cdot}85 =)\ 50^\circ\ 5'\ 47''{\cdot}15$, and $c = 114^\circ\ 7'\ 30''{\cdot}14$.

$$\begin{array}{rl}
\log\cos c = \log\cos 114^\circ\ 7'\ 30''{\cdot}14 = & 9{\cdot}6114359 \\
\log\tan b = \log\tan\ \ 50\ \ 5\ \ 47\ {\cdot}15 = & 10{\cdot}0776713 \\
\hline
\log\tan\varphi = \log\tan -26^\circ\ 2'\ 53''{\cdot}3 = & 9{\cdot}6891072
\end{array}$$

Because φ is negative, $a - \varphi = 41^\circ\ 9'\ 45''{\cdot}9 + 26^\circ\ 2'\ 53''{\cdot}3 = 67^\circ\ 12'\ 39''{\cdot}2$.

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* According to the French mathematicians ¼ of the whole terrestrial meridian (viz. 100°) = 5130740 toises in length, the *ten millionth part* of which is the *metre*; therefore the *metre* = $\frac{1}{10}$ of a second = ·5130740 toises = 3·078444 French feet; and because 107 French feet are equal to 114 English feet nearly, the *French metre* = 3·28 English feet. If a French toise = 6·3945 English feet, a metre will be = 3·280852 English feet.

† Tables Portatives, par Callet, édition Stéréotype, An. 3.

$$\begin{array}{lll}
\log\cos(a-\varphi) = \log\cos 67^\circ\ 12'\ 39''{\cdot}2 & = & 9{\cdot}5880928 \\
\log\cos b \quad = \log\cos 50\ \ 5\ 47\ {\cdot}15 & = & 9{\cdot}8071949 \\
\hline
 & & 19{\cdot}3952877 \\
\log\cos\varphi = \log\cos 26^\circ\ 2'\ 53''{\cdot}3 & = & 9{\cdot}9534821 \\
\hline
\log\cos c = \log\cos 73^\circ\ 56'\ 39''{\cdot}6 & = & 9{\cdot}4418066 \\
\hline
\end{array}$$

The length of a degree by the Trigonometrical Survey of England and Wales, is 364950 feet*; hence, the distance between Paris and Pekin is 26985984 English feet.

73° 56′ 39″·6 English = 82° 16′ 4″ French, as above (639).

PROPOSITION III. (*Plate IV. Fig.* 7.)

(641) *If from the sum of the three angles of any spherical triangle two right angles be deducted, the remainder will be to two right angles, as the area of the triangle is to one fourth of the surface of the sphere.*

Let ABC be the triangle; complete the circle ABED, and produce AC, BC to meet in the point F, cutting the circle ABED in the points E and D.

Then because BD and CF are each semicircles (285), BC = DF, and for the same reason AC = EF.

The angle ACB = (295) ∠ DCE′ = (298) ∠ DFE, therefore the triangle ABC is equal to the triangle DEF (310); let the surface of each of these triangles be represented by m, and let w, x, and y represent the surfaces of the triangles in which they are situated. Now it is plain, that if any great circle of the sphere (as 1, 2, 3.) be divided into any number of equal parts, and through the points of division (1, 2, 3.) great circles be drawn, so as to pass through the poles (B and D) of the divided great circle, the surface of the sphere will be divided into as many equal parts, or *lunulæ*, as there are parts in the divided great circle †; therefore the whole circumference of the sphere will be to any arc of this great circle, as the whole surface of the sphere is to the surface of the lune comprehended between two great circles, whose greatest distance is measured by the aforesaid arc.

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[*] Vol. II. part ii. page 113.

[†] This may be farther illustrated by considering the manner in which the several meridians, on a terrestrial globe, divide the equator and the surface of the globe.

$$\text{Hence } 180^\circ : \angle\text{A} :: \tfrac{1}{2} \text{ surface of the sphere} : m+y$$
$$180^\circ : \angle\text{B} :: \tfrac{1}{2} \text{ surface of the sphere} : m+w$$
$$180^\circ : \angle\text{C} :: \tfrac{1}{2} \text{ surface of the sphere} : m+x$$

By composition,

$$180^\circ : \angle\text{A}+\angle\text{B}+\angle\text{C} :: \tfrac{1}{2} \text{ surf. of the sphere} : 3m+y+w+x$$

and, by division,

$$180^\circ : \angle\text{A}+\angle\text{B}+\angle\text{C}-180^\circ :: \tfrac{1}{2} \text{ surface of the sphere} : 2m$$
$$\text{or, } 180^\circ : \angle\text{A}+\angle\text{B}+\angle\text{C}-180^\circ :: \tfrac{1}{4} \text{ surface of the sphere} : m$$
$$\therefore \angle\text{A}+\angle\text{B}+\angle\text{C}-180^\circ : 180^\circ :: m : \tfrac{1}{4} \text{ surf. of the sphere.}$$

(642) COROLLARY. *The measure of the surface of a spherical triangle is the difference between the sum of its three angles and two right angles.*

For if $\text{S}=\frac{1}{4}$ of the surface of the sphere,

$$180^\circ \times m = \text{S} \times (\text{A}+\text{B}+\text{C}-180^\circ)$$
$$\therefore m = \frac{\text{S}}{180^\circ}(\text{A}+\text{B}+\text{C}-180^\circ)$$

But the whole surface of the sphere is equal to four times the area of one of its great circles*, and the area of a great circle $=\frac{1}{2}$ circumference × radius $=180^\circ \times$ radius; if therefore radius $=1$, the surface of one fourth of the sphere may be expressed by 180°; therefore $m=\text{A}+\text{B}+\text{C}-180^\circ$.

PROPOSITION IV.

(643) The sum of the three angles of every spherical triangle being greater than 180° (303), the sum of the three observed angles of a triangle, on the surface of the earth †, ought to exceed 180°; which excess may be found by the following rule. *From the logarithm of the area of the triangle taken as a plane one, in feet, subtract the constant logarithm* 9·3267737, *and the remainder is the logarithm of the excess above* 180° *in seconds nearly.*‡

Put $\frac{1}{4}$ of the surface of the sphere $=\text{S}$, then

$\angle\text{A}+\angle\text{B}+\angle\text{C}-180^\circ = \dfrac{m \times 180^\circ}{\text{S}}$ (642). But, by mensuration, $\frac{1}{4}$ of the surface of the sphere $=\text{rad}^2 \times 3{\cdot}14159265$, therefore

$$\angle\text{A}+\angle\text{B}+\angle\text{C}-180^\circ = \frac{180^\circ \times m}{\text{rad}^2 \times 3\ 14159265}.$$

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* Simpson's Fluxions, page 189.

† The sides of the triangles which connect the successive stations of a trigonometrical survey, may be considered as formed by arcs of great circles, whose radii are equal to the radius of the sphere; and each observed angle, being formed by the tangents of two of these arcs, will be the measure of the spherical angle contained between them, at each successive station (299).

‡ Trigonometrical Survey of England and Wales, vol. i. page 138.

Now the length of a degree, supposing the earth to be a sphere, is 60859·1 × 6 * feet, hence the earth's radius is $\frac{180}{3·14159265} \times 60859·1 \times 6$ feet.

$\therefore \angle A + \angle B + \angle C - 180^\circ = \frac{m \times 3·14159265}{(365154·6)^2 \times 180}$ the excess of the three angles above two right angles; and if A, B, and C be expressed in seconds, the excess in *seconds* will be $\frac{m \times 3·14159265 \times 60^2}{(365154·6)^2 \times 180} = m \times \frac{62·831853}{(365154·6)^2}$, ∴ log of the excess in *seconds* = log m + log 62·831853 − 2 log 365154·6 = log m + 1·7981799 − 11·1249536 = log m − 9·3267737.

EXAMPLE TO THE FOREGOING PROPOSITION. (*Trigonometrical Survey.*)

Names of the Stations.		*Observed angles.*
† Butser-hill	-	76° 12′ 22″
Dean-hill	-	48 4 32·25
Dunnose	-	55 43 7
		180 0 1·25

Distance of Dunnose from	
Butser-hill	140580·4 feet.
Dean-hill	183496·2 feet.

Here we have two sides of a plane triangle, and the angle contained between them, given, to find the area, viz. $a = 140580·4$ feet, $b = 183496·2$ feet, and the contained $\angle C = 55^\circ\ 43'\ 7''$. By mensuration the area of the triangle $= \frac{1}{2} ab \times$ nat. sin $\angle C$.

$\frac{1}{2} a = 70290·2$ - - - log = 4·8468947
$b = 183496·2$ - - - log = 5·2636271
$\angle C = 55^\circ\ 43'\ 7''$ - - log sin = 9·9171279

log m = 10·0276497
Deduct the constant log = 9·3267737

Log of the spherical excess = 0·7008760

the number corresponding to which is 5″·022 or 5″. Hence the error in the three observed angles above, is 5″ − 1″·25 = 3″·75.

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* Trigonometrical Survey, vol. i. page 138. Faden's edition.

† Philosophical Transactions, 1803, page 428. Dr. Rees's New Cyclopædia, second series of triangles, word Degree. A variety of examples will be met with in the works here referred to, and in the Trigonometrical Survey, published by Mr. Faden.

By the assistance of this spherical excess the observer will be enabled to examine the accuracy of his observations; and, having corrected the angles, the sides of the triangle, being arcs of great circles (supposing the earth to be a sphere) may be calculated by the rules of spherical trigonometry.

Legendre, in his Geometry, page 416, has given the rule demonstrated in the following proposition for correcting the observed angles, and though the method is only an approximation, it is sufficiently accurate and commodious for practice.

PROPOSITION V.

(644) *A spherical triangle, whose sides are very small when compared with the radius of the sphere, being proposed; if from each of its angles you subtract one third of the excess of the sum of its three angles above two right angles, the angles thus diminished may be taken for the angles of a rectilineal triangle, whose sides are equal in length to those of the proposed spherical triangle.*

Let A, B, and C represent the three angles of a plane triangle, of which the opposite sides a, b, and c are small relatively to the radius of the sphere, which we shall suppose *unity.*

Since $a : b :: \sin A : \sin B$ (144), there must exist some minute arc z, such that, considering the triangle to be spherical, $\sin a : \sin b :: \sin (A+z) : \sin (B+z)$. But $\sin (A+z) = \sin A \,.\, \cos z + \cos A \,.\, \sin z$ (244); now z being very small, its sine is equal to the arc, and its cosine equal to the radius, therefore $\sin (A+z) = \sin A + z \cos A$, and for the same reason $\sin (B+z) = \sin B + z \cos B$.

Also, by art. (270), $\sin a = a - \frac{a^3}{6}$, and $\sin b = b - \frac{b^3}{6}$;

$$\therefore a - \frac{a^3}{6} : b - \frac{b^3}{6} :: \sin A + z \cos A : \sin B + z \cos B;$$

but $b : a :: \sin B : \sin A$; hence, compounding these proportions, we get

$$1 - \frac{a^2}{6} : 1 - \frac{b^2}{6} :: \sin A \,.\, \sin B + z \cos A \,.\, \sin B : \sin A \,.\, \sin B + z \sin A \,.\, \cos B.$$

But the first and second terms being very nearly equal, and likewise the third and fourth, it is obvious that the analogy will not be disturbed by adding $\frac{a^2}{6}$ to the first and second terms, and deducting $z \cos A \,.\, \sin B$ from the third and fourth. Hence

$1 : 1+\frac{a^2-b^2}{6} :: \sin A . \sin B : \sin A . \sin B + z \sin A . \cos B - z \cos A . \sin B$.

But by (244) $z (\sin A . \cos B - \cos A . \sin B) = z \sin (A-B)$;

$$\therefore 1 : 1+\frac{a^2-b^2}{6} :: \sin A . \sin B : \sin A . \sin B + z \sin (A-B),$$

and (17 Euc. V.) $1 : \frac{a^2-b^2}{6} :: \sin A . \sin B : \sin z (A-B)$;

$$\text{hence } z \sin (A-B) = \sin A . \sin B . \frac{a^2-b^2}{6}.$$

Now, since $a : b :: \sin A : \sin B$,

by composition $a+b : b :: \sin A + \sin B : \sin B$,

and by division $a-b : b :: \sin A - \sin B : \sin B$.

By multiplying the corresponding terms, we obtain

$$a^2-b^2 : b^2 :: \sin^2 A - \sin^2 B : \sin^2 B,$$

and because $\frac{a}{b} = \frac{\sin A}{\sin B}$, if the two consequents be multiplied by these quantities, we get

$$a^2-b^2 : ab :: \sin^2 A - \sin^2 B : \sin A . \sin B;$$

$$\therefore \sin A . \sin B . (a^2-b^2) = ab . (\sin^2 A - \sin^2 B);$$

hence, by substitution,

$$z \sin (A-B) = \frac{ab}{6} . (\sin^2 A - \sin^2 B).$$

But $\sin (A+B) . \sin (A-B) = \sin^2 A - \sin^2 B$ (244);

$$\therefore z \sin (A-B) = \frac{ab}{6} . \sin (A+B) . \sin (A-B),$$

$$\text{and } z = \frac{ab}{6} . \sin (A+B).$$

Now, because the sum of two of the angles, A and B, of a rectilineal triangle is equal to the supplement of the third angle C, and any arc and its supplement have the same sine, hence $z = \frac{1}{6} ab . \sin C$, or $3z = \frac{1}{2} ab . \sin C$. But the three angles of the spherical triangle are $A+z$, $B+z$, $C+z$, and the three angles of the plane triangle are A, B, C; hence

$$(A+z)+(B+z)+(C+z) = A+B+C+3z:$$

or, $(A+z)+(B+z)+(C+z)-180° = 3z = \frac{1}{2} ab . \sin C$, the area of a plane triangle whereof the two sides are a and b, and the contained angle C, or the three sides a, b, and c, and the three angles A, B, and C; and since the three sides a, b, c of the plane triangle differ insensibly from the three sides of the spherical triangle, the surface of the one may be taken from the surface of the other without material error: but the surface of the spherical triangle is the difference between the sum of its three angles and two right angles (642), therefore the truth of the proposition is evident.*

EXAMPLE TO THE PRECEDING PROPOSITION. (*Trigonometrical Survey.*)

Names of Stations.	*Observed angles.*	*Distance of Hundred Acres from*	
Hundred Acres -	53° 58′ 35″·75		
Hanger-hill tower	68 24 44	Hanger-hill	71932·8 feet.
St. Ann's hill - -	57 36 39 ·5	St. Ann's hill	79209·7 feet.
	179 59 59 ·25		

Here $a = 71932{\cdot}8$, $b = 79209{\cdot}7$, and $\text{C} = 53°\ 58'\ 35''{\cdot}75$.

$\frac{1}{2}a = 35966{\cdot}4$ - - log = 4·5558969
$b = 79209{\cdot}7$ - - log = 4·8987783
53° 58′ 35″·75 - log sin = 9·9078287

log of the area of the triangle = 9·3625039
deduct the constant log - = 9·3267737

log of the spherical excess = ·0357302, the number corresponding to which is 1·0858 seconds $= 3d$, hence $d = 0''{\cdot}36$ nearly. But the observed angles are spherical angles, hence C corrected becomes $(53°\ 58'\ 35''{\cdot}75 - 0''{\cdot}36 =)$ 53° 58′ 35″·39; with this as an included angle, and the two sides given above, find the angles and the third side of a plane triangle. By the formulæ, Case III., page 43,

$$\tan\theta = \frac{b}{a}\,.\,\text{rad, and } \tan\tfrac{1}{2}(\text{B}-\text{A}) = \frac{\cot\frac{1}{2}\text{C}\,.\,\tan(\theta-45°)}{\text{rad}}.$$

log b - - 4·8987783
log a - - 4·8569270

0·0418513
log rad - 10·

tan θ - - 10·0418513

Hence θ - 47° 45′ 23″·1
$\theta - 45°$ - 2 45 23 ·1

$\theta - 45° =$ 2° 45′ 23″·1 tan 8·6825575
$\frac{1}{2}\text{C}$ = 26 59 17 ·695 cot 10·2930544

sum (rejecting rad = 10) - 8·9756119
Hence $\frac{1}{2}(\text{B}-\text{A}) =$ 5° 24′ 2″·327
and $\frac{1}{2}(\text{B}+\text{A}) =$ 63 0 42 ·305

Therefore B = 68 24 44 ·632
and A = 57 36 39 ·978

The third side c may be found by the formula page 42, viz. $c = \frac{a \sin \text{C}}{\sin \text{A}}$; hence by logarithms, 4·8569270 + 9·9078281 − 9·9265643 = 4·8381908 log of c, and $c = 68895{\cdot}49$.

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* This demonstration, which is more simple than Legendre's, is grounded on the same principles as that given in page 418 of LESLIE's Elements of Geometry, fourth edition, 1820.

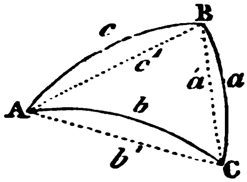
This example is solved exactly by the same rules as the example on page 417 of Legendre's Geometry, and by adding to each of the angles one third of the spherical excess, a complete solution will be given to the spherical triangle.

Angles of the spherical triangle.	Sides of the spherical triangle.
$C+d=53^\circ\ 58'\ 35''{\cdot}75$	$c=68895{\cdot}49$
$B+d=68\ \ 24\ \ 44\ {\cdot}99$	$b=79209{\cdot}7$
$A+d=57\ \ 36\ \ 40\ {\cdot}34$	$a=71932{\cdot}8$
$180\ \ 0\ \ 1\ {\cdot}08$	

The arcs and angles which occur in a trigonometrical survey may be calculated by the exact rules of spherical trigonometry in the manner of Boscovich, by the approximating rules of Legendre, or the observed angles may be reduced to the angles formed by the chords of arcs, and an arc of the meridian may be considered as formed by the chords of curves. This last method has been successfully practised by Delambre in France, and by the late General Mudge, in the trigonometrical survey of England and Wales.

PROPOSITION VI.

(645) *Given two sides of a spherical triangle, and the angle comprehended between them, to find the angle contained between the chords of these sides, supposing the chords not to differ materially from the arcs which they subtend.*



Let the three angles of the spherical triangle be represented by A, B, C, and their opposite sides by a, b, c; and let a', b', c' represent the chords of these sides, which chords are supposed not to differ essentially from the arcs.

Taking the radius *unity*, we have, by art. (420),

$$\cos a=\cos A\,.\,\sin b\,.\,\sin c+\cos b\,.\,\cos c;$$

but, by art. (249), $\cos a=1-2\sin^2\frac{1}{2}a$, $\cos b=1-2\sin^2\frac{1}{2}b$, and $\cos c=1-2\sin^2\frac{1}{2}c$, the radius being 1; therefore by substitution

$$1-2\sin^2\tfrac{1}{2}a=\cos A\,.\,\sin b\,.\,\sin c-(1-2\sin^2\tfrac{1}{2}b)\,.\,(1-2\sin^2\tfrac{1}{2}c).$$

By transposition, &c. we obtain

$$2\sin^2\tfrac{1}{2}b+2\sin^2\tfrac{1}{2}c-2\sin^2\tfrac{1}{2}a=\cos A\,.\,\sin b\,.\,\sin c+4\sin^2\tfrac{1}{2}b\,.\,\sin^2\tfrac{1}{2}c.$$

But by art. (119) $a'^2 = b'^2 + c'^2 - 2b'c' \,.\, \cos\angle \text{A}'$, and because a', b', c' do not differ essentially from a, b, c (by hypothesis), and that small arcs are nearly equal to their sines, $a' = 2 \sin \frac{1}{2}a$, $b' = 2 \sin \frac{1}{2}b$, and $c' = 2 \sin \frac{1}{2}c$; therefore by substitution
$4 \sin^2 \frac{1}{2}a = 4 \sin^2 \frac{1}{2}b + 4 \sin^2 \frac{1}{2}c - 4 \sin \frac{1}{2}b \,.\, 2 \sin^2 \frac{1}{2}c \,.\, \cos\angle \text{A}'$,
or $2 \sin^2 \frac{1}{2}a = 2 \sin^2 \frac{1}{2}b + 2 \sin^2 \frac{1}{2}c - 2 \sin \frac{1}{2}b \,.\, 2 \sin \frac{1}{2}c \,.\, \cos\angle \text{A}$;
by substituting this value of $2 \sin^2 \frac{1}{2}a$ in the spherical equation above, we get
$2 \sin \frac{1}{2}b \,.\, 2 \sin \frac{1}{2}c \,.\, \cos \text{A}' = \cos \text{A} \,.\, \sin b \,.\, \sin c + 4 \sin^2 \frac{1}{2}b \,.\, \sin^2 \frac{1}{2}c$.

$$\therefore \cos \tfrac{1}{2}\angle \text{A}' = \frac{\cos \text{A} \,.\, \sin b \,.\, \sin c + 4 \sin^2 \frac{1}{2}c \,.\, \sin^2 \frac{1}{2}b}{2 \sin \frac{1}{2}c \,.\, 2 \sin \frac{1}{2}b}$$

$$= \cos \text{A}. \frac{\sin b}{2 \sin \frac{1}{2}b} \,.\, \frac{\sin c}{2 \sin \frac{1}{2}c} + \sin \tfrac{1}{2}b \,.\, \sin \tfrac{1}{2}c.$$

But $\sin b = 2 \cos \frac{1}{2}b \,.\, \sin \frac{1}{2}b$, and $\sin c = 2 \cos \frac{1}{2}c \,.\, \sin \frac{1}{2}c$ (229), therefore by substitution,

$$\cos\angle \text{A}' = \cos \text{A} \,.\, \cos \tfrac{1}{2}b \,.\, \cos \tfrac{1}{2}c + \sin \tfrac{1}{2}b \,.\, \sin \tfrac{1}{2}c.$$

SCHOLIUM. In the investigation of the preceding formula, the radius 1 has been used in order to lessen the number of symbols, but formulæ constructed for the radius 1 are easily converted into formulæ which involve a radius $= r$, by substituting for sin A, cos A, &c., $\frac{\sin \text{A}}{r}$, $\frac{\cos \text{A}}{r}$, &c. Hence the foregoing formula by the introduction of r becomes

$$\cos \text{A}' = \frac{1}{r^2}[\cos \text{A} \,.\, \cos \tfrac{1}{2}b \,.\, \cos \tfrac{1}{2}c + r \,.\, \sin \tfrac{1}{2}b \,.\, \sin \tfrac{1}{2}c].$$

EXAMPLE.

At Calais, the correct angle between Watten and Fiennes, was 66° 30′ 36″·875, the distance from Calais to Watten 15′ 59″ of the earth's meridian, and from Calais to Fiennes 7′ 26″; required the angle formed by the chords of these arcs.

log cos A = log cos	66° 30′ 36″·875	= 9·6005211	Reject the tens in the index, and subtract the resulting index from 10, prefixing the sign minus to the remainder.
log cos $\frac{1}{2}b$ = log cos	7 59 ·5	= 9·9999988	
log cos $\frac{1}{2}c$ = log cos	3 43 ·0	= 9·9999998	
		9·6005197	

The natural number answering to the common log − 1·6005197 is ·3985839.

$\log \sin \tfrac{1}{2}b = \log \sin 7' \, 59''{\cdot}5 = 7{\cdot}36636$
$\log \sin \tfrac{1}{2}c = \log \sin 3 \;\, 43 \;\, {\cdot}0 = 7{\cdot}03388$
$4{\cdot}40024$

Nat. num. answering to $-6{\cdot}40024$ is ·0000025.

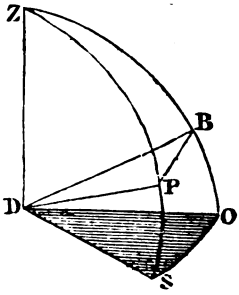
Hence $\cdot 3985839 + \cdot 0000025 = \cdot 3985864$ the natural cosine of the rectilinear ∠ A′, answering to 66° 30′ 36″·59.

The remainder of the calculation is performed by the rules of Plane Trigonometry, and the criterion of the accuracy of the observations is at once determined; for the sum of the three angles formed by the chords, and deduced from the spherical angles, will always be equal to two right angles, if the spherical angles have been correctly observed.

In all cases the rectilinear angle A′ differs but little from the spherical angle A, and to enable the calculator to obtain the necessary correction with as little trouble as possible, M. Delambre has investigated a formula, from which he has constructed a set of tables for determining, in an easy manner, the angles formed by the chords from the spherical angles.

PROPOSITION VII.

(646) *The angles of elevation of two distant objects on the surface of the earth being given, together with the oblique angle contained between the objects, to find the horizontal angle.*



Let SP and OB be two objects elevated above the surface of the earth, D the place of the observer, Z his zenith, and S and O two points on the surface of the earth.

The observed ∠ will be PDB, and the horizontal or required ∠ will be SDO, measured by the arc SO.

In the spherical triangle PZB, PZ, BZ, and PB are given to find the ∠ PZB, the measure of which is the arc SO = ∠ SDO. By art. (422)

$$\sin \tfrac{1}{2} \angle \text{PZB} = \sqrt{\frac{\sin \frac{1}{2}(\text{PB}+\text{PZ}-\text{BZ}) \, . \, \sin \frac{1}{2}(\text{PB}+\text{BZ}-\text{PZ})}{\sin \text{PZ} \, . \, \sin \text{BZ}}}$$

the radius being 1; but PB is the measure of the observed ∠ PDB, PZ is the complement of the arc SP which is the measure of the ∠ PDS, BZ is the complement of the arc OB which is the measure of the ∠ BDO, and the difference between any two arcs PZ and BZ is equal to the difference between their complements SP and OB. Therefore

$$\sin \tfrac{1}{2} \angle \text{SDO} = \sqrt{\frac{\sin\frac{1}{2}(\text{PDB}+\text{PDS}-\text{BDO}).\sin\frac{1}{2}(\text{PDB}+\text{BDO}-\text{PDS})}{\cos \text{PDS} \, . \, \cos \text{PDO}}}$$

the same as M. Delambre's formula already given at page 94, which formula is there illustrated by an example.

(647) In a trigonometrical survey, where a formula of the same nature as that given above is usually applied, ZP and ZB do not materially differ from 90°, or the altitudes or depressións PS and BO do not exceed 2° or 3°; in such cases, the problem will admit of a convenient and useful approximation, such as that given by Legendre at page 413, of his Eléments de Géométrie, sixiéme èdition, and also by other authors.

Let PS and BO be represented by H and h, and let D be the observed angle BDP and x the correction, viz. let $D+x$ be the required angle $=$ ODS, for the $\angle$ ODS will always be greater than the $\angle$ BDP (page 91). Now $\cos Z = \frac{\cos PB - \cos PZ \cdot \cos BZ}{\sin PZ \cdot \sin BZ}$ (420) the radius being 1, or $\cos(D+x) = \frac{\cos D - \sin H \cdot \sin h}{\cos H \cdot \cos h}$; but $\cos H = 1 - \frac{H^2}{2} + \frac{H^4}{2.3.4} - \&c.$ and $\cos h = 1 - \frac{h^2}{2} + \frac{h^4}{2.3.4} - \&c.$ (271), and in very small arcs, the *sines* do not differ essentially from the arcs; hence $\cos(D+x) = \frac{\cos D - Hh}{(1-\frac{1}{2}H^2) \cdot (1-\frac{1}{2}h^2)} = \frac{\cos D - Hh}{1-\frac{1}{2}H^2 - \frac{1}{2}h^2}$, rejecting all the powers of H and h above the second, as being small. But $\cos(D+x) = \cos D \cdot \cos x - \sin D \cdot \sin x$, the radius being 1 (237), and as x is very small, by supposition, $\cos x = \text{rad} = 1$, and $\sin x = x$,

$$\therefore \cos D - x \sin D = \frac{\cos D - Hh}{1-\frac{1}{2}H^2-\frac{1}{2}h^2} = (\cos D - Hh) \cdot (1+\frac{1}{2}H^2+\frac{1}{2}h^2)$$

nearly, or

$$\cos D - x \sin D = \cos D + \tfrac{1}{2} \cos D \cdot (H^2+h^2) - Hh,$$

by rejecting the smaller terms after multiplication;

hence $x = \frac{Hh - \frac{1}{2} \cos D \cdot (H^2+h^2)}{\sin D}$, the correction of the observed angle D, the same as Legendre's formula.

(648) The preceding formula may be simplified, by following the method of Legendre; thus, let $\frac{1}{2}(H+h) = p$, and $\frac{1}{2}(H-h) = q$; then $p^2 - q^2 = Hh$, $p^2+q^2 = \frac{1}{2}(H^2+h^2)$, and therefore $x =$

$$\frac{(p^2-q^2)-(p^2+q^2) \cdot \cos D}{\sin D} = \frac{p^2-p^2 \cdot \cos D - q^2 - q^2 \cdot \cos D}{\sin D} =$$

$p^2\left(\frac{1-\cos D}{\sin D}\right) - q^2\left(\frac{1+\cos D}{\sin D}\right)$ by reduction. Again,

$\tan \frac{1}{2}D = \frac{1 - \cos D}{\sin D}$ the radius being 1 (2 , and $\cot \frac{1}{2}D = \frac{1 + \cos D}{\sin D}$; therefore $x = p^2 . \tan \frac{1}{2}D - q^2 . \cot \frac{1}{2}D$. In applying this formula to practice, p and q are generally given in seconds, x should therefore be expressed in seconds; hence if r be the number of seconds contained in the radius, the number of seconds contained in $x = \frac{p^2}{r} . \tan \frac{1}{2}D - \frac{q^2}{r} . \cot \frac{1}{2}D$.

SCHOLIUM.

The five preceding propositions, viz. from the 3d to the 7th inclusive, comprehend the principal rules and formulæ used in the calculation of a trigonometrical survey, considering the earth as a sphere. The spheroidal form of the earth will occasion a small correction, and require formulæ somewhat different, though of no great difficulty, but they are here omitted because they do not properly belong to spherical trigonometry.

THE END OF SPHERICAL TRIGONOMETRY.

BOOK IV.

THE THEORY OF NAVIGATION.

CHAPTER I.

DEFINITIONS AND PLANE SAILING.

(649) NAVIGATION is the art of finding the latitude and longitude of a ship at sea, and her course and distance from that place to any other given place.

(650) *The Earth* is considered as a perfect sphere or globe, revolving on an imaginary line called its axis, from west to east, in twenty-four hours. This rotation towards the east causes all the heavenly bodies to have an apparent motion from east to west.

(651) *The equator*, generally called the line by seamen, divides the globe into two equal parts, called the northern and southern hemispheres.

(652) *Meridians* are great circles cutting the equator at right angles, and passing through its poles. Every point upon the surface of the earth is supposed to have a meridian passing through it. That meridian passing through *Greenwich* is called the *first* * meridian.

(653) *Longitude* of places on the earth is reckoned on the equator from the *first* meridian. If they be situated eastward of the first meridian, they are said to be in east longitude; if westward, they are in west longitude. The greatest longitude on the earth is 180 degrees.

(654) *The difference of longitude* between two places is an arc of the equator, contained between the two meridians passing through these places.

(655) *The latitude* of a place on the earth is reckoned from the equator, upon a meridian passing through the place. The greatest latitude a place can have is 90 degrees.

(656) *Parallels of latitude* are small circles parallel to the

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* It is necessary for the purposes of Geography and Navigation, to call the meridian of some remarkable place the *first* meridian, and to estimate the longitudes of all other places from that meridian. And, as all the tables in the Nautical Almanac, and other English astronomical tables, are adapted to the meridian passing over the Royal Observatory at Greenwich, our seamen always reckon their longitude from that meridian.

equator. Every place upon the surface of the earth is supposed to have a parallel of latitude passing through it.

(657) *The difference of latitude* between two places, is an arc of a meridian contained between the parallels of latitude which pass through these places.

(658) *Meridional distance* is the distance between the meridian sailed from and that arrived at, and is reckoned on that parallel of latitude which the ship is in.

(659) *The Mariner's compass* is a representation of the horizon; and is divided into 32 points, each point 11° 15′.

(660) *The variation of the compass* is the deviation of its points from the corresponding points of the horizon. When the north point of the compass is to the east of the true north point of the horizon, the variation is east; if it be to the west, the variation is west.

(661) If a ship be steered due north or due south, her distance sailed is equal to her difference of latitude; and her track will be on some meridian.

(662) If a ship be steered due east or west, her track will be either on the equator or some parallel of latitude; and the distance sailed will be equal to her departure, or meridional distance.

(663) If a ship be steered towards any point of the horizon between the north and east, north and west, south and east, or south and west, the track she describes will be a *Rhumb line.*

(664) *A rhumb line* is a curve upon the surface of the sphere, cutting all the meridians in equal angles.

(665) *The course* of a ship is the angle in which the track she describes cuts the meridians.

(666) *The bearing between two places* on the same parallel of latitude is east and west, on the same meridian north and south; in all other situations it is a rhumb line, continually approaching the pole.

(667) *The departure* is the whole easting, or westing, the ship makes in any single course.

PROPOSITION I. (*Plate III. Fig.* 2.)

(668) *In sailing upon a rhumb line the differences of latitudes are proportional to the distances sailed.*

Let P represent the pole, WOQE a portion of the equator, A*bczeu*L a rhumb line, or the track described by a ship sailing from A to L; AP, *d*P, *f*P, *g*P, *q*P, &c. meridians; *ib*, *hc*, *kz*, &c. parallels of latitude; and let the elementary triangles A*ib*, *bhc*, *ckz*, *zte*, &c. be conceived so indefinitely small as to differ insensibly from plane triangles.

Then, the angles iAb, hbc, kcz, &c. are equal (664); and the angles Aib, bhc, ckz, &c. are right angles, for the parallels of latitude cut the meridians at right angles. Therefore all the elementary triangles Aib, bhc, ckz, zte, &c. are equiangular and similar.

Hence, Ab : Ai :: bc : bh :: cz : ck :: ze : zt, &c. (4 Euclid VI). Therefore Ab : Ai :: Ab + bc + cz + ze, &c. : Ai + bh + ck + zt, &c. (12 Euclid V). That is,

Ab : Ai :: AL : Al, where Ab and AL are distances, and Ai and Al corresponding differences of latitude.

PROPOSITION II. (*Plate III. Fig.* 2.)

(669) *In sailing upon a rhumb line, the departure corresponding to any course and distance is equal to the sum of all the intermediate departures.*

For, as in the preceding proposition,

Ab : ib :: bc : hc :: cz : kz :: ze : te, &c. (4 Euclid VI); therefore Ab : ib :: Ab + bc + cz + ze, &c. : ib + hc + kz + te, &c. (12 Euclid V). But the whole distance AL is equal to the sum of all the intermediate distances ab + bc + cz, &c.; hence Ab : ib :: AL : ib + hc + kz + te, &c.

(670) SCHOLIUM. Hence it appears that the *meridional distance*, *departure*, and *difference of longitude*, are essentially different. Let a ship sail from A to L, when she arrives at L her meridional distance will be Ll, her departure ib + hc + kz + te, &c. and her difference of longitude WE. But the meridional distance is evidently less than the departure (which is equal to the sum of all the arcs ib + hc + kz, &c.); because the several meridians converge towards the pole; and for the same reason the difference of longitude WE is greater than the departure. Again, let the ship return from L to A along the rhumb line LA, her meridional distance will then be Aa, and her departure xu + we + qz + gc, &c. the same as before; for the elementary triangles are equal, an equal portion of the ship's track being the hypothenuse of each. Here the meridional distance Aa is greater than the departure; hence in the same course, or track, backward and forward, the departure and difference of longitude remain the same, but the meridional distance is variable.

(671) While the course remains the same, it has been shown that the departure is greater than the meridional distance Ll, and less than the meridional distance Aa; yet it is very nearly equal to the meridional distance MN, in the middle latitude, between the latitude sailed from, and the latitude arrived at. This was probably a casual discovery; and when the places

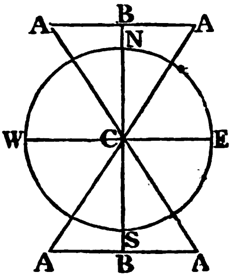
are near the equator, or when their parallels of latitude are not very far distant from each other, the nautical conclusions, drawn from a supposition that the departure is equal to the meridional distance in the middle latitude, between the latitude sailed from and the latitude arrived at, are very nearly the same as the conclusions derived from *Mercator's* sailing. But in high latitudes, or in a long run, when the course is not near some of the four cardinal points, this method is not sufficiently accurate. And, if a ship sail upon *several* courses, she makes a less departure near the pole, and a greater departure near the equator, from one place to another, than if she were to sail in a direct course; yet in such small distances, as a day's run, the difference is almost insensible.

PROPOSITION III. (*Plate III. Fig.* 2.)

(672) *Straight lines equal in length to the distance run, difference of latitude, and departure, from a right-angled plane triangle, having the angle opposite to the departure equal to the ship's course.*

For it is shown in art. (668) that
$Ab : Ai :: AL : Al$; or $Ab : AL :: Ai : Al$; and by art. (669) we have $Ab : AL :: ib : ib+hc+kz+te$, &c. therefore $Ai : Al :: ib : ib+hc+kz+te$, &c. But the small elementary triangle is considered as plane, and is right-angled at i; therefore the triangle to which it is similar, may be considered as plane; AL will be the hypothenuse, Al the difference of latitude, and the departure $ib+hc+kz+te+yu+50\ L$.

(673) All problems solved by the preceding propositions are said to be in plane sailing; because the very same conclusions would be drawn if the earth were a plane, and all the meridians parallel to each other. Hence it appears that plane sailing is true, so far as course, distance, difference of latitude, and departure, are concerned.



(674) Scholium. Since by this proposition, the distance run, difference of latitude, and departure from a plane triangle, let CA in the annexed figure represent the distance, CB the difference of latitude, AB the departure, and the angle ACB the course (129), then will CAB be the complement of the course. Hence are deduced the following proportions for solving *all* the cases that can occur in the practice of plane sailing.

1. *Radius*
: distance
:: sine of the course
: departure.

2. *Radius*
: distance
:: cosine of the course
: difference of latitude.

3. *Cosine of course*
: diff. of latitude
:: radius
: distance.

4. *Cosine of course*
: diff. of latitude
:: sine of course
: departure.

5. *Distance*
: radius
:: difference of latitude
: cosine of course.

6. *Distance*
: radius
:: departure
: sine of course.

7. *Sine of course*
: departure
:: radius
: distance.

8. *Sine of course*
: departure
:: cosine of course
: difference of latitude.

9. *Difference of latitude*
: radius
:: the departure
: tangent of the course.

For,

1. $\text{rad} : AC :: \sin C : AB$	(109)
2. $\text{rad} : AC :: \sin A = \cos C : BC$	
3. $\cos C : BC :: \text{rad} : AC$	(*From the* 2*d.*)
4. $\cos C : BC :: \sin C : AB$	(1*st and* 2*d.*)
5. $AC : \text{rad} :: BC : \cos C$	(2*d inverted.*)
6. $AC : \text{rad} :: AB : \sin C$	(1*st inverted.*)
7. $\sin C : AB :: \text{rad} : AC$	(*From the first.*)
8. $\sin C : AB :: \cos C : BC$	(1*st and* 2*d.*)
9. $BC : \text{rad} :: AB : \tan C$	(109).

CHAP. II.

PARALLEL AND MIDDLE LATITUDE SAILING.

(675) When a ship sails directly east or west, upon any parallel of latitude, the method of finding her difference of longitude, distance, &c. is called parallel sailing.

(676) Middle latitude sailing is founded on a *supposition*, that the meridional distance, half way between the latitude sailed from, and that bound to, is equal to the departure which the ship makes in sailing from one latitude to the other. It is a compound of plane and parallel sailing.

PROPOSITION I. (*Plate III. Fig.* 2.)

(677) *In sailing upon any parallel of latitude, or directly east, or west, the cosine of the latitude is to radius, as the distance run is to the difference of longitude.*

Let C be the centre of the sphere, P the north pole, WOQE an arc of the equator, P*m*W and P*n*O meridians, *mn* the distance run on the parallel *mns*, and WO the difference of longitude.

In the plane of the parallel *mns*, draw *mr*, *nr*, meeting PC the axis of the sphere in *r*; and in the plane of the equator WOQE, draw WC, OC, meeting the axis of the sphere in C; then the angle *mrn* is equal to the angle WCO (299); therefore the arcs *mn* and WO are similar. Hence

mn : parallel of lat. : : WO : equator.
Or *mn* : WO : : parallel of lat. : equator.

But the circumferences of circles are as their radii (*Keith*'s *Geom.* Prop. 196, Book VII.); therefore

mr : WC : : parallel of lat. : equator.
Consequently *mr* : WC : : *mn* : WO.

But *mr* is the sine of the arc P*m*, or the cosine of the latitude W*m*, and WC is the radius of the sphere.

Therefore cos lat. : rad : : dist. : diff. long.

(678) COROLLARY I. *Radius is to the cosine of the latitude, as the difference of longitude is to the distance run.*

(679) COROLLARY II. *The distance between any two meridians on one parallel of latitude, is to the distance between the same two meridians on any other parallel, as the cosine of the latitude of the first parallel, is to the cosine of the latitude of the second.*

(SCHOL.) This proposition and Corollary I. will solve all the cases that can occur in parallel sailing. Examples exercising these cases have been given already at page 297.

PROPOSITION II. (*Plate III. Fig.* 2.)

(680) *Cosine of the middle latitude* (viz. *the latitude half way between the latitude sailed from and that bound to is to radius, as the departure is to the difference of longitude* (*nearly*).

Let A*da* represent the parallel of latitude sailed from, *l*L that bound to, M*kzq*N the parallel of the middle latitude between these parallels, and A*bczeu*L the rhumb line, or track of the ship, in sailing from A to L.

Now whether the ship sail from A to L, or from L to A, it has been shown in art. (670) that the departure will be

$ib+hc+kz+te$, &c., and (671) this departure is *nearly* equal to the meridional distance M$kzq$N. But

cos WM : rad : : M$kzq$N : WOQE (677),

or cos mid. lat. : rad. : : departure : diff. long. (*nearly*).

PROPOSITION III. (*Plate III. Fig.* 2.)

(681) *Difference of latitude is to the difference of longitude, as the cosine of middle latitude is to the tangent of the course* (*nearly*).

For diff. lat. : rad : : dep. : tan course (674),
And cos mid. lat. : rad : : dep. : diff. long. (680).
Hence diff. lat. × tan course = rad × dep.,
And cos mid. lat. × diff. long. = rad × dep.
∴ diff. lat. × tan course = cos mid. lat. × diff. long.;
or diff. lat. : diff. long. : : cos mid. lat. : tan course.

(682) COR. *Cos mid. lat. : tan course : : diff. lat. : diff. long* (*nearly*).

PROPOSITION IV. (*Plate III. Fig.* 2.)

(683) *Distance sailed is to the difference of longitude, as cosine of the middle latitude is to sine of the course* (*nearly*).

For rad : dist. : : sin course : dep. (674),
And cos mid. lat. : rad : : dep. : diff. long. (680).
Hence rad × dep. = dist. × sin course,
And rad × dep. = cos mid. lat. × diff. long.
Consequently dist. × sin course = cos mid. lat. × diff. long.
∴ Dist. : diff. long. : : cos mid. lat. : sin course.

(684) SCHOLIUM. The second, third, and fourth propositions include all the cases that can occur in the practice of *middle latitude* sailing; the several proportions for the sake of uniformity, are here collected.

1. *Radius*
: *difference of longitude*
: : *cosine of middle latitude*
: *departure.*

2. *Cosine of middle latitude*
: *departure*
: : *radius*
: *difference of longitude.*

3. *Difference of latitude*
: *difference of longitude*
: : *cosine of middle latitude*
: *tangent of the course.*

4. *Cosine of middle latitude*
: *tangent of the course*
: : *difference of latitude*
: *difference of longitude.*

5. *Cosine of middle latitude*
: *sine of the course*
: : *distance*
: *difference of longitude.*

See pages 297 and 298.

CHAP. III.

MERCATOR'S SAILING.

(685) Mercator's sailing is the art of finding on a plane surface the motion of a ship upon any assigned course by the compass, which shall be true in latitude, longitude, and distance sailed.*

(686) In a Mercator's chart, from which this method of sailing is derived, the degrees of *longitude* are every where equal, the degrees of latitude increase as you approach the poles, and the rhumb line, or track the ship describes, is represented by a straight line.

(687) On the globe the degrees of *latitude* are every where equal, and the degrees of longitude decrease as you approach the poles: that is, the distance between any two meridians in any latitude, is to the difference of longitude between these meridians, as the cosine of that latitude is to the radius (677). But on a Mercator's chart, the distance between any two meridians is made equal to their difference of longitude, and consequently the parallels of latitude are enlarged, in the ratio of the cosine of the latitude to the radius: now, in order that the angle which the rhumb line makes with the several meridians on the chart may correspond with the same angle on the globe, it will be necessary to increase the meridians in the same ratio, by which the parallels of latitude are increased, viz. as the cosine of the latitude is to the radius, or, which is the same thing, as the radius is to the secant of the latitude. Hence the degrees of the meridians on a Mercator's chart increase towards the poles, as the secant of the latitude increases; likewise all the parallels of latitude, and every part of them, are larger than they are on the globe, in the ratio of the radius to the secant of the latitude. Hence, though the latitudes, longitudes, and bearings of places are truly represented on a Mercator's chart, the distances are distorted in various proportions. To render these observations more clear, let us suppose WEL*l* (*Plate V. Fig.* 24.) a Mercator's chart, constructed to represent the spherical surface WEL*l* (*Plate III. Fig.* 2.). Then W*l*, 10-10, 20-20, &c. are meridians; ABCZ*e*UL a rhumb line; IB, HC, KZ, T*e*, YU, &c.

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* This includes the whole theory and practice of navigation; and if any method could be devised for measuring a ship's course and distance truly, nothing more would be wanted to complete the art.

parallels of latitude, each straight line on the chart representing its corresponding arc on the spherical surface.

The meridians in this projection being parallel to each other, IB is equal to W–10; hence IB is the difference of longitude on the chart, corresponding with *ib* on the sphere. Therefore cos W*i* : radius : : *ib* : IB (677).

Now to make the elementary triangle AIB on the chart equiangular with A*ib* on the sphere, the difference of latitude A*i* must be increased in the same ratio with the departure *ib*; hence cos W*i* : rad : : A*i* : AI. But the cosine of any arc is to the radius, as the radius is to the secant; therefore rad : sec W*i* : : A*i* : AI.

But A*i* is an indefinitely small portion of the sphere which is increased to AI on the chart. Let us suppose the radius of the sphere *unity*, and the difference of latitude A*i* 1 minute*; then the last proportion will be 1 : sec W*i* : : 1′ : AI; consequently in this case, the increased minute will be equal to the secant of the latitude.

Hence it follows that the natural secant of any latitude, will be equal to the increase of the next minute on the chart, nearly.

(688) *The meridional line on a Mercator's chart (constructed on the above principles) is equal in length to the sum of all the natural secants of every minute of latitude contained in it.*

Thus suppose the difference of latitude A*l* on the sphere to be 6 minutes, then the enlarged meridian A*l* on the chart is equal to AI+BH+CK+ZT+*e*Y+U-50, and these are the secants of WA, 10-B, 20-C, 30-Z, &c.

Hence is derived Mr. Wright's method of constructing a table of meridional parts.

(689) The difference of latitude (in geographical miles) between two places on the globe, is generally called the proper difference of latitude, to distinguish it from the difference of latitude (in geographical miles) on the chart, which is called the meridional difference of latitude.

(690) *A plane right-angled triangle formed by the ship's course, distance, difference of latitude, and departure* (672), *will be similar to a plane triangle formed upon a Mercator's chart, by the enlarged distance, meridional difference of latitude and difference of longitude.*

For all the elementary right-angled triangles on the sphere, (*Plate III. Fig.* 2.) viz. A*ib*, *bhc*, *ckz*, *zte*, &c. are equiangular and similar (668); and from the very nature of the chart, the elementary triangles (*Plate V. Fig.* 24.), AIB, BHC, CKZ, ZTE, &c. are likewise equiangular and similar, not only to each other, but to the elementary triangles *aib*, *bhc*, &c. on the sphere.

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* It is evident that if A*i* were to represent a smaller portion of the meridian than one minute, the conclusions derived from such a supposition would be more accurate.

Now the elementary triangles on the sphere may be truly represented by a plane triangle (672); therefore the elementary triangles on the chart may likewise be truly represented by a plane triangle, equiangular and similar.

(691) I. MR. WRIGHT'S *method of constructing a table of meridional parts.*

Merid. Parts of 1′=nat sec 1′
M. P. of 2′= sec 1′+sec 2′
M. P. of 3′= sec 1′+sec 2′+sec 3′
M. P. of 4′= sec 1′+sec 2′+sec 3′+sec 4′
M. P. of 5′= sec 1′+sec 2′+sec 3′+sec 4′+sec 5′

Hence by a table of natural secants, as Sherwin's or Dr. Hutton's Tables.

Nat sec 1′=1·0000000 M. P. of 1′=1·0000000
Nat sec 2′=1·0000002 M. P. of 2′=2·0000002
Nat sec 3′=1·0000004 M. P. of 3′=3·0000006
Nat sec 4′=1·0000007 M. P. of 4′=4·0000013
Nat sec 5′=1·0000011 M. P. of 5′=5·0000024

And by adding together the natural secants of a few pages, in either of the above-mentioned books, you will find.

The meridional parts of 1 degree = 60·0031231
M. P. of 2 degrees=120·0246423
M. P. of 3 degrees=180·0829505
M. P. of 4 degrees=240·1963752
M. P. of 5 degrees=300·3833699

In this manner (by a continual addition of the secants) Mr. Edward Wright formed the FIRST table of meridional parts, which is contained in his celebrated work entitled, "*Certain Errors in Navigation Detected and Corrected.*" This book was first published in 1599. Mr. William Oughtred was the next who constructed a table of meridional parts, by the continual addition of the intermediate secants of ½, 1½, 2½, &c. minutes; and these tables were again corrected and extended by Sir Jonas Moor.

It must be admitted that *Wright's* method of constructing a table of meridional parts is not *strictly* geometrical; for the cosines of any two parallels of latitude are not precisely the same, even if you suppose the common difference between these parallels to be less than *one* minute of a degree, and the secants of such parallels do not increase in any regular ratio. A table constructed on *Wright's* principles, in high latitudes, wil exceed the truth a small matter. This, the learned author was sufficiently aware of; for he says he rejected some of the decimal parts of the secants in making his table. "Because "that indeed, at every point of latitude, a minute of the meridian in this nautical planisphære [*Mercator's Chart*] hath "somewhat lesse proportion to a minute of the parallel adjoyning towards the *Æquinoctial*, than the secans of that "parallels latitude hath to the whole sine. But in this table "it was thought sufficient to use such exactness as that thereby (in drawing the lineaments of the nautical planisphære) "sensible error might be avoided. He that listeth to be "more precise may make the like tables to decades or tennes "of seconds, out of *Ioachimus Rhæticus* his *Canon magnus* "*triangulorum*. Notwithstanding the *Geometrician* that desireth *exact* truth, cannot be so satisfied neither."

(692) II. ANOTHER *Method of constructing a table of meridional parts from the secant of an arc.*

It is shown in art. (273) that if $z =$ the length of any arc, the radius being 1, the secant will be

$$1 + \frac{z}{2} + \frac{5z^4}{24} + \frac{61z^6}{720} + \frac{277z^8}{8064} + \frac{50521z^{10}}{3628800} + \frac{540553z^{12}}{95800320} + \text{\&c.}$$

This multiplied by dz, the differential of the arc, gives $dz + \frac{z^2dz}{2} + \frac{5z^4dz}{24} + \frac{61z^6dz}{720} + \frac{277z^8dz}{8064} + \frac{50521z^{10}dz}{362880}$, &c., the differential of the sum of the secants in the arc z, the integral of which is $z + \frac{z^3}{6} + \frac{z^5}{24} + \frac{61z^7}{5040} + \frac{277z^9}{72576} + \frac{50521z^{11}}{8991680} +$, &c., the sum of all the secants contained in z.

EXAMPLE.

Let it be required to find the meridional parts corresponding to 5 degrees. The length of the arc of one minute being ·0002908882086657, &c.*, supposing the radius unity.

First $5 \times 60 \times {\cdot}000290888208666 = {\cdot}087266462600$, &c. $=$ z, the length of an arc of 5 degrees.

$$\begin{array}{rr}
z = & +\,{\cdot}087266462600 \\
\frac{1}{6}z^3 = & +\quad 110762019 \\
\frac{1}{24}z^5 = & +\quad 210875 \\
\frac{61}{5040}z^7 = & +\quad 466 \\
\hline
\end{array}$$

Hence $z + \frac{1}{6}z^3 + \frac{1}{24}z^3 + \frac{61}{5040}z^7$, &c. $=$ ·087377435960

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* Gardiner's edition of Sherwin's Logarithms, page 44. (edition 1742). Traité de Trigonométrie, par Cagnoli (Table AA), page 474.

This result divided by ·000290888208666, the length of the arc of 1′, will give the meridional parts of 5° = 300·381498, true to the last place of decimals.

By a similar process you will find the meridional parts of 10° to be 603·0695795. Proceed in the same manner to find the meridional parts of any other degree of latitude.

(693) III. *The method of constructing a table of meridional parts, by a table of logarithmic tangents.*

PROPOSITION I. (*Plate V. Fig.* 25.)

If the logarithmic tangent of half the complement of any degree of latitude be subtracted from 10, *and the remainder be multiplied by* 7915·7044679 &c., *the product will give the meridional parts* (*in minutes*) *corresponding to that latitude.*

Demonstration. Let P be the pole, and EQ a portion of the equator; Q*v* the latitude, O*v* its sine and OE = *vw* its cosine; and E*v* the radius of the sphere.

Put z = Q*v* the latitude, m = the length of Q*v*, on a Mercator's chart, called the meridional parts; y = O*v* = E*w* the sine of the latitude, and r = E*v* the radius of the sphere. Then will EO = *wv* = $\sqrt{r^2-y^2}$; and because the degrees of latitude on a Mercator's chart increase as the cosine of latitude is to radius, we have

$$\sqrt{r^2-y^2} : r :: dz : dm, \text{ hence } dm = \frac{rdz}{\sqrt{r^2-y^2}};$$

$$\text{But } \sqrt{r^2-y^2} : r :: dy : dz, \text{ conseq. } dz = \frac{rdy}{\sqrt{r^2-y^2}}.$$

Hence we have $dm = \frac{r^2dy}{r^2-y}$, and integrating

$m = r \times 2·30258509299405 \times \frac{1}{2} \times \log\frac{r+y}{r-y}$ + correction

$= r \times 2·302585$ &c. $\times \log\sqrt{\frac{r+y}{r-y}}$, the correction being *nothing*, because when $y = 0$, $m = 0$, as is obviously the case.

But by Plane Trigonometry, in the triangle *vws*,

vw : rad : : *ws* : tan *wvs* or cot *wsv*,

$$\textit{Viz.} \sqrt{r^2-y^2} : r :: r+y : \frac{r(r+y)}{\sqrt{r^2-y^2}} = r\sqrt{\frac{r+y}{r-y} \times \frac{r+y}{r+y}}$$

$= r\sqrt{\frac{r+y}{r-y}}$ = cot *wsv*, measured by half the arc P*v* the complement of the latitude. Hence we have $m = r \times 2{\cdot}302585 \text{ \&c.} \times \log \frac{\cot \frac{1}{2} \text{ comp. lat.}}{r} = r \times 2{\cdot}3025 \text{ \&c.} \times \log \frac{r}{\tan \frac{1}{2} \text{ comp. lat.}}$; for the cotangent of an arc divided by the radius, is equal to the radius divided by the tangent.

But the tables of meridional parts are generally expressed in geographical miles, therefore we must express the radius of the sphere in geographical miles. If the diameter of the earth be 1, the circumference will be 3·141592653589 &c.; hence 3·141592653589 : $\frac{1}{2}$: : 360° × 60 : 3437·74677 &c.

$$\therefore m = 2{\cdot}3025 \text{ \&c.} \times 3437{\cdot}74677 \text{ \&c.} \times \log \frac{r}{\tan \frac{1}{2} \text{ comp. lat.}}$$

$$= 7915{\cdot}7044679 \times \log \frac{r}{\tan \frac{1}{2} \text{ comp. lat.}}.$$

Note. It would be improper to make the value of r in the latter part of the expression = 3437·7467, &c. unless the table of logarithmic tangents was calculated to the same radius.

EXAMPLE.

Let it be required to find the meridional parts corresponding to 5 degrees.

$$\begin{array}{rr} & 90^\circ \\ & 5 \\ \hline 2\,| & 85 \\ \hline \end{array}$$

Log radius = 10·

Half comp. lat. 42° 30′, tan by Ulacq's tables = 9·9620524617

$$\begin{array}{r} \hline 0{\cdot}0379475383 \\ \hline \end{array}$$

this multiplied by 7915·7044679 produces 300·38149846 &c., exactly agreeing with the answer before found by the secants.

In the same manner you will find the meridional parts of 10° to be 603·069579 &c., the logarithmic tangent of 40° being 9·9238135302.

(694) The following table comprehends the meridional parts answering to every five degrees of latitude. The first column contains the degrees, the second the meridional parts copied from *Wright's* original treatise, and the third the meridional parts truly calculated by this problem. As our tables of meridional parts are seldom carried to decimals, the reader will

readily perceive that *Wright's* tables may be used without sensible error.

Degrees.	*Mr. Wright's* Merid. Pts	*True* Merid. Parts.	Degrees.	*Mr. Wright's* Merid. Pts	*True* Merid. Parts.	Degrees.	*Mr. Wright's* Merid. Pts.	*True* Merid. Parts.
5	300·3694	300·3815	35	2244·3047	2244·2868	65	5179·3079	5178·8081
10	603·0475	603·0696	40	2622·7559	2622·6902	70	5966·6811	5965·9179
15	910·4325	910·4606	45	3030·1271	3029·9392	75	6971·5485	6970·3890
20	1225·1292	1225·1390	50	3474·6045	3474·4720	80	8377·3416	8375·1970
25	1549·9878	1549·9952	55	3968·1879	3967·9661	85	10769·6200	10764·6210
30	1888·3768	1888·3754	60	4527·7106	4527·3677	89	16317·5324	16299·5563

(695) A table of meridional parts being formed, by any of the preceding methods, to every degree and minute of the quadrant, a Mercator's chart may be readily constructed therefrom.

Example. Let it be required to construct a Mercator's chart containing 60 degrees of east longitude, and extending from the equator to 40 degrees of north latitude.

I. Draw WE to represent the equator (*Plate V. Fig.* 24.) on which set off WE = 3600 miles (the number of miles contained in 60 degrees, the extent of the chart), from any scale of equal parts.

II. Divide the line WE into six equal parts in the points 10, 20, 30, 40, &c., each part containing 10 degrees; and each of these parts may be subdivided into single degrees, half degrees, &c. if necessary. Then straight lines drawn through the points W, 10, 20, 30, &c. perpendicular to WE, will represent the meridians passing through every 10 degrees.

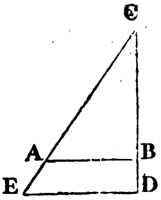
III. From the same scale of equal parts take 2622·7 (the meridional parts answering to 40 degrees) and set them off from W to l and from E to L, and join Ll, which will represent the parallel of 40 degrees of latitude, and determine the extent of the chart northward.

IV. From the scale of equal parts take 603 (the meridional parts answering to 10 degrees) and set them off from W to A, and from E to 10, and join A–10, this will represent the parallel of 10 degrees. In the same manner W–20 = 1225; and W–30 = 1888; and draw the parallels of latitude; or make A–20 = 1225 – 603, 20–30 = 1888 – 1225, &c.

If the chart does not begin at the equator, it is evident that the meridional parts corresponding to the least latitude contained in it, must be subtracted from the meridional parts of each point of greater latitude.

PROPOSITION II.

(696) *Radius is to the tangent of the course, as the meridional difference of latitude is to the difference of longitude.*



In the annexed figure BC is the proper difference of latitude, DC the meridional difference of latitude, the angle ACB the course, AC the distance sailed, AB the departure, and ED the difference of longitude. Then EDC is a right-angled triangle, and similar to ABC (690).

Now DC : rad : : ED : tan ACB; therefore rad : tan ACB : : DC : ED.

PROPOSITION III.

(697) *The number* ·00012633114 &c.

Is to the natural tangent of the course,

As the difference between the logarithmic tangents of half the complements of the latitudes sailed from and bound to

Is to the differences of longitudes between these places, in geographical miles.

Let the less latitude be represented by l, and the greater by L; then by art. (693) the meridional parts corresponding to $l = 7915{\cdot}7044$ &c. $\times \log\frac{r}{\tan\frac{1}{2}\text{comp. } l}$, and the meridional parts corresponding to $\text{L} = 7915{\cdot}7044$ &c. $\times \log\frac{r}{\tan\frac{1}{2}\text{ comp. L}}$.

The difference between these quantities is $\left(\log\frac{r}{\tan\frac{1}{2}\text{ comp. } l} \sim \log\frac{r}{\tan\frac{1}{2}\text{ comp. L}}\right) \times 7915{\cdot}7044$ &c., the meridional difference of latitude in miles. Now (696) rad : tan course : : $\left(\log\frac{r}{\tan\frac{1}{2}\text{ comp. } l} \sim \log\frac{r}{\tan\frac{1}{2}\text{ comp. L}}\right)$ $\times\, 7915{\cdot}7044$ &c. : diff. longitude.

But $\log\frac{r}{\tan\frac{1}{2}\text{ comp. } l} \sim \log\frac{r}{\tan\frac{1}{2}\text{ comp. L}}$ is evidently = log tan $\frac{1}{2}$ comp. $l \sim$ log tan $\frac{1}{2}$ comp. L; hence rad : tan course : : (log tan $\frac{1}{2}$ comp. $l \sim$ log tan $\frac{1}{2}$ comp. L) $\times$ 7915·7044 &c. : diff. longitude.

Divide the antecedents by 7915·7044 &c., then

$\frac{\text{rad}}{7915{\cdot}7044\text{ \&c.}}$: tan course : : log tan $\frac{1}{2}$ comp. $l \sim$ log tan $\frac{1}{2}$ comp. L : diff. longitude; but $\frac{\text{rad}}{7915{\cdot}7044\text{ \&c.}} = \frac{1}{7915{\cdot}7044\text{ \&c.}} =$

·000126331143874 &c., hence it follows that
·000126331143874 &c. : tan course :: log tan $\frac{1}{2}$ comp. $l \sim$ log tan $\frac{1}{2}$ comp. L : diff. longitude.

(698) Corollary. *Find the logarithmic tangent of half the complement of the latitude sailed from, the logarithmic tangent of half the complement of the latitude bound to, multiply their difference by* 10000, *and find the common logarithm of the product.*

Then,

The logarithmic tangent of the angle $51^\circ\ 38'\ 9''\ 14''' =$ 10·1015103
Is to the logarithm found above,
As the logarithmic tangent of the course
Is to the logarithm of the difference of longitude in miles.

For, ·00012633114 &c. : the nat tan course ::
log tan $\frac{1}{2}$ comp. $l \sim$ log tan $\frac{1}{2}$ comp. L : diff. longitude in miles.

Multiply the antecedents by 10000, then
1·2633114 &c. : the nat tan course ::
log tan $\frac{1}{2}$ comp. $l \sim$ log tan $\frac{1}{2}$ comp. $L \times 10000$: diff. longitude; but 1·2633114, &c. is the natural tangent of $51^\circ\ 38'\ 9''\ 14'''$, the logarithmic tangent of which is 10·1015103; hence by taking the logarithms of all the terms in the last proportion, we deduce the above rule.

SCHOLIUM.

The second proposition (696) and the following, which is immediately deduced therefrom (viz. Meridional difference of latitude is to difference of longitude, as radius is to the tangent of the course), will solve all the cases that can occur in Mercator's sailing by the help of a table of meridional parts. And the corollary (698) will solve the whole with the assistance of a table of logarithmic tangents (independent of a table of meridional parts) by varying the proportion.

THE END OF NAVIGATION.

TABLES.

I. A table of *the Logarithms* of Numbers, from a unit to ten thousand.

II. A table of *natural sines* to every degree and minute of the quadrant.

III. A table of *Logarithmic sines and tangents* to every degree and minute of the quadrant.

IV. A table of the Refraction in altitude, of the heavenly bodies.

V. A table of the depression, or dip, of the horizon of the sea.

VI. A table of the sun's parallax in altitude.

VII. A table of the augmentation of the moon's semi-diameter.

VIII. A table of the right ascensions and declinations of forty principal fixed stars, corrected to the beginning of the year 1840.

TABLE I. LOGARITHMS OF NUMBERS FROM 1 TO 10,000.

No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
1	00000	21	32222	41	61278	61	78533	81	90849
2	30103	22	34242	42	62325	62	79239	82	91381
3	47712	23	36173	43	63347	63	79934	83	91908
4	60206	24	38021	44	64345	64	80618	84	92428
5	69897	25	39794	45	65321	65	81291	85	92942
6	77815	26	41497	46	66276	66	81954	86	93450
7	84510	27	43136	47	67210	67	82607	87	93952
8	90309	28	44716	48	68124	68	83251	88	94448
9	95424	29	46240	49	69020	69	83885	89	94939
10	00000	30	47712	50	69897	70	84510	90	95424
11	04139	31	49136	51	70757	71	85126	91	95904
12	07918	32	50515	52	71600	72	85733	92	96379
13	11394	33	51851	53	72428	73	86332	93	96848
14	14613	34	53148	54	73239	74	86923	94	97313
15	17609	35	54407	55	74036	75	87506	95	97772
16	20412	36	55630	56	74819	76	88081	96	98227
17	23045	37	56820	57	75587	77	88649	97	98677
18	25527	38	57978	58	76343	78	89209	98	99123
19	27875	39	59106	59	77085	79	89763	99	99564
20	30103	40	60206	60	77815	80	90309	100	00000

No.	0	1	2	3	4	5	6	7	8	9
100	00000	00043	00087	00130	00173	00217	00260	00303	00346	00889
101	00432	00475	00518	00561	00604	00647	00689	00732	00775	00817
102	00860	00903	00945	00988	01030	01072	01115	01157	01199	01242
103	01284	01326	01368	01410	01452	01494	01536	01578	01620	01662
104	01703	01745	01787	01828	01870	01912	01953	01995	02036	02078
105	02119	02160	02202	02243	02284	02325	02366	02407	02449	02490
106	02531	02572	02612	02653	02694	02735	02776	02816	02857	02898
107	02938	02979	03019	03060	03100	03141	03181	03222	03262	03302
108	03342	03383	03423	03463	03503	03543	03583	03623	03663	03703
109	03743	03782	03822	03862	03902	03941	03981	04021	04060	04100
110	04139	04179	04218	04258	04297	04336	04376	04415	04454	04493
111	04532	04571	04610	04650	04689	04727	04766	04805	04844	04883
112	04922	04961	04999	05038	05077	05115	05154	05192	05231	05269
113	05308	05346	05385	05423	05461	05500	05538	05576	05614	05652
114	05690	05729	05767	05805	05843	05881	05918	05956	05994	06032
115	06070	06108	06145	06183	06221	06258	06296	06333	06371	06408
116	06446	06483	06521	06558	06595	06633	06670	06707	06744	06781
117	06819	06856	06893	06930	06967	07004	07041	07078	07115	07151
118	07188	07225	07262	07298	07335	07372	07408	07445	07482	07518
119	07555	07591	07628	07664	07700	07737	07773	07809	07846	07882
120	07918	07954	07990	08027	08063	08099	08135	08171	08207	08243
121	08279	08314	08350	08386	08422	08458	08493	08529	08565	08600
122	08636	08672	08707	08743	08778	08814	08849	08884	08920	08955
123	08991	09026	09061	09096	09132	09167	09202	09237	09272	09307
124	09342	09377	09412	09447	09482	09517	09552	09587	09621	09656
125	09691	09726	09760	09795	09830	09864	09899	09934	09968	10003
126	10037	10072	10106	10140	10175	10209	10243	10278	10312	10346
127	10380	10415	10449	10483	10517	10551	10585	10619	10653	10687
128	10721	10755	10789	10823	10857	10890	10924	10958	10992	11025
129	11059	11093	11126	11160	11193	11227	11261	11294	11327	11361
130	11394	11428	11461	11494	11528	11561	11594	11628	11661	11694
131	11727	11760	11793	11826	11860	11893	11926	11959	11992	12024
132	12057	12090	12123	12156	12189	12222	12254	12287	12320	12352
133	12385	12418	12450	12483	12516	12548	12581	12613	12646	12678
134	12710	12743	12775	12808	12840	12872	12905	12937	12969	13001
135	13033	13066	13098	13130	13162	13194	13226	13258	13290	13322
136	13354	13386	13418	13450	13481	13513	13545	13577	13609	13640
137	13672	13704	13735	13767	13799	13830	13862	13893	13925	13956
138	13988	14019	14051	14082	14114	14145	14176	14208	14239	14270
139	14301	14333	14364	14395	14426	14457	14489	14520	14551	14582
140	14613	14644	14675	14706	14737	14768	14799	14829	14860	14891
141	14922	14953	14983	15014	15045	15076	15106	15137	15168	15198
142	15229	15259	15290	15320	15351	15381	15412	15442	15473	15503
143	15534	15564	15594	15625	15655	15685	15715	15746	15776	15806
144	15836	15866	15897	15927	15957	15987	16017	16047	16077	16107
145	16137	16167	16197	16227	16256	16286	16316	16346	16376	16406
146	16435	16465	16495	16524	16554	16584	16613	16643	16673	16702
147	16732	16761	16791	16820	16850	16879	16909	16938	16967	16997
148	17026	17056	17085	17114	17143	17173	17202	17231	17260	17289
149	17319	17348	17377	17406	17435	17464	17493	17522	17551	17580
150	17609	17638	17667	17696	17725	17754	17782	17811	17840	17869
151	17898	17926	17955	17984	18013	18041	18070	18099	18127	18156
152	18184	18213	18241	18270	18298	18327	18355	18384	18412	18441
153	18469	18498	18526	18554	18583	18611	18639	18667	18696	18724
154	18752	18780	18808	18837	18865	18893	18921	18949	18977	19005
155	19033	19061	19089	19117	19145	19173	19201	19229	19257	19285
156	19312	19340	19368	19396	19424	19451	19479	19507	19535	19562
157	19590	19618	19645	19673	19700	19728	19756	19783	19811	19838
158	19866	19893	19921	19948	19976	20003	20030	20058	20085	20112
159	20140	20167	20194	20222	20249	20276	20303	20330	20358	20385

No.	0	1	2	3	4	5	6	7	8	9
160	20412	20439	20466	20493	20520	20548	20575	20602	20629	20656
161	20683	20710	20737	20763	20790	20817	20844	20871	20898	20925
162	20952	20978	21005	21032	21059	21085	21112	21139	21165	21192
163	21219	21245	21272	21299	21325	21352	21378	21405	21431	21458
164	21484	21511	21537	21564	21590	21617	21643	21669	21696	21722
165	21748	21775	21801	21827	21854	21880	21906	21932	21958	21985
166	22011	22037	22063	22089	22115	22141	22167	22194	22220	22246
167	22272	22298	22324	22350	22376	22401	22427	22453	22479	22505
168	22531	22557	22583	22608	22634	22660	22686	22712	22737	22763
169	22789	22814	22840	22866	22891	22917	22943	22968	22994	23019
170	23045	23070	23096	23121	23147	23172	23198	23223	23249	23274
171	23300	23325	23350	23376	23401	23426	23452	23477	23502	23528
172	23553	23578	23603	23629	23654	23679	23704	23729	23754	23779
173	23805	23830	23855	23880	23905	23930	23955	23980	24005	24030
174	24055	24080	24105	24130	24155	24180	24204	24229	24254	24279
175	24304	24329	24353	24378	24403	24428	24452	24477	24502	24527
176	24551	24576	24601	24625	24650	24674	24699	24724	24748	24773
177	24797	24822	24846	24871	24895	24920	24944	24969	24993	25018
178	25042	25066	25091	25115	25139	25164	25188	25212	25237	25261
179	25285	25310	25334	25358	25382	25406	25431	25455	25479	25503
180	25527	25551	25575	25600	25624	25648	25672	25696	25720	25744
181	25768	25792	25816	25840	25864	25888	25912	25935	25959	25983
182	26007	26031	26055	26079	26102	26126	26150	26174	26198	26221
183	26245	26269	26293	26316	26340	26364	26387	26411	26435	26458
184	26482	26505	26529	26553	26576	26600	26623	26647	26670	26694
185	26717	26741	26764	26788	26811	26834	26858	26881	26905	26928
186	26951	26975	26998	27021	27045	27068	27091	27114	27138	27161
187	27184	27207	27231	27254	27277	27300	27323	27346	27370	27393
188	27416	27439	27462	27485	27508	27531	27554	27577	27600	27623
189	27646	27669	27692	27715	27738	27761	27784	27807	27830	27852
190	27875	27898	27921	27944	27967	27989	28012	28035	28058	28081
191	28103	28126	28149	28171	28194	28217	28240	28262	28285	28307
192	28330	28353	28375	28398	28421	28443	28466	28488	28511	28533
193	28556	28578	28601	28623	28646	28668	28691	28713	28735	28758
194	28780	28803	28825	28847	28870	28892	28914	28937	28959	28981
195	29003	29026	29048	29070	29092	29115	29137	29159	29181	29203
196	29226	29248	29270	29292	29314	29336	29358	29380	29403	29425
197	29447	29469	29491	29513	29535	29557	29579	29601	29623	29645
198	29667	29688	29710	29732	29754	29776	29798	29820	29842	29863
199	29885	29907	29929	29951	29973	29994	30016	30038	30060	30081
200	30103	30125	30146	30168	30190	30211	30233	30255	30276	30298
201	30320	30341	30363	30384	30406	30428	30449	30471	30492	30514
202	30535	30557	30578	30600	30621	30643	30664	30685	30707	30728
203	30750	30771	30792	30814	30835	30856	30878	30899	30920	30942
204	30963	30984	31006	31027	31048	31069	31091	31112	31133	31154
205	31175	31197	31218	31239	31260	31281	31302	31323	31345	31366
206	31387	31408	31429	31450	31471	31492	31513	31534	31555	31576
207	31597	31618	31639	31660	31681	31702	31723	31744	31765	31785
208	31806	31827	31848	31869	31890	31911	31931	31952	31973	31994
209	32015	32035	32056	32077	32098	32118	32139	32160	32181	32201
210	32222	32243	32263	32284	32305	32325	32346	32366	32387	32408
211	32428	32449	32469	32490	32510	32531	32552	32572	32593	32613
212	32634	32654	32675	32695	32715	32736	32756	32777	32797	32818
213	32838	32858	32879	32899	32919	32940	32960	32980	33001	33021
214	33041	33062	33082	33102	33122	33143	33163	33183	33203	33224
215	33244	33264	33284	33304	33325	33345	33365	33385	33405	33425
216	33445	33465	33486	33506	33526	33546	33566	33586	33606	33626
217	33646	33666	33686	33706	33726	33746	33766	33786	33806	33826
218	33846	33866	33885	33905	33925	33945	33965	33985	34005	34025
219	34044	34064	34084	34104	34124	34143	34163	34183	34203	34223

No.	0	1	2	3	4	5	6	7	8	9
220	34242	34262	34282	34301	34321	34341	34361	34380	34400	34420
221	34439	34459	34479	34498	34518	34537	34557	34577	34596	34616
222	34635	34655	34674	34694	34713	34733	34753	34772	34792	34811
223	34830	34850	34869	34889	34908	34928	34947	34967	34986	35005
224	35025	35044	35064	35083	35102	35122	35141	35160	35180	35199
225	35218	35238	35257	35276	35295	35315	35334	35353	35372	35392
226	35411	35430	35449	35468	35488	35507	35526	35545	35564	35583
227	35603	35622	35641	35660	35679	35698	35717	35736	35755	35774
228	35793	35813	35832	35851	35870	35889	35908	35927	35946	35965
229	35984	36003	36021	36040	36059	36078	36097	36116	36135	36154
230	36173	36192	36211	36229	36248	36267	36286	36305	36324	36342
231	36361	36380	36399	36418	36436	36455	36474	36493	36511	36530
232	36549	36568	36586	36605	36624	36642	36661	36680	36698	36717
233	36736	36754	36773	36791	36810	36829	36847	36866	36884	36903
234	36922	36940	36959	36977	36996	37014	37033	37051	37070	37088
235	37107	37125	37144	37162	37181	37199	37218	37236	37254	37273
236	37291	37310	37328	37346	37365	37383	37401	37420	37438	37457
237	37475	37493	37511	37530	37548	37566	37585	37603	37621	37639
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247	39270	39287	39305	39322	39340	39358	39375	39393	39410	39428
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296	47129	47144	47159	47173	47188	47202	47217	47232	47246	47261
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414	61700	61711	61721	61731	61742	61752	61763	61773	61784	61794
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705	84819	84825	84831	84837	84844	84850	84856	84862	84868	84874
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882	94547	94552	94557	94562	94567	94571	94576	94581	94586	94591
883	94596	94601	94606	94611	94616	94621	94626	94630	94635	94640
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925	96614	96619	96624	96628	96633	96638	96642	96647	96652	96656
926	96661	96666	96670	96675	96680	96685	96689	96694	96699	96703
927	96708	96713	96717	96722	96727	96731	96736	96741	96745	96750
928	96755	96759	96764	96769	96774	96778	96783	96788	96792	96797
929	96802	96806	96811	96816	96820	96825	96830	96834	96839	96844
930	96848	96853	96858	96862	96867	96872	96876	96881	96886	96890
931	96895	96900	96904	96909	96914	96918	96923	96928	96932	96937
932	96942	96946	96951	96956	96960	96965	96970	96974	96979	96984
933	96988	96993	96997	97002	97007	97011	97016	97021	97025	97030
934	97035	97039	97044	97049	97053	97058	97063	97067	97072	97077
935	97081	97086	97090	97095	97100	97104	97109	97114	97118	97123
936	97128	97132	97137	97142	97146	97151	97155	97160	97165	97169
937	97174	97179	97183	97188	97192	97197	97202	97206	97211	97216
938	97220	97225	97230	97234	97239	97243	97248	97253	97257	97262
939	97267	97271	97276	97280	97285	97290	97294	97299	97304	97308

No.	0	1	2	3	4	5	6	7	8	9
940	97313	97317	97322	97327	97331	97336	97340	97345	97350	97354
941	97359	97364	97368	97373	97377	97382	97387	97391	97396	97400
942	97405	97410	97414	97419	97424	97428	97433	97437	97442	97447
943	97451	97456	97460	97465	97470	97474	97479	97483	97488	97493
944	97497	97502	97506	97511	97516	97520	97525	97529	97534	97539
945	97543	97548	97552	97557	97562	97566	97571	97575	97580	97585
946	97589	97594	97598	97603	97607	97612	97617	97621	97626	97630
947	97635	97640	97644	97649	97653	97658	97663	97667	97672	97676
948	97681	97685	97690	97695	97699	97704	97708	97713	97717	97722
949	97727	97731	97736	97740	97745	97749	97754	97759	97763	97768
950	97772	97777	97782	97786	97791	97795	97800	97804	97809	97813
951	97818	97823	97827	97832	97836	97841	97845	97850	97855	97859
952	97864	97868	97873	97877	97882	97886	97891	97896	97900	97905
953	97909	97914	97918	97923	97928	97932	97937	97941	97946	97950
954	97955	97959	97964	97968	97973	97978	97982	97987	97991	97996
955	98000	98005	98009	98014	98019	98023	98028	98032	98037	98041
956	98046	98050	98055	98059	98064	98068	98073	98078	98082	98087
957	98091	98096	98100	98105	98109	98114	98118	98123	98127	98132
958	98137	98141	98146	98150	98155	98159	98164	98168	98173	98177
959	98182	98186	98191	98195	98200	98204	98209	98214	98218	98223
960	98227	98232	98236	98241	98245	98250	98254	98259	98263	98268
961	98272	98277	98281	98286	98290	98295	98299	98304	98308	98313
962	98318	98322	98327	98331	98336	98340	98345	28349	98354	98358
963	98363	98367	98372	98376	98381	98385	98390	98394	98399	98403
964	98408	98412	98417	98421	98426	98430	98435	98439	98444	98448
965	98453	98457	98462	98466	98471	98475	98480	98484	98489	98493
966	98498	98502	98507	98511	98516	98520	98525	98529	98534	98538
967	98543	98547	98552	98556	98561	98565	98570	98574	98579	98583
968	98588	98592	98597	98601	98605	98610	98614	98619	98623	98628
969	98632	98637	98641	98646	98650	98655	98659	98664	98668	98673
970	98677	98682	98686	98691	98695	98700	98704	98709	98713	98717
971	98722	98726	98731	98735	98740	98744	98749	98753	98758	98762
972	98767	98771	98776	98780	98784	98789	98793	98798	98802	98807
973	98811	98816	98820	98825	98829	98834	98838	98843	98847	98851
974	98856	98860	98865	98869	98874	98878	98883	98887	98892	98896
975	98900	98905	98909	98914	98918	98923	98927	98932	98936	98941
976	98945	98949	98954	98958	98963	98967	98972	98976	98981	98985
977	98989	98994	98998	99003	99007	99012	99016	99021	99025	99029
978	99034	99038	99043	99047	99052	99056	99061	99065	99069	99074
979	99078	99083	99087	99092	99096	99100	99105	99109	99114	99118
980	99123	99127	99131	99136	99140	99145	99149	99154	99158	99162
981	99167	99171	99176	99180	99185	99189	99193	99198	99202	99207
982	99211	99216	99220	99224	99229	99233	99238	99242	99247	99251
983	99255	99260	99264	99269	99273	99277	99282	99286	99291	99295
984	99300	99304	99308	99313	99317	99322	99326	99330	99335	99339
985	99344	99348	99352	99357	99361	99366	99370	99374	99379	99383
986	99388	99392	99396	99401	99405	99410	99414	99419	99423	99427
987	99432	99436	99441	99445	99449	99454	99458	99463	99467	99471
988	99476	99480	99484	99489	99493	99498	99502	99506	99511	99515
989	99520	99524	99528	99533	99537	99542	99546	99550	99555	99559
990	99564	99568	99572	99577	99581	99585	99590	99594	99599	99603
991	99607	99612	99616	99621	99625	99629	99634	99638	99642	99647
992	99651	99656	99660	99664	99669	99673	99677	99682	99686	99691
993	99695	99699	99704	99708	99712	99717	99721	99726	99730	99734
994	99739	99743	99747	99752	99756	99760	99765	99769	99774	99778
995	99782	99787	99791	99795	99800	99804	99808	99813	99817	99822
996	99826	99830	99835	99839	99843	99848	99852	99856	99861	99865
997	99870	99874	99878	99883	99887	99891	99896	99900	99904	99909
998	99913	99917	99922	99926	99930	99935	99939	99944	99948	99952
999	99957	99961	99965	99970	99974	99978	99983	99987	99991	99996

	0°		1°		2°		3°		4°		
′	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	′
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1	00029	1·00000	01774	99984	03519	99938	05263	99861	07005	99754	59
2	00058	1·00000	01803	99984	03548	99937	05292	99860	07034	99752	58
3	00087	1·00000	01832	99983	03577	99936	05321	99858	07063	99750	57
4	00116	1·00000	01862	99983	03606	99935	05350	99857	07092	99748	56
5	00145	1·00000	01891	99982	03635	99934	05379	99855	07121	99746	55
6	00175	1·00000	01920	99982	03664	99933	05408	99854	07150	99744	54
7	00204	1·00000	01949	99981	03693	99932	05437	99852	07179	99742	53
8	00233	1·00000	01978	99980	03723	99931	05466	99851	07208	99740	52
9	00262	1·00000	02007	99980	03752	99930	05495	99849	07237	99738	51
10	00291	1·00000	02036	99979	03781	99929	05524	99847	07266	99736	50
11	00320	·99999	02065	99979	03810	99927	05553	99846	07295	99734	49
12	00349	·99999	02094	99978	03839	99926	05582	99844	07324	99731	48
13	00378	·99999	02123	99977	03868	99925	05611	99842	07353	99729	47
14	00407	·99999	02152	99977	03897	99924	05640	99841	07382	99727	46
15	00436	·99999	02181	99976	03926	99923	05669	99839	07411	99725	45
16	00465	·99999	02211	99976	03955	99922	05698	99838	07440	99723	44
17	00495	·99999	02240	99975	03984	99921	05727	99836	07469	99721	43
18	00524	·99999	02269	99974	04013	99919	05756	99834	07498	99719	42
19	00553	·99998	02298	99974	04042	99918	05785	99833	07527	99716	41
20	00582	·99998	02327	99973	04071	99917	05814	99831	07556	99714	40
21	00611	·99998	02356	99972	04100	99916	05844	99829	07585	99712	39
22	00640	·99998	02385	99972	04129	99915	05873	99827	07614	99710	38
23	00669	·99998	02414	99971	04159	99913	05902	99826	07643	99708	37
24	00698	·99998	02443	99970	04188	99912	05931	99824	07672	99705	36
25	00727	·99997	02472	99969	04217	99911	05960	99822	07701	99703	35
26	00756	·99997	02501	99969	04246	99910	05989	99821	07730	99701	34
27	00785	·99997	02530	99968	04275	99909	06018	99819	07759	99699	33
28	00814	·99997	02560	99967	04304	99907	06047	99817	07788	99696	32
29	00844	·99996	02589	99966	04333	99906	06076	99815	07817	99694	31
30	00873	·99996	02618	99966	04362	99905	06105	99813	07846	99692	30
31	00902	·99996	02647	99965	04391	99904	06134	99812	07875	99689	29
32	00931	·99996	02676	99964	04420	99902	06163	99810	07904	99687	28
33	00960	·99995	02705	99963	04449	99901	06192	99808	07933	99685	27
34	00989	·99995	02734	99963	04478	99900	06221	99806	07962	99683	26
35	01018	·99995	02763	99962	04507	99898	06250	99804	07991	99680	25
36	01047	·99995	02792	99961	04536	99897	06279	99803	08020	99678	24
37	01076	·99994	02821	99960	04565	99896	06308	99801	08049	99676	23
38	01105	·99994	02850	99959	04594	99894	06337	99799	08078	99673	22
39	01134	·99994	02879	99959	04623	99893	06366	99797	08107	99671	21
40	01164	·99993	02908	99958	04653	99892	06395	99795	08136	99668	20
41	01193	·99993	02938	99957	04682	99890	06424	99793	08165	99666	19
42	01222	·99993	02967	99956	04711	99889	06453	99792	08194	99664	18
43	01251	·99992	02996	99955	04740	99888	06482	99790	08223	99661	17
44	01280	·99992	03025	99954	04769	99886	06511	99788	08252	99659	16
45	01309	·99991	03054	99953	04798	99885	06540	99786	08281	99657	15
46	01338	·99991	03083	99952	04827	99883	06569	99784	08310	99654	14
47	01367	·99991	03112	99952	04856	99882	06598	99782	08339	99652	13
48	01396	·99990	03141	99951	04885	99881	06627	99780	08368	99649	12
49	01425	·99990	03170	99950	04914	99879	06656	99778	08397	99647	11
50	01454	·99989	03199	99949	04943	99878	06685	99776	08426	99644	10
51	01483	·99989	03228	99948	04972	99876	06714	99774	08455	99642	9
52	01513	·99989	03257	99947	05001	99875	06743	99772	08484	99639	8
53	01542	·99988	03286	99946	05030	99873	06773	99770	08513	99637	7
54	01571	·99988	03316	99945	05059	99872	06802	99768	08542	99635	6
55	01600	·99987	03345	99944	05088	99870	06831	99766	08571	99632	5
56	01629	·99987	03374	99943	05117	99869	06860	99764	08600	99630	4
57	01658	·99986	03403	99942	05146	99867	06889	99762	08629	99627	3
58	01687	·99986	03432	99941	05175	99866	06918	99760	08658	99625	2
59	01716	·99985	03461	99940	05205	99864	06947	99758	08687	99622	1
60	01745	·99985	03490	99939	05234	99863	06976	99756	08716	99619	0
′	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	′
	89°		88°		87°		86°		85°		

	5°		6°		7°		8°		9°		
′	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	′
0	08716	99619	10453	99452	12187	99255	13917	99027	15643	98769	60
1	08745	99617	10482	99449	12216	99251	13946	99023	15672	98764	59
2	08774	99614	10511	99446	12245	99248	13975	99019	15701	98760	58
3	08803	99612	10540	99443	12274	99244	14004	99015	15730	98755	57
4	08831	99609	10569	99440	12302	99240	14033	99011	15758	98751	56
5	08860	99607	10597	99437	12331	99237	14061	99006	15787	98746	55
6	08889	99604	10626	99434	12360	99233	14090	99002	15816	98741	54
7	08918	99602	10655	99431	12389	99230	14119	98998	15845	98737	53
8	08947	99599	10684	99428	12418	99226	14148	98994	15873	98732	52
9	08976	99596	10713	99424	12447	99222	14177	98990	15902	98728	51
10	09005	99594	10742	99421	12476	99219	14205	98986	15931	98723	50
11	09034	99591	10771	99418	12504	99215	14234	98982	15959	98718	49
12	09063	99588	10800	99415	12533	99211	14263	98978	15988	98714	48
13	09092	99586	10829	99412	12562	99208	14292	98973	16017	98709	47
14	09121	99583	10858	99409	12591	99204	14320	98969	16046	98704	46
15	09150	99580	10887	99406	12620	99200	14349	98965	16074	98700	45
16	09179	99578	10916	99402	12649	99197	14378	98961	16103	98695	44
17	09208	99575	10945	99399	12678	99193	14407	98957	16132	98690	43
18	09237	99572	10973	99396	12706	99189	14436	98953	16160	98686	42
19	09266	99570	11002	99393	12735	99186	14464	98948	16189	98681	41
20	09295	99567	11031	99390	12764	99182	14493	98944	16218	98676	40
21	09324	99564	11060	99386	12793	99178	14522	98940	16246	98671	39
22	09353	99562	11089	99383	12822	99175	14551	98936	16275	98667	38
23	09382	99559	11118	99380	12851	99171	14580	98931	16304	98662	37
24	09411	99556	11147	99377	12880	99167	14608	98927	16333	98657	36
25	09440	99553	11176	99374	12908	99163	14637	98923	16361	98652	35
26	09469	99551	11205	99370	12937	99160	14666	98919	16390	98648	34
27	09498	99548	11234	99367	12966	99156	14695	98914	16419	98643	33
28	09527	99545	11263	99364	12995	99152	14723	98910	16447	98638	32
29	09556	99542	11291	99360	13024	99148	14752	98906	16476	98633	31
30	09585	99540	11320	99357	13053	99144	14781	98902	16505	98629	30
31	09614	99537	11349	99354	13081	99141	14810	98897	16533	98624	29
32	09642	99534	11378	99351	13110	99137	14838	98893	16562	98619	28
33	09671	99531	11407	99347	13139	99133	14867	98889	16591	98614	27
34	09700	99528	11436	99344	13168	99129	14896	98884	16620	98609	26
35	09729	99526	11465	99341	13197	99125	14925	98880	16648	98604	25
36	09758	99523	11494	99337	13226	99122	14954	98876	16677	98600	24
37	09787	99520	11523	99334	13254	99118	14982	98871	16706	98595	23
38	09816	99517	11552	99331	13283	99114	15011	98867	16734	98590	22
39	09845	99514	11580	99327	13312	99110	15040	98863	16763	98585	21
40	09874	99511	11609	99324	13341	99106	15069	98858	16792	98580	20
41	09903	99508	11638	99320	13370	99102	15097	98854	16820	98575	19
42	09932	99506	11667	99317	13399	99098	15126	98849	16849	98570	18
43	09961	99503	11696	99314	13427	99094	15155	98845	16878	98565	17
44	09990	99500	11725	99310	13456	99091	15184	98841	16906	98561	16
45	10019	99497	11754	99307	13485	99087	15212	98836	16935	98556	15
46	10048	99494	11783	99303	13514	99083	15241	98832	16964	98551	14
47	10077	99491	11812	99300	13543	99079	15270	98827	16992	98546	13
48	10106	99488	11840	99297	13572	99075	15299	98823	17021	98541	12
49	10135	99485	11869	99293	13600	99071	15327	98818	17050	98536	11
50	10164	99482	11898	99290	13629	99067	15356	98814	17078	98531	10
51	10192	99479	11927	99286	13658	99063	15385	98809	17107	98526	9
52	10221	99476	11956	99283	13687	99059	15414	98805	17136	98521	8
53	10250	99473	11985	99279	13716	99055	15442	98800	17164	98516	7
54	10279	99470	12014	99276	13744	99051	15471	98796	17193	98511	6
55	10308	99467	12043	99272	13773	99047	15500	98791	17222	98506	5
56	10337	99464	12071	99269	13802	99043	15529	98787	17250	98501	4
57	10366	99461	12100	99265	13831	99039	15557	98782	17279	98496	3
58	10395	99458	12129	99262	13860	99035	15586	98778	17308	98491	2
59	10424	99455	12158	99258	13889	99031	15615	98773	17336	98486	1
60	10453	99452	12187	99255	13917	99027	15643	98769	17365	98481	0
′	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	′
	84°		83°		82°		81°		80°		

	10°		11°		12°		13°		14°		
′	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	′
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1	17393	98476	19109	98157	20820	97809	22523	97430	24220	97023	59
2	17422	98471	19138	98152	20848	97803	22552	97424	24249	97015	58
3	17451	98466	19167	98146	20877	97797	22580	97417	24277	97008	57
4	17479	98461	19195	98140	20905	97791	22608	97411	24305	97001	56
5	17508	98455	19224	98135	20933	97784	22637	97404	24333	96994	55
6	17537	98450	19252	98129	20962	97778	22665	97398	24362	96987	54
7	17565	98445	19281	98124	20990	97772	22693	97391	24390	96980	53
8	17594	98440	19309	98118	21019	97766	22722	97384	24418	96973	52
9	17623	98435	19338	98112	21047	97760	22750	97378	24446	96966	51
10	17651	98430	19366	98107	21076	97754	22778	97371	24474	96959	50
11	17680	98425	19395	98101	21104	97748	22807	97365	24503	96952	49
12	17708	98420	19423	98096	21132	97742	22835	97358	24531	96945	48
13	17737	98414	19452	98090	21161	97735	22863	97351	24559	96937	47
14	17766	98409	19481	98084	21189	97729	22892	97345	24587	96930	46
15	17794	98404	19509	98079	21218	97723	22920	97338	24615	96923	45
16	17823	98399	19538	98073	21246	97717	22948	97331	24644	96916	44
17	17852	98394	19566	98067	21275	97711	22977	97325	24672	96909	43
18	17880	98389	19595	98061	21303	97705	23005	97318	24700	96902	42
19	17909	98383	19623	98056	21331	97698	23033	97311	24728	96894	41
20	17937	98378	19652	98050	21360	97692	23062	97304	24756	96887	40
21	17966	98373	19680	98044	21388	97686	23090	97298	24784	96880	39
22	17995	98368	19709	98039	21417	97680	23118	97291	24813	96873	38
23	18023	98362	19737	98033	21445	97673	23146	97284	24841	96866	37
24	18052	98357	19766	98027	21474	97667	23175	97278	24869	96858	36
25	18081	98352	19794	98021	21502	97661	23203	97271	24897	96851	35
26	18109	98347	19823	98016	21530	97655	23231	97264	24925	96844	34
27	18138	98341	19851	98010	21559	97648	23260	97257	24953	96837	33
28	18166	98336	19880	98004	21587	97642	23288	97251	24982	96829	32
29	18195	98331	19908	97998	21616	97636	23316	97244	25010	96822	31
30	18224	98325	19937	97992	21644	97630	23345	97237	25038	96815	30
31	18252	98320	19965	97987	21672	97623	23373	97230	25066	96807	29
32	18281	98315	19994	97981	21701	97617	23401	97223	25094	96800	28
33	18309	98310	20022	97975	21729	97611	23429	97217	25122	96793	27
34	18338	98304	20051	97969	21758	97604	23458	97210	25151	96786	26
35	18367	98299	20079	97963	21786	97598	23486	97203	25179	96778	25
36	18395	98294	20108	97958	21814	97592	23514	97196	25207	96771	24
37	18424	98288	20136	97952	21843	97585	23542	97189	25235	96764	23
38	18452	98283	20165	97946	21871	97579	23571	97182	25263	96756	22
39	18481	98277	20193	97940	21899	97573	23599	97176	25291	96749	21
40	18509	98272	20222	97934	21928	97566	23627	97169	25320	96742	20
41	18538	98267	20250	97928	21956	97560	23656	97162	25348	96734	19
42	18567	98261	20279	97922	21985	97553	23684	97155	25376	96727	18
43	18595	98256	20307	97916	22013	97547	23712	97148	25404	96719	17
44	18624	98250	20336	97910	22041	97541	23740	97141	25432	96712	16
45	18652	98245	20364	97905	22070	97534	23769	97134	25460	96705	15
46	18681	98240	20393	97899	22098	97528	23797	97127	25488	96697	14
47	18710	98234	20421	97893	22126	97521	23825	97120	25516	96690	13
48	18738	98229	20450	97887	22155	97515	23853	97113	25545	96682	12
49	18767	98223	20478	97881	22183	97508	23882	97106	25573	96675	11
50	18795	98218	20507	97875	22212	97502	23910	97100	25601	96667	10
51	18824	98212	20535	97869	22240	97496	23938	97093	25629	96660	9
52	18852	98207	20563	97863	22268	97489	23966	97086	25657	96653	8
53	18881	98201	20592	97857	22297	97483	23995	97079	25685	96645	7
54	18910	98196	20620	97851	22325	97476	24023	97072	25713	96638	6
55	18938	98190	20649	97845	22353	97470	24051	97065	25741	96630	5
56	18967	98185	20677	97839	22382	97463	24079	97058	25769	96623	4
57	18995	98179	20706	97833	22410	97457	24108	97051	25798	96615	3
58	19024	98174	20734	97827	22438	97450	24136	97044	25826	96608	2
59	19052	98168	20763	97821	22467	97444	24164	97037	25854	96600	1
60	19081	98163	20791	97815	22495	97437	24192	97030	25882	96593	0
′	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N.cos	N. sin	′
	79°		78°		77°		76°		75°		

	15°		16°		17°		18°		19°		
′	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	′
0	25882	96593	27564	96126	29237	95630	30902	95106	32557	94552	60
1	25910	96585	27592	96118	29265	95622	30929	95097	32584	94542	59
2	25938	96578	27620	96110	29293	95613	30957	95088	32612	94533	58
3	25966	96570	27648	96102	29321	95605	30985	95079	32639	94523	57
4	25994	96562	27676	96094	29348	95596	31012	95070	32667	94514	56
5	26022	96555	27704	96086	29376	95588	31040	95061	32694	94504	55
6	26050	96547	27731	96078	29404	95579	31068	95052	32722	94495	54
7	26079	96540	27759	96070	29432	95571	31095	95043	32749	94485	53
8	26107	96532	27787	96062	29460	95562	31123	95033	32777	94476	52
9	26135	96524	27815	96054	29487	95554	31151	95024	32804	94466	51
10	26163	96517	27843	96046	29515	95545	31178	95015	32832	94457	50
11	26191	96509	27871	96037	29543	95536	31206	95006	32859	94447	49
12	26219	96502	27899	96029	29571	95528	31233	94997	32887	94438	48
13	26247	96494	27927	96021	29599	95519	31261	94988	32914	94428	47
14	26275	96486	27955	96013	29626	95511	31289	94979	32942	94418	46
15	26303	96479	27983	96005	29654	95502	31316	94970	32969	94409	45
16	26331	96471	28011	95997	29682	95493	31344	94961	32997	94399	44
17	26359	96463	28039	95989	29710	95485	31372	94952	33024	94390	43
18	26387	96456	28067	95981	29737	95476	31399	94943	33051	94380	42
19	26415	96448	28095	95972	29765	95467	31427	94933	33079	94370	41
20	26443	96440	28123	95964	29793	95459	31454	94924	33106	94361	40
21	26471	96433	28150	95956	29821	95450	31482	94915	33134	94351	39
22	26500	96425	28178	95948	29849	95441	31510	94906	33161	94342	38
23	26528	96417	28206	95940	29876	95433	31537	94897	33189	94332	37
24	26556	96410	28234	95931	29904	95424	31565	94888	33216	94322	36
25	26584	96402	28262	95923	29932	95415	31593	94878	33244	94313	35
26	26612	96394	28290	95915	29960	95407	31620	94869	33271	94303	34
27	26640	96386	28318	95907	29987	95398	31648	94860	33298	94293	33
28	26668	96379	28346	95898	30015	95389	31675	94851	33326	94284	32
29	26696	96371	28374	95890	30043	95380	31703	94842	33353	94274	31
30	26724	96363	28402	95882	30071	95372	31730	94832	33381	94264	30
31	26752	96355	28429	95874	30098	95363	31758	94823	33408	94254	29
32	26780	96347	28457	95865	30126	95354	31786	94814	33436	94245	28
33	26808	96340	28485	95857	30154	95345	31813	94805	33463	94235	27
34	26836	96332	28513	95849	30182	95337	31841	94795	33490	94225	26
35	26864	96324	28541	95841	30209	95328	31868	94786	33518	94216	25
36	26892	96316	28569	95832	30237	95319	31896	94777	33545	94206	24
37	26920	96308	28597	95824	30265	95310	31923	94768	33573	94196	23
38	26948	96301	28625	95816	30292	95301	31951	94758	33600	94186	22
39	26976	96293	28652	95807	30320	95293	31979	94749	33627	94176	21
40	27004	96285	28680	95799	30348	95284	32006	94740	33655	94167	20
41	27032	96277	28708	95791	30376	95275	32034	94730	33682	94157	19
42	27060	96269	28736	95782	30403	95266	32061	94721	33710	94147	18
43	27088	96261	28764	95774	30431	95257	32089	94712	33737	94137	17
44	27116	96253	28792	95766	30459	95248	32116	94702	33764	94127	16
45	27144	96246	28820	95757	30486	95240	32144	94693	33792	94118	15
46	27172	96238	28847	95749	30514	95231	32171	94684	33819	94108	14
47	27200	96230	28875	95740	30542	95222	32199	94674	33846	94098	13
48	27228	96222	28903	95732	30570	95213	32227	94665	33874	94088	12
49	27256	96214	28931	95724	30597	95204	32254	94656	33901	94078	11
50	27284	96206	28959	95715	30625	95195	32282	94646	33929	94068	10
51	27312	96198	28987	95707	30653	95186	32309	94637	33956	94058	9
52	27340	96190	29015	95698	30680	95177	32337	94627	33983	94049	8
53	27368	96182	29042	95690	30708	95168	32364	94618	34011	94039	7
54	27396	96174	29070	95681	30736	95159	32392	94609	34038	94029	6
55	27424	96166	29098	95673	30763	95150	32419	94599	34065	94019	5
56	27452	96158	29126	95664	30791	95142	32447	94590	34093	94009	4
57	27480	96150	29154	95656	30819	95133	32474	94580	34120	93999	3
58	27508	96142	29182	95647	30846	95124	32502	94571	34147	93989	2
59	27536	96134	29209	95639	30874	95115	32529	94561	34175	93979	1
60	27564	96126	29237	95630	30902	95106	32557	94552	34202	93969	0
′	N. cos	N. sin	N. cos	N. sin	N.cos	N. sin	N.cos	N. sin	N. cos	N. sin	′
	75°		72°		71°		70°		69°		

	20°		21°		22°		23°		24°		
′	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	′
0	34202	93969	35837	93358	37461	92718	39073	92050	40674	91355	60
1	34229	93959	35864	93348	37488	92707	39100	92039	40700	91343	59
2	34257	93949	35891	93337	37515	92697	39127	92028	40727	91331	58
3	34284	93939	35918	93327	37542	92686	39153	92016	40753	91319	57
4	34311	93929	35945	93316	37569	92675	39180	92005	40780	91307	56
5	34339	93919	35973	93306	37595	92664	39207	91994	40806	91295	55
6	34366	93909	36000	93295	37622	92653	39234	91982	40833	91283	54
7	34393	93899	36027	93285	37649	92642	39260	91971	40860	91272	53
8	34421	93889	36054	93274	37676	92631	39287	91959	40886	91260	52
9	34448	93879	36081	93264	37703	92620	39314	91948	40913	91248	51
10	34475	93869	36108	93253	37730	92609	39341	91936	40939	91236	50
11	34503	93859	36135	93243	37757	92598	39367	91925	40966	91224	49
12	34530	93849	36162	93232	37784	92587	39394	91914	40992	91212	48
13	34557	93839	36190	93222	37811	92576	39421	91902	41019	91200	47
14	34584	93829	36217	93211	37838	92565	39448	91891	41045	91188	46
15	34612	93819	36244	93201	37865	92554	39474	91879	41072	91176	45
16	34639	93809	36271	93190	37892	92543	39501	91868	41098	91164	44
17	34666	93799	36298	93180	37919	92532	39528	91856	41125	91152	43
18	34694	93789	36325	93169	37946	92521	39555	91845	41151	91140	42
19	34721	93779	36352	93159	37973	92510	39581	91833	41178	91128	41
20	34748	93769	36379	93148	37999	92499	39608	91822	41204	91116	40
21	34775	93759	36406	93137	38026	92488	39635	91810	41231	91104	39
22	34803	93748	36434	93127	38053	92477	39661	91799	41257	91092	38
23	34830	93738	36461	93116	38080	92466	39688	91787	41284	91080	37
24	34857	93728	36488	93106	38107	92455	39715	91775	41310	91068	36
25	34884	93718	36515	93095	38134	92444	39741	91764	41337	91056	35
26	34912	93708	36542	93084	38161	92432	39768	91752	41363	91044	34
27	34939	93698	36569	93074	38188	92421	39795	91741	41390	91032	33
28	34966	93688	36596	93063	38215	92410	39822	91729	41416	91020	32
29	34993	93677	36623	93052	38241	92399	39848	91718	41443	91008	31
30	35021	93667	36650	93042	38268	92388	39875	91706	41469	90996	30
31	35048	93657	36677	93031	38295	92377	39902	91694	41496	90984	29
32	35075	93647	36704	93020	38322	92366	39928	91683	41522	90972	28
33	35102	93637	36731	93010	38349	92355	39955	91671	41549	90960	27
34	35130	93626	36758	92999	38376	92343	39982	91660	41575	90948	26
35	35157	93616	36785	92988	38403	92332	40008	91648	41602	90936	25
36	35184	93606	36812	92978	38430	92321	40035	91636	41628	90924	24
37	35211	93596	36839	92967	38456	92310	40062	91625	41655	90911	23
38	35239	93585	36867	92956	38483	92299	40088	91613	41681	90899	22
39	35266	93575	36894	92945	38510	92287	40115	91601	41707	90887	21
40	35293	93565	36921	92935	38537	92276	40141	91590	41734	90875	20
41	35320	93555	36948	92924	38564	92265	40168	91578	41760	90863	19
42	35347	93544	36975	92913	38591	92254	40195	91566	41787	90851	18
43	35375	93534	37002	92902	38617	92243	40221	91555	41813	90839	17
44	35402	93524	37029	92892	38644	92231	40248	91543	41840	90826	16
45	35429	93514	37056	92881	38671	92220	40275	91531	41866	90814	15
46	35456	93503	37083	92870	38698	92209	40301	91519	41892	90802	14
47	35484	93493	37110	92859	38725	92198	40328	91508	41919	90790	13
48	35511	93483	37137	92849	38752	92186	40355	91496	41945	90778	12
49	35538	93472	37164	92838	38778	92175	40381	91484	41972	90766	11
50	35565	93462	37191	92827	38805	92164	40408	91472	41998	90753	10
51	35592	93452	37218	92816	38832	92152	40434	91461	42024	90741	9
52	35619	93441	37245	92805	38859	92141	40461	91449	42051	90729	8
53	35647	93431	37272	92794	38886	92130	40488	91437	42077	90717	7
54	35674	93420	37299	92784	38912	92119	40514	91425	42104	90704	6
55	35701	93410	37326	92773	38939	92107	40541	91414	42130	90692	5
56	35728	93400	37353	92762	38966	92096	40567	91402	42156	90680	4
57	35755	93389	37380	92751	38993	92085	40594	91390	42183	90668	3
58	35782	93379	37407	92740	39020	92073	40621	91378	42209	90655	2
59	35810	93368	37434	92729	39046	92062	40647	91366	42235	90643	1
60	35837	93358	37461	92718	39073	92050	40674	91355	42262	90631	0
′	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	′
	69°		68°		67°		66°		65°		

	25°		26°		27°		28°		29°		
′	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	′
0	42262	90631	43837	89879	45399	89101	46947	88295	48481	87462	60
1	42288	90618	43863	89867	45425	89087	46973	88281	48506	87448	59
2	42315	90606	43889	89854	45451	89074	46999	88267	48532	87434	58
3	42341	90594	43916	89841	45477	89061	47024	88254	48557	87420	57
4	42367	90582	43942	89828	45503	89048	47050	88240	48583	87406	56
5	42394	90569	43968	89816	45529	89035	47076	88226	48608	87391	55
6	42420	90557	43994	89803	45554	89021	47101	88213	48634	87377	54
7	42446	90545	44020	89790	45580	89008	47127	88199	48659	87363	53
8	42473	90532	44046	89777	45606	88995	47153	88185	48684	87349	52
9	42499	90520	44072	89764	45632	88981	47178	88172	48710	87335	51
10	42525	90507	44098	89752	45658	88968	47204	88158	48735	87321	50
11	42552	90495	44124	89739	45684	88955	47229	88144	48761	87306	49
12	42578	90483	44151	89726	45710	88942	47255	88130	48786	87292	48
13	42604	90470	44177	89713	45736	88928	47281	88117	48811	87278	47
14	42631	90458	44203	89700	45762	88915	47306	88103	48837	87264	46
15	42657	90446	44229	89687	45787	88902	47332	88089	48862	87250	45
16	42683	90433	44255	89674	45813	88888	47358	88075	48888	87235	44
17	42709	90421	44281	89662	45839	88875	47383	88062	48913	87221	43
18	42736	90408	44307	89649	45865	88862	47409	88048	48938	87207	42
19	42762	90396	44333	89636	45891	88848	47434	88034	48964	87193	41
20	42788	90383	44359	89623	45917	88835	47460	88020	48989	87178	40
21	42815	90371	44385	89610	45942	88822	47486	88006	49014	87164	39
22	42841	90358	44411	89597	45968	88808	47511	87993	49040	87150	38
23	42867	90346	44437	89584	45994	88795	47537	87979	49065	87136	37
24	42894	90334	44464	89571	46020	88782	47562	87965	49090	87121	36
25	42920	90321	44490	89558	46046	88768	47588	87951	49116	87107	35
26	42946	90309	44516	89545	46072	88755	47614	87937	49141	87093	34
27	42972	90296	44542	89532	46097	88741	47639	87923	49166	87079	33
28	42999	90284	44568	89519	46123	88728	47665	87909	49192	87064	32
29	43025	90271	44594	89506	46149	88715	47690	87896	49217	87050	31
30	43051	90259	44620	89493	46175	88701	47716	87882	49242	87036	30
31	43077	90246	44646	89480	46201	88688	47741	87868	49268	87021	29
32	43104	90233	44672	89467	46226	88674	47767	87854	49293	87007	28
33	43130	90221	44698	89454	46252	88661	47793	87840	49318	86993	27
34	43156	90208	44724	89441	46278	88647	47818	87826	49344	86978	26
35	43182	90196	44750	89428	46304	88634	47844	87812	49369	86964	25
36	43209	90183	44776	89415	46330	88620	47869	87798	49394	86949	24
37	43235	90171	44802	89402	46355	88607	47895	87784	49419	86935	23
38	43261	90158	44828	89389	46381	88593	47920	87770	49445	86921	22
39	43287	90146	44854	89376	46407	88580	47946	87756	49470	86906	21
40	43313	90133	44880	89363	46433	88566	47971	87743	49495	86892	20
41	43340	90120	44906	89350	46458	88553	47997	87729	49521	86878	19
42	43366	90108	44932	89337	46484	88539	48022	87715	49546	86863	18
43	43392	90095	44958	89324	46510	88526	48048	87701	49571	86849	17
44	43418	90082	44984	89311	46536	88512	48073	87687	49596	86834	16
45	43445	90070	45010	89298	46561	88499	48099	87673	49622	86820	15
46	43471	90057	45016	89285	46587	88485	48124	87659	49647	86805	14
47	43497	90045	45062	89272	46613	88472	48150	87645	49672	86791	13
48	43523	90032	45088	89259	46639	88458	48175	87631	49697	86777	12
49	43549	90019	45114	89245	46664	88445	48201	87617	49723	86762	11
50	43575	90007	45140	89232	46690	88431	48226	87603	49748	86748	10
51	43602	89994	45166	89219	46716	88417	48252	87589	49773	86733	9
52	43628	89981	45192	89206	46742	88404	48277	87575	49798	86719	8
53	43654	89968	45218	89193	46767	88390	48303	87561	49824	86704	7
54	43680	89956	45243	89180	46793	88377	48328	87546	49849	86690	6
55	43706	89943	45269	89167	46819	88363	48354	87532	49874	86675	5
56	43733	89930	45295	89153	46844	88349	48379	87518	49899	86661	4
57	43759	89918	45321	89140	46870	88336	48405	87504	49924	86646	3
58	43785	89905	45347	89127	46896	88322	48430	87490	49950	86632	2
59	43811	89892	45373	89114	46921	88308	48456	87476	49975	86617	1
60	43837	89879	45399	89101	46947	88295	48481	87462	50000	86603	0
′	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	′
	64°		63°		62°		61°		60°		

	30°		31°		32°		33°		34°		
′	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	′
0	50000	86603	51504	85717	52992	84805	54464	83867	55919	82904	60
1	50025	86588	51529	85702	53017	84789	54488	83851	55943	82887	59
2	50050	86573	51554	85687	53041	84774	54513	83835	55968	82871	58
3	50076	86559	51579	85672	53066	84759	54537	83819	55992	82855	57
4	50101	86544	51604	85657	53091	84743	54561	83804	56016	82839	56
5	50126	86530	51628	85642	53115	84728	54586	83788	56040	82822	55
6	50151	86515	51653	85627	53140	84712	54610	83772	56064	82806	54
7	50176	86501	51678	85612	53164	84697	54635	83756	56088	82790	53
8	50201	86486	51703	85597	53189	84681	54659	83740	56112	82773	52
9	50227	86471	51728	85582	53214	84666	54683	83724	56136	82757	51
10	50252	86457	51753	85567	53238	84650	54708	83708	56160	82741	50
11	50277	86442	51778	85551	53263	84635	54732	83692	56184	82724	49
12	50302	86427	51803	85536	53288	84619	54756	83676	56208	82708	48
13	50327	86413	51828	85521	53312	84604	54781	83660	56232	82692	47
14	50352	86398	51852	85506	53337	84588	54805	83645	56256	82675	46
15	50377	86384	51877	85491	53361	84573	54829	83629	56280	82659	45
16	50403	86369	51902	85476	53386	84557	54854	83613	56305	82643	44
17	50428	86354	51927	85461	53411	84542	54878	83597	56329	82626	43
18	50453	86340	51952	85446	53435	84526	54902	83581	56353	82610	42
19	50478	86325	51977	85431	53460	84511	54927	83565	56377	82593	41
20	50503	86310	52002	85416	53484	84495	54951	83549	56401	82577	40
21	50528	86295	52026	85401	53509	84480	54975	83533	56425	82561	39
22	50553	86281	52051	85385	53534	84464	54999	83517	56449	82544	38
23	50578	86266	52076	85370	53558	84448	55024	83501	56473	82528	37
24	50603	86251	52101	85355	53583	84433	55048	83485	56497	82511	36
25	50628	86237	52126	85340	53607	84417	55072	83469	56521	82495	35
26	50654	86222	52151	85325	53632	84402	55097	83453	56545	82478	34
27	50679	86207	52175	85310	53656	84386	55121	83437	56569	82462	33
28	50704	86192	52200	85294	53681	84370	55145	83421	56593	82446	32
29	50729	86178	52225	85279	53705	84355	55169	83405	56617	82429	31
30	50754	86163	52250	85264	53730	84339	55194	83389	56641	82413	30
31	50779	86148	52275	85249	53754	84324	55218	83373	56665	82396	29
32	50804	86133	52299	85234	53779	84308	55242	83356	56689	82380	28
33	50829	86119	52324	85218	53804	84292	55266	83340	56713	82363	27
34	50854	86104	52349	85203	53828	84277	55291	83324	56736	82347	26
35	50879	86089	52374	85188	53853	84261	55315	83308	56760	82330	25
36	50904	86074	52399	85173	53877	84245	55339	83292	56784	82314	24
37	50929	86059	52423	85157	53902	84230	55363	83276	56808	82297	23
38	50954	86045	52448	85142	53926	84214	55388	83260	56832	82281	22
39	50979	86030	52473	85127	53951	84198	55412	83244	56856	82264	21
40	51004	86015	52498	85112	53975	84182	55436	83228	56880	82248	20
41	51029	86000	52522	85096	54000	84167	55460	83212	56904	82231	19
42	51054	85985	52547	85081	54024	84151	55484	83195	56928	82214	18
43	51079	85970	52572	85066	54049	84135	55509	83179	56952	82198	17
44	51104	85956	52597	85051	54073	84120	55533	83163	56976	82181	16
45	51129	85941	52621	85035	54097	84104	55557	83147	57000	82165	15
46	51154	85926	52646	85020	54122	84088	55581	83131	57024	82148	14
47	51179	85911	52671	85005	54146	84072	55605	83115	57047	82132	13
48	51204	85896	52696	84989	54171	84057	55630	83098	57071	82115	12
49	51229	85881	52720	84974	54195	84041	55654	83082	57095	82098	11
50	51254	85866	52745	84959	54220	84025	55678	83066	57119	82082	10
51	51279	85851	52770	84943	54244	84009	55702	83050	57143	82065	9
52	51304	85836	52794	84928	54269	83994	55726	83034	57167	82048	8
53	51329	85821	52819	84913	54293	83978	55750	83017	57191	82032	7
54	51354	85806	52844	84897	54317	83962	55775	83001	57215	82015	6
55	51379	85792	52869	84882	54342	83946	55799	82985	57238	81999	5
56	51404	85777	52893	84866	54366	83930	55823	82969	57262	81982	4
57	51429	85762	52918	84851	54391	83915	55847	82953	57286	81965	3
58	51454	85747	52943	84836	54415	83899	55871	82936	57310	81949	2
59	51479	85732	52967	84820	54440	83883	55895	82920	57334	81932	1
60	51504	85717	52992	84805	54464	83867	55919	82904	57358	81915	0
′	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	′
	59°		58°		57°		56°		55°		

	35°		36°		37°		38°		39°		
′	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	′
0	57358	81915	58779	80902	60182	79864	61566	78801	62932	77715	60
1	57381	81899	58802	80885	60205	79846	61589	78783	62955	77696	59
2	57405	81882	58826	80867	60228	79829	61612	78765	62977	77678	58
3	57429	81865	58849	80850	60251	79811	61635	78747	63000	77660	57
4	57453	81848	58873	80833	60274	79793	61658	78729	63022	77641	56
5	57477	81832	58896	80816	60298	79776	61681	78711	63045	77623	55
6	57501	81815	58920	80799	60321	79758	61704	78694	63068	77605	54
7	57524	81798	58943	80782	60344	79741	61726	78676	63090	77586	53
8	57548	81782	58967	80765	60367	79723	61749	78658	63113	77568	52
9	57572	81765	58990	80748	60390	79706	61772	78640	63135	77550	51
10	57596	81748	59014	80730	60414	79688	61795	78622	63158	77531	50
11	57619	81731	59037	80713	60437	79671	61818	78604	63180	77513	49
12	57643	81714	59061	80696	60460	79653	61841	78586	63203	77494	48
13	57667	81698	59084	80679	60483	79635	61864	78568	63225	77476	47
14	57691	81681	59108	80662	60506	79618	61887	78550	63248	77458	46
15	57715	81664	59131	80644	60529	79600	61909	78532	63271	77439	45
16	57738	81647	59154	80627	60553	79583	61932	78514	63293	77421	44
17	57762	81631	59178	80610	60576	79565	61955	78496	63316	77402	43
18	57786	81614	59201	80593	60599	79547	61978	78478	63338	77384	42
19	57810	81597	59225	80576	60622	79530	62001	78460	63361	77366	41
20	57833	81580	59248	80558	60645	79512	62024	78442	63383	77347	40
21	57857	81563	59272	80541	60668	79494	62046	78424	63406	77329	39
22	57881	81546	59295	80524	60691	79477	62069	78405	63428	77310	38
23	57904	81530	59318	80507	60714	79459	62092	78387	63451	77292	37
24	57928	81513	59342	80489	60738	79441	62115	78369	63473	77273	36
25	57952	81496	59365	80472	60761	79424	62138	78351	63496	77255	35
26	57976	81479	59389	80455	60784	79406	62160	78333	63518	77236	34
27	57999	81462	59412	80438	60807	79388	62183	78315	63540	77218	33
28	58023	81445	59436	80420	60830	79371	62206	78297	63563	77199	32
29	58047	81428	59459	80403	60853	79353	62229	78279	63585	77181	31
30	58070	81412	59482	80386	60876	79335	62251	78261	63608	77162	30
31	58094	81395	59506	80368	60899	79318	62274	78243	63630	77144	29
32	58118	81378	59529	80351	60922	79300	62297	78225	63653	77125	28
33	58141	81361	59552	80334	60945	79282	62320	78206	63675	77107	27
34	58165	81344	59576	80316	60968	79264	62342	78188	63698	77088	26
35	58189	81327	59599	80299	60991	79247	62365	78170	63720	77070	25
36	58212	81310	59622	80282	61015	79229	62388	78152	63742	77051	24
37	58236	81293	59646	80264	61038	79211	62411	78134	63765	77033	23
38	58260	81276	59669	80247	61061	79193	62433	78116	63787	77014	22
39	58283	81259	59693	80230	61084	79176	62456	78098	63810	76996	21
40	58307	81242	59716	80212	61107	79158	62479	78079	63832	76977	20
41	58330	81225	59739	80195	61130	79140	62502	78061	63854	76959	19
42	58354	81208	59763	80178	61153	79122	62524	78043	63877	76940	18
43	58378	81191	59786	80160	61176	79105	62547	78025	63899	76921	17
44	58401	81174	59809	80143	61199	79087	62570	78007	63922	76903	16
45	58425	81157	59832	80125	61222	79069	62592	77988	63944	76884	15
46	58449	81140	59856	80108	61245	79051	62615	77970	63966	76866	14
47	58472	81123	59879	80091	61268	79033	62638	77952	63989	76847	13
48	58496	81106	59902	80073	61291	79016	62660	77934	64011	76828	12
49	58519	81089	59926	80056	61314	78998	62683	77916	64033	76810	11
50	58543	81072	59949	80038	61337	78980	62706	77897	64056	76791	10
51	58567	81055	59972	80021	61360	78962	62728	77879	64078	76772	9
52	58590	81038	59995	80003	61383	78944	62751	77861	64100	76754	8
53	58614	81021	60019	79986	61406	78926	62774	77843	64123	76735	7
54	58637	81004	60042	79968	61429	78908	62796	77824	64145	76717	6
55	58661	80987	60065	79951	61451	78891	62819	77806	64167	76698	5
56	58684	80970	60089	79934	61474	78873	62842	77788	64190	76679	4
57	58708	80953	60112	79916	61497	78855	62864	77769	64212	76661	3
58	58731	80936	60135	79899	61520	78837	62887	77751	64234	76642	2
59	58755	80919	60158	79881	61543	78819	62909	77733	64256	76623	1
60	58779	80902	60182	79864	61566	78801	62932	77715	64279	76604	0
′	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	N. cos	N. sin	′
	54°		53°		52°		51°		50°		

	40°		41°		42°		43°		44°		
	N.sin	N.cos	N.sin	N.cos	N.sin	N.cos	N.sin	N.cos	N.sin	N.cos	′
0	64279	76604	65606	75471	66913	74314	68200	73135	69466	71934	60
1	64301	76586	65628	75452	66935	74295	68221	73116	69487	71914	59
2	64323	76567	65650	75433	66956	74276	68242	73096	69508	71894	58
3	64346	76548	65672	75414	66978	74256	68264	73076	69529	71873	57
4	64368	76530	65694	75395	66999	74237	68285	73056	69549	71853	56
5	64390	76511	65716	75375	67021	74217	68306	73036	69570	71833	55
6	64412	76492	65738	75356	67043	74198	68327	73016	69591	71813	54
7	64435	76473	65759	75337	67064	74178	68349	72996	69612	71792	53
8	64457	76455	65781	75318	67086	74159	68370	72976	69633	71772	52
9	64479	76436	65803	75299	67107	74139	68391	72957	69654	71752	51
10	64501	76417	65825	75280	67129	74120	68412	72937	69675	71732	50
11	64524	76398	65847	75261	67151	74100	68434	72917	69696	71711	49
12	64546	76380	65869	75241	67172	74080	68455	72897	69717	71691	48
13	64568	76361	65891	75222	67194	74061	68476	72877	69737	71671	47
14	64590	76342	65913	75203	67215	74041	68497	72857	69758	71650	46
15	64612	76323	65935	75184	67237	74022	68518	72837	69779	71630	45
16	64635	76304	65956	75165	67258	74002	68539	72817	69800	71610	44
17	64657	76286	65978	75146	67280	73983	68561	72797	69821	71590	43
18	64679	76267	66000	75126	67301	73963	68582	72777	69842	71569	42
19	64701	76248	66022	75107	67323	73944	68603	72757	69862	71549	41
20	64723	76229	66044	75088	67344	73924	68624	72737	69883	71529	40
21	64746	76210	66066	75069	67366	73904	68645	72717	69904	71508	39
22	64768	76192	66088	75050	67387	73885	68666	72697	69925	71488	38
23	64790	76173	66109	75030	67409	73865	68688	72677	69946	71468	37
24	64812	76154	66131	75011	67430	73846	68709	72657	69966	71447	36
25	64834	76135	66153	74992	67452	73826	68730	72637	69987	71427	35
26	64856	76116	66175	74973	67473	73806	68751	72617	70008	71407	34
27	64878	76097	66197	74953	67495	73787	68772	72597	70029	71386	33
28	64901	76078	66218	74934	67516	73767	68793	72577	70049	71366	32
29	64923	76059	66240	74915	67538	73747	68814	72557	70070	71345	31
30	64945	76041	66262	74896	67559	73728	68835	72537	70091	71325	30
31	64967	76022	66284	74876	67580	73708	68857	72517	70112	71305	29
32	64989	76003	66306	74857	67602	73688	68878	72497	70132	71284	28
33	65011	75984	66327	74838	67623	73669	68899	72477	70153	71264	27
34	65033	75965	66349	74818	67645	73649	68920	72457	70174	71243	26
35	65055	75946	66371	74799	67666	73629	68941	72437	70195	71223	25
36	65077	75927	66393	74780	67688	73610	68962	72417	70215	71203	24
37	65100	75908	66414	74760	67709	73590	68983	72397	70236	71182	23
38	65122	75889	66436	74741	67730	73570	69004	72377	70257	71162	22
39	65144	75870	66458	74722	67752	73551	69025	72357	70277	71141	21
40	65166	75851	66480	74703	67773	73531	69046	72337	70298	71121	20
41	65188	75832	66501	74683	67795	73511	69067	72317	70319	71100	19
42	65210	75813	66523	74664	67816	73491	69088	72297	70339	71080	18
43	65232	75794	66545	74644	67837	73472	69109	72277	70360	71059	17
44	65254	75775	66566	74625	67859	73452	69130	72257	70381	71039	16
45	65276	75756	66588	74606	67880	73432	69151	72236	70401	71019	15
46	65298	75738	66610	74586	67901	73413	69172	72216	70422	70998	14
47	65320	75719	66632	74567	67923	73393	69193	72196	70443	70978	13
48	65342	75700	66653	74548	67944	73373	69214	72176	70463	70957	12
49	65364	75680	66675	74528	67965	73353	69235	72156	70484	70937	11
50	65386	75661	66697	74509	67987	73333	69256	72136	70505	70916	10
51	65408	75642	66718	74489	68008	73314	69277	72116	70525	70896	9
52	65430	75623	66740	74470	68029	73294	69298	72095	70546	70875	8
53	65452	75604	66762	74451	68051	73274	69319	72075	70567	70855	7
54	65474	75585	66783	74431	68072	73254	69340	72055	70587	70834	6
55	65496	75566	66805	74412	68093	73234	69361	72035	70608	70813	5
56	65518	75547	66827	74392	68115	73215	69382	72015	70628	70793	4
57	65540	75528	66848	74373	68136	73195	69403	71995	70649	70772	3
58	65562	75509	66870	74353	68157	73175	69424	71974	70670	70752	2
59	65584	75490	66891	74334	68179	73155	69445	71954	70690	70731	1
60	65606	75471	66913	74314	68200	73135	69466	71934	70711	70711	0
′	N.cos	N.sin	N.cos	N.sin	N.cos	N.sin	N.cos	N.sin	N.cos	N.sin	′
	49°		48°		47°		46°		45°		

III. A Table of Logarithmic Sines and Tangents, to every degree and minute of the Quadrant; the natural radius being *Ten thousand millions*, and the logarithmic radius 10.

From which the secants and versed sines are readily found; for,

The cosine subtracted from 20 leaves the secant, and the sine subtracted from 20 leaves the cosecant (page 131).

Also,

If from double the sine of the *half* of any *given Arc*, you subtract the number 9·69897, the remainder will be the versed sine of *that arc* (page 131).

0°					
′	Sine	Cosine	Tang.	Cotang.	′
0	Inf. neg.	10·00000	Inf. neg.	Inf. pos.	60
1	6·46373	10·00000	6·46373	13·53627	59
2	6·76476	10·00000	6·76476	13·23524	58
3	6·94085	10·00000	6·94085	13·05915	57
4	7·06579	10·00000	7·06579	12·93421	56
5	7·16270	10·00000	7·16270	12·83730	55
6	7·24188	10·00000	7·24188	12·75812	54
7	7·30882	10·00000	7·30882	12·69118	53
8	7·36682	10·00000	7·36682	12·63318	52
9	7·41797	10·00000	7·41797	12·58203	51
10	7·46373	10·00000	7·46373	12·53627	50
11	7·50512	10·00000	7·50512	12·49488	49
12	7·54291	10·00000	7·54291	12·45709	48
13	7·57767	10·00000	7·57767	12·42233	47
14	7·60985	10·00000	7·60986	12·39014	46
15	7·63982	10·00000	7·63982	12·36018	45
16	7·66784	10·00000	7·66785	12·33215	44
17	7·69417	9·99999	7·69418	12·30582	43
18	7·71900	9·99999	7·71900	12·28100	42
19	7·74248	9·99999	7·74248	12·25752	41
20	7·76475	9·99999	7·76476	12·23524	40
21	7·78594	9·99999	7·78595	12·21405	39
22	7·80615	9·99999	7·80615	12·19385	38
23	7·82545	9·99999	7·82546	12·17454	37
24	7·84393	9·99999	7·84394	12·15606	36
25	7·86166	9·99999	7·86167	12·13833	35
26	7·87870	9·99999	7·87871	12·12129	34
27	7·89509	9·99999	7·89510	12·10490	33
28	7·91088	9·99999	7·91089	12·08911	32
29	7·92612	9·99998	7·92613	12·07387	31
′	Cosine	Sine	Cotang.	Tang.	′
89°					

0°					
′	Sine	Cosine	Tang.	Cotang.	′
30	7·94084	9·99998	7·94086	12·05914	30
31	7·95508	9·99998	7·95510	12·04490	29
32	7·96887	9·99998	7·96889	12·03111	28
33	7·98223	9·99998	7·98225	12·01775	27
34	7·99520	9·99998	7·99522	12·00478	26
35	8·00779	9·99998	8·00781	11·99219	25
36	8·02002	9·99998	8·02004	11·97996	24
37	8·03192	9·99997	8·03194	11·96806	23
38	8·04350	9·99997	8·04353	11·95647	22
39	8·05478	9·99997	8·05481	11·94519	21
40	8·06578	9·99997	8·06581	11·93419	20
41	8·07650	9·99997	8·07653	11·92347	19
42	8·08696	9·99997	8·08700	11·91300	18
43	8·09718	9·99997	8·09722	11·90278	17
44	8·10717	9·99996	8·10720	11·89280	16
45	8·11693	9·99996	8·11696	11·88304	15
46	8·12647	9·99996	8·12651	11·87349	14
47	8·13581	9·99996	8·13585	11·86415	13
48	8·14495	9·99996	8·14500	11·85500	12
49	8·15391	9·99996	8·15395	11·84605	11
50	8·16268	9·99995	8·16273	11·83727	10
51	8·17128	9·99995	8·17133	11·82867	9
52	8·17971	9·99995	8·17976	11·82024	8
53	8·18798	9·99995	8·18804	11·81196	7
54	8·19610	9·99995	8·19616	11·80384	6
55	8·20407	9·99994	8·20413	11·79587	5
56	8·21189	9·99994	8·21195	11·78805	4
57	8·21958	9·99994	8·21964	11·78036	3
58	8·22713	9·99994	8·22720	11·77280	2
59	8·23456	9·99994	8·23462	11·76538	1
60	8·24186	9·99993	8·24192	11·75808	0
′	Cosine	Sine	Cotang.	Tang.	′
89°					

	1°				2°				
′	Sine	Co sine	Tang.	Co tang.	Sine	Co sine	Tang.	Co tang.	′
0	8·24186	9·99993	8·24192	11·75808	8·54282	9·99974	8·54308	11·45692	60
1	8·24903	9·99993	8·24910	11·75090	8·54642	9·99973	8·54669	11·45331	59
2	8·25609	9·99993	8·25616	11·74384	8·54999	9·99973	8·55027	11·44973	58
3	8·26304	9·99993	8·26312	11·73688	8·55354	9·99972	8·55382	11·44618	57
4	8·26988	9·99992	8·26996	11·73004	8·55705	9·99972	8·55734	11·44266	56
5	8·27661	9·99992	8·27669	11·72331	8·56054	9·99971	8·56083	11·43917	55
6	8·28324	9·99992	8·28332	11·71668	8·56400	9·99971	8·56429	11.43571	54
7	8·28977	9·99992	8·28986	11·71014	8·56743	9·99970	8·56773	11·43227	53
8	8·29621	9·99992	8·29629	11·70371	8·57084	9·99970	8·57114	11·42886	52
9	8·30255	9·99991	8·30263	11·69737	8·57421	9·99969	8·57452	11·42548	51
10	8·30879	9·99991	8·30888	11·69112	8·57757	9·99969	8·57788	11·42212	50
11	8·31495	9·99991	8·31505	11·68495	8·58089	9·99968	8·58121	11·41879	49
12	8·32103	9·99990	8·32112	11·67888	8·58419	9·99968	8·58451	11·41549	48
13	8·32702	9·99990	8·32711	11·67289	8·58747	9·99967	8·58779	11·41221	47
14	8·33292	9·99990	8·33302	11·66698	8·59072	9·99967	8·59105	11·40895	46
15	8·33875	9·99990	8·33886	11·66114	8·59395	9·99967	8·59428	11·40572	45
16	8·34450	9·99989	8·34461	11·65539	8·59715	9·99966	8·59749	11·40251	44
17	8·35018	9·99989	8·35029	11·64971	8·60033	9·99966	8·60068	11·39932	43
18	8·35578	9·99989	8·35590	11·64410	8·60349	9·99965	8·60384	11·39616	42
19	8·36131	9·99989	8·36143	11·63857	8·60662	9·99964	8·60698	11·39302	41
20	8·36678	9·99988	8·36689	11·63311	8·60973	9·99964	8·61009	11·38991	40
21	8·37217	9·99988	8·37229	11·62771	8·61282	9·99963	8·61319	11·38681	39
22	8·37750	9·99988	8·37762	11·62238	8·61589	9·99963	8·61626	11·38374	38
23	8·38276	9·99987	8·38289	11·61711	8·61894	9·99962	8·61931	11·38069	37
24	8·38796	9·99987	8·38809	11·61191	8·62196	9·99962	8·62234	11·37766	36
25	8·39310	9·99987	8·39323	11·60677	8·62497	9·99961	8·62535	11·37465	35
26	8·39818	9·99986	8·39832	11·60168	8·62795	9·99961	8·62834	11·37166	34
27	8·40320	9·99986	8·40334	11·59666	8·63091	9·99960	8·63131	11·36869	33
28	8·40816	9·99986	8·40830	11·59170	8·63385	9·99960	8·63426	11·36574	32
29	8·41307	9·99985	8·41321	11·58679	8·63678	9·99959	8·63718	11·36282	31
30	8·41792	9·99985	8·41807	11·58193	8·63968	9·99959	8·64009	11·35991	30
31	8·42272	9·99985	8·42287	11·57713	8·64256	9·99958	8·64298	11·35702	29
32	8·42746	9·99984	8·42762	11·57238	8·64543	9·99958	8·64585	11·35415	28
33	8·43216	9·99984	8·43232	11·56768	8·64827	9·99957	8·64870	11·35130	27
34	8·43680	9·99984	8·43696	11·56304	8·65110	9·99956	8·65154	11·34846	26
35	8·44139	9·99983	8·44156	11·55844	8·65391	9·99956	8·65435	11·34565	25
36	8·44594	9·99983	8·44611	11·55389	8·65670	9·99955	8·65715	11·34285	24
37	8·45044	9·99983	8·45061	11·54939	8·65947	9·99955	8·65993	11·34007	23
38	8·45489	9·99982	8·45507	11·54493	8·66223	9·99954	8·66269	11·33731	22
39	8·45930	9·99982	8·45948	11·54052	8·66497	9·99954	8·66543	11·33457	21
40	8·46366	9·99982	8·46385	11·53615	8·66769	9·99953	8·66816	11·33184	20
41	8·46799	9·99981	8·46817	11·53183	8·67039	9·99952	8·67087	11·32913	19
42	8·47226	9·99981	8·47245	11·52755	8·67308	9·99952	8·67356	11·32644	18
43	8·47650	9·99981	8·47669	11·52331	8·67575	9·99951	8·67624	11·32376	17
44	8·48069	9·99980	8·48089	11·51911	8·67841	9·99951	8·67890	11·32110	16
45	8·48485	9·99980	8·48505	11·51495	8·68104	9·99950	8·68154	11·31846	15
46	8·48896	9·99979	8·48917	11·51083	8·68367	9·99949	8·68417	11·31583	14
47	8·49304	9·99979	8·49325	11·50675	8·68627	9·99949	8·68678	11·31322	13
48	8·49708	9·99979	8·49729	11·50271	8·68886	9·99948	8·68938	11·31062	12
49	8·50108	9·99978	8·50130	11·49870	8·69144	9·99948	8·69196	11·30804	11
50	8·50504	9·99978	8·50527	11·49473	8·69400	9·99947	8·69453	11·30547	10
51	8·50897	9·99977	8·50920	11·49080	8·69654	9·99946	8·69708	11·30292	9
52	8·51287	9·99977	8·51310	11·48690	8·69907	9·99946	8·69962	11·30038	8
53	8·51673	9·99977	8·51696	11·48304	8·70159	9·99945	8·70214	11·29786	7
54	8·52055	9·99976	8·52079	11·47921	8·70409	9·99944	8·70465	11·29535	6
55	8·52434	9·99976	8·52459	11·47541	8·70658	9·99944	8·70714	11·29286	5
56	8·52810	9·99975	8·52835	11·47165	8·70905	9·99943	8·70962	11·29038	4
57	8·53183	9·99975	8·53208	11·46792	8·71151	9·99942	8·71208	11·28792	3
58	8·53552	9·99974	8·53578	11·46422	8·71395	9·99942	8·71453	11·28547	2
59	8·53919	9·99974	8·53945	11·46055	8·71638	9·99941	8·71697	11·28303	1
60	8·54282	9·99974	8·54308	11·45692	8·71880	9·99940	8·71940	11·28060	0
′	Co sine	Sine	Cotang.	Tang.	Co sine	Sine	Cotang.	Tang.	′
	88°				87°				

	3°				4°				
′	Sine	Co sine	Tang.	Co tang.	Sine	Co sine	Tang.	Co tang.	′
0	8·71880	9·99940	8·71940	11·28060	8·84358	9·99894	8·84464	11·15536	60
1	8·72120	9·99940	8·72181	11·27819	8·84539	9·99893	8·84646	11·15354	59
2	8·72359	9·99939	8·72420	11·27580	8·84718	9·99892	8·84826	11·15174	58
3	8·72597	9·99938	8·72659	11·27341	8·84897	9·99891	8·85006	11·14994	57
4	8·72834	9·99938	8·72896	11·27104	8·85075	9·99891	8·85185	11·14815	56
5	8·73069	9·99937	8·73132	11·26868	8·85252	9·99890	8·85363	11·14637	55
6	8·73303	9·99936	8·73366	11·26634	8·85429	9·99889	8·85540	11·14460	54
7	8·73535	9·99936	8·73600	11·26400	8·85605	9·99888	8·85717	11·14283	53
8	8·73767	9·99935	8·73832	11·26168	8·85780	9·99887	8·85893	11·14107	52
9	8·73997	9·99934	8·74063	11·25937	8·85955	9·99886	8·86069	11·13931	51
10	8·74226	9·99934	8·74292	11·25708	8·86128	9·99885	8·86243	11·13757	50
11	8·74454	9·99933	8·74521	11·25479	8·86301	9·99884	8·86417	11·13583	49
12	8·74680	9·99932	8·74748	11·25252	8·86474	9·99883	8·86591	11·13409	48
13	8·74906	9·99932	8·74974	11·25026	8·86645	9·99882	8·86763	11·13237	47
14	8·75130	9·99931	8·75199	11·24801	8·86816	9·99881	8·86935	11·13065	46
15	8·75353	9·99930	8·75423	11·24577	8·86987	9·99880	8·87106	11·12894	45
16	8·75575	9·99929	8·75645	11·24355	8·87156	9·99879	8·87277	11·12723	44
17	8·75795	9·99929	8·75867	11·24133	8·87325	9·99879	8·87447	11·12553	43
18	8·76015	9·99928	8·76087	11·23913	8·87494	9·99878	8·87616	11·12384	42
19	8·76234	9·99927	8·76306	11·23694	8·87661	9·99877	8·87785	11·12215	41
20	8·76451	9·99926	8·76525	11·23475	8·87829	9·99876	8·87953	11·12047	40
21	8·76667	9·99926	8·76742	11·23258	8·87995	9·99875	8·88120	11·11880	39
22	8·76883	9·99925	8·76958	11·23042	8·88161	9·99874	8·88287	11·11713	38
23	8·77097	9·99924	8·77173	11·22827	8·88326	9·99873	8·88453	11·11547	37
24	8·77310	9·99923	8·77387	11·22613	8·88490	9·99872	8·88618	11·11382	36
25	8·77522	9·99923	8·77600	11·22400	8·88654	9·99871	8·88783	11·11217	35
26	8·77733	9·99922	8·77811	11·22189	8·88817	9·99870	8·88948	11·11052	34
27	8·77943	9·99921	8·78022	11·21978	8·88980	9·99869	8·89111	11·10889	33
28	8·78152	9·99920	8·78232	11·21768	8·89142	9·99868	8·89274	11·10726	32
29	8·78360	9·99920	8·78441	11·21559	8·89304	9·99867	8·89437	11·10563	31
30	8·78568	9·99919	8·78649	11·21351	8·89464	9·99866	8·89598	11·10402	30
31	8·78774	9·99918	8·78855	11·21145	8·89625	9·99865	8·89760	11·10240	29
32	8·78979	9·99917	8·79061	11·20939	8·89784	9·99864	8·89920	11·10080	28
33	8·79183	9·99917	8·79266	11·20734	8·89943	9·99863	8·90080	11·09920	27
34	8·79386	9·99916	8·79470	11·20530	8·90102	9·99862	8·90240	11·09760	26
35	8·79588	9·99915	8·79673	11·20327	8·90260	9·99861	8·90399	11·09601	25
36	8·79789	9·99914	8·79875	11·20125	8·90417	9·99860	8·90557	11·09443	24
37	8·79990	9·99913	8·80076	11·19924	8·90574	9·99859	8·90715	11·09285	23
38	8·80189	9·99913	8·80277	11·19723	8·90730	9·99858	8·90872	11·09128	22
39	8·80388	9·99912	8·80476	11·19524	8·90885	9·99857	8·91029	11·08971	21
40	8·80585	9·99911	8·80674	11·19326	8·91040	9·99856	8·91185	11·08815	20
41	8·80782	9·99910	8·80872	11·19128	8·91195	9·99855	8·91340	11·08660	19
42	8·80978	9·99909	8·81068	11·18932	8·91349	9·99854	8·91495	11·08505	18
43	8·81173	9·99909	8·81264	11·18736	8·91502	9·99853	8·91650	11·08350	17
44	8·81367	9·99908	8·81459	11·18541	8·91655	9·99852	8·91803	11·08197	16
45	8·81560	9·99907	8·81653	11·18347	8·91807	9·99851	8·91957	11·08043	15
46	8·81752	9·99906	8·81846	11·18154	8·91959	9·99850	8·92110	11·07890	14
47	8·81944	9·99905	8·82038	11·17962	8·92110	9·99848	8·92262	11·07738	13
48	8·82134	9·99904	8·82230	11·17770	8·92261	9·99847	8·92414	11·07586	12
49	8·82324	9·99904	8·82420	11·17580	8·92411	9·99846	8·92565	11·07435	11
50	8·82513	9·99903	8·82610	11·17390	8·92561	9·99845	8·92716	11·07284	10
51	8·82701	9·99902	8·82799	11·17201	8·92710	9·99844	8·92866	11·07134	9
52	8·82888	9·99901	8·82987	11·17013	8·92859	9·99843	8·93016	11·06984	8
53	8·83075	9·99900	8·83175	11·16825	8·93007	9·99842	8·93165	11·06835	7
54	8·83261	9·99899	8·83361	11·16639	8·93154	9·99841	8·93313	11·06687	6
55	8·83446	9·99898	8·83547	11·16453	8·93301	9·99840	8·93462	11·06538	5
56	8·83630	9·99898	8·83732	11·16268	8·93448	9·99839	8·93609	11·06391	4
57	8·83813	9·99897	8·83916	11·16084	8·93594	9·99838	8·93756	11·06244	3
58	8·83996	9·99896	8·84100	11·15900	8·93740	9·99837	8·93903	11·06097	2
59	8·84177	9·99895	8·84282	11·15718	8·93885	9·99836	8·94049	11·05951	1
60	8·84358	9·99894	8·84464	11.15536	8·94030	9·99834	8·94195	11·05805	0
′	Co sine	Sine	Co tang.	Tang.	Co sine	Sine	Co tang.	Tang.	′
	86°				85°				

	5°				6°				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	′
0	8·94030	9·99834	8·94195	11·05805	9·01923	9·99761	9·02162	10·97838	60
1	8·94174	9·99833	8·94340	11·05660	9·02043	9·99760	9·02283	10.97717	59
2	8·94317	9·99832	8·94485	11·05515	9·02163	9·99759	9·02404	10·97596	58
3	8·94461	9·99831	8·94630	11·05370	9·02283	9·99757	9·02525	10·97475	57
4	8·94603	9·99830	8·94773	11·05227	9·02402	9·99756	9·02645	10·97355	56
5	8·94746	9·99829	8·94917	11·05083	9·02520	9·99755	9·02765	10·97234	55
6	8·94887	9·99828	8·95060	11·04940	9·02639	9·99753	9·02885	10·97115	54
7	8·95029	9·99827	8·95202	11·04798	9·02757	9·99752	9·03005	10·96995	53
8	8·95170	9·99825	8·95344	11·04656	9·02874	9·99751	9·03124	10·96876	52
9	8·95310	9·99824	8·95486	11·04514	9·02992	9·99749	9·03242	10·96758	51
10	8·95450	9·99823	8·95627	11·04373	9·03109	9·99748	9·03361	10·96639	50
11	8·95589	9·99822	8·95767	11·04233	9·03226	9·99747	9·03479	10·96521	49
12	8·95728	9·99821	8·95908	11·04092	9·03342	9·99745	9·03597	10·96403	48
13	8·95867	9·99820	8·96047	11·03953	9·03458	9·99744	9·03714	10·96286	47
14	8·96005	9·99819	8·96187	11·03813	9·03574	9·99742	9·03832	10·96168	46
15	8·96143	9·99817	8·96325	11·03675	9·03690	9·99741	9·03948	10·96052	45
16	8·96280	9·99816	8·96464	11·03536	9·03805	9·99740	9·04065	10·95935	44
17	8·96417	9·99815	8·96602	11·03398	9·03920	9·99738	9·04181	10·95819	43
18	8·96553	9·99814	8·96739	11·03261	9·04034	9·99737	9·04297	10·95703	42
19	8·96689	9·99813	8·96877	11·03123	9·04149	9·99736	9·04413	10·95587	41
20	8·96825	9·99812	8·97013	11·02987	9·04262	9·99734	9·04528	10·95472	40
21	8·96960	9·99810	8·97150	11·02850	9·04376	9·99733	9·04643	10·95357	39
22	8·97095	9·99809	8·97285	11·02715	9·04490	9·99731	9·04758	10·95242	38
23	8·97229	9·99808	8·97421	11·02579	9·04603	9·99730	9·04873	10·95127	37
24	8·97363	9·99807	8·97556	11·02444	9·04715	9·99728	9·04987	10·95013	36
25	8·97496	9·99806	8·97691	11·02309	9·04828	9·99727	9·05101	10·94899	35
26	8·97629	9·99804	8·97825	11·02175	9·04940	9·99726	9·05214	10·94786	34
27	8·97762	9·99803	8·97959	11·02041	9·05052	9·99724	9·05328	10·94672	33
28	8·97894	9·99802	8·98092	11·01908	9·05164	9·99723	9·05441	10·94559	32
29	8·98026	9·99801	8·98225	11·01775	9.05275	9·99721	9·05553	10·94447	31
30	8·98157	9·99800	8·98358	11·01642	9·05386	9·99720	9·05666	10·94334	30
31	8·98288	9·99798	8·98490	11·01510	9·05497	9·99718	9·05778	10·94222	29
32	8·98419	9·99797	8·98622	11·01378	9·05607	9·99717	9·05890	10·94110	28
33	8·98549	9·99796	8·98753	11·01247	9·05717	9·99716	9·06002	10.93998	27
34	8·98679	9·99795	8·98884	11·01116	9·05827	9·99714	9·06113	10·93887	26
35	8·98808	9·99793	8·99015	11·00985	9·05937	9·99713	9·06224	10·93776	25
36	8·98937	9·99792	8·99145	11·00855	9·06046	9·99711	9·06335	10·93665	24
37	8·99066	9·99791	8·99275	11·00725	9·06155	9·99710	9·06445	10·93555	23
38	8·99194	9·99790	8·99405	11·00595	9·06264	9·99708	9·06556	10·93444	22
39	8·99322	9·99788	8·99534	11·00466	9·06372	9·99707	9·06666	10·93334	21
40	8·99450	9·99787	8·99662	11·00338	9·06481	9·99705	9·06775	10·93225	20
41	8·99577	9·99786	8·99791	11·00209	9·06589	9·99704	9·06885	10·93115	19
42	8·99704	9·99785	8·99919	11·00081	9·06696	9·99702	9·06994	10·93006	18
43	8·99830	9·99783	9·00046	10·99954	9·06804	9·99701	9·07103	10·92897	17
44	8·99956	9·99782	9·00174	10·99826	9·06911	9·99599	9·07211	10·92789	16
45	9·00082	9·99781	9·00301	10·99699	9·07018	9·99698	9·07320	10·92680	15
46	9·00207	9·99780	9·00427	10·99573	9·07124	9·99696	9·07428	10·92572	14
47	9·00332	9·99778	9·00553	10·99447	9·07231	9·99695	9·07536	10·92464	13
48	9·00456	9·99777	9·00679	10·99321	9·07337	9·99693	9·07643	10·92357	12
49	9·00581	9·99776	9·00805	10·99195	9·07442	9·99692	9·07751	10·92249	11
50	9·00704	9·99775	9·00930	10·99070	9·07548	9·99690	9·07858	10·92142	10
51	9·00828	9·99773	9·01055	10·98945	9·07653	9·99689	9·07964	10·92036	9
52	9·00951	9·99772	9·01179	10·98821	9·07758	9·99687	9.08071	10·91929	8
53	9·01074	9·99771	9·01303	10·98697	9·07863	9·99686	9·08177	10·91823	7
54	9·01196	9·99769	9.01427	10·98573	9·07968	9·99684	9·08283	10·91717	6
55	9·01318	9·99768	9·01550	10·98450	9·08072	9·99683	9·08389	10·91611	5
56	9·01440	9·99767	9·01673	10·98327	9·08176	9·99681	9·08495	10·91505	4
57	9·01561	9·99765	9·01796	10·98204	9·08280	9·99680	9·08600	10·91400	3
58	9·01682	9·99764	9·01918	10·98082	9·08383	9·99678	9·08705	10·91295	2
59	9·01803	9·99763	9·02040	10·97960	9·08486	9·99677	9·08810	10·91190	1
60	9·01923	9·99761	9·02162	10·97838	9·08589	9·99675	9·08914	10·91086	0
′	Cosine	Sine	Cotang.	Tang.	Cosine	Sine	Cotang.	Tang.	′
	84°				83°				

	7°				8°				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	′
0	9·08589	9·99675	9·08914	10·91086	9·14356	9·99575	9·14780	10·85220	60
1	9·08692	9·99674	9·09019	10.90981	9·14445	9·99574	9·14872	10·85128	59
2	9·08795	9·99672	9·09123	10·90877	9·14535	9·99572	9·14963	10·85037	58
3	9·08897	9·99670	9·09227	10·90773	9·14624	9·99570	9·15054	10·84946	57
4	9·08999	9·99669	9·09330	10·90670	9·14714	9·99568	9·15145	10·84855	56
5	9·09101	9·99667	9·09434	10·90566	9·14803	9·99566	9·15236	10·84764	55
6	9·09202	9·99666	9·09537	10·90463	9·14891	9·99565	9·15327	10·84673	54
7	9·09304	9·99664	9·09640	10·90360	9·14980	9·99563	9·15417	10·84583	53
8	9·09405	9·99663	9·09742	10·90258	9·15069	9·99561	9·15508	10·84492	52
9	9·09506	9·99661	9·09845	10·90155	9·15157	9·99559	9·15598	10·84402	51
10	9·09606	9·99659	9·09947	10·90053	9·15245	9·99557	9·15688	10·84312	50
11	9·09707	9·99658	9·10049	10·89951	9·15333	9·99556	9·15777	10·84223	49
12	9·09807	9·99656	9·10150	10·89850	9·15421	9·99554	9·15867	10·84133	48
13	9·09907	9·99655	9·10252	10·89748	9·15508	9·99552	9·15956	10·84044	47
14	9·10006	9·99653	9·10353	10·89647	9·15596	9·99550	9·16046	10·83954	46
15	9·10106	9·99651	9·10454	10·89546	9·15683	9·99548	9·16135	10·83865	45
16	9·10205	9·99650	9·10555	10·89445	9·15770	9·99546	9·16224	10·83776	44
17	9·10304	9·99648	9·10656	10·89344	9·15857	9·99545	9·16312	10·83688	43
18	9·10402	9·99647	9·10756	10·89244	9·15944	9·99543	9·16401	10·83599	42
19	9·10501	9·99645	9·10856	10·89144	9·16030	9·99541	9·16489	10·83511	41
20	9·10599	9·99643	9·10956	10·89044	9·16116	9·99539	9·16577	10·83423	40
21	9·10697	9·99642	9·11056	10·88944	9·16203	9·99537	9·16665	10·83335	39
22	9·10795	9·99640	9·11155	10·88845	9·16289	9·99535	9·16753	10·83247	38
23	9·10893	9·99638	9·11254	10·88746	9·16374	9·99533	9·16841	10·83159	37
24	9·10990	9·99637	9·11355	10·88647	9·16460	9·99532	9·16928	10·83072	36
25	9·11087	9·99635	9·11452	10·88548	9·16545	9·99530	9·17016	10·82984	35
26	9·11184	9·99633	9·11551	10·88449	9·16631	9·99528	9·17103	10·82897	34
27	9·11281	9·99632	9·11649	10·88351	9·16716	9·99526	9·17190	10·82810	33
28	9·11377	9·99630	9·11747	10·88253	9·16801	9·99524	9·17277	10·82723	32
29	9·11474	9·99629	9·11845	10·88155	9·16886	9·99522	9·17363	10·82637	31
30	9·11570	9·99627	9·11943	10·88057	9·16970	9·99520	9·17450	10·82550	30
31	9·11666	9·99625	9·12040	10·87960	9·17055	9·99518	9·17536	10·82464	29
32	9·11761	9·99624	9·12138	10·87862	9·17139	9·99517	9·17622	10·82378	28
33	9·11857	9·99622	9·12235	10·87765	9·17223	9·99515	9·17708	10·82292	27
34	9·11952	9·99620	9·12332	10·87668	9·17307	9·99513	9·17794	10·82206	26
35	9·12047	9·99618	9·12428	10·87572	9·17391	9·99511	9·17880	10·82120	25
36	9·12142	9·99617	9·12525	10·87475	9·17474	9·99509	9·17965	10·82035	24
37	9·12236	9·99615	9·12621	10·87379	9·17558	9·99507	9·18051	10·81949	23
38	9·12331	9·99613	9·12717	10·87283	9·17641	9·99505	9·18136	10·81864	22
39	9·12425	9·99612	9·12813	10·87187	9·17724	9·99503	9·18221	10·81779	21
40	9·12519	9·99610	9·12909	10·87091	9·17807	9·99501	9·18306	10·81694	20
41	9·12612	9·99608	9·13004	10·86996	9·17890	9·99499	9·18391	10·81609	19
42	9·12706	9·99607	9·13099	10·86901	9·17973	9·99497	9·18475	10·81525	18
43	9·12799	9·99605	9·13194	10·86806	9·18055	9·99495	9·18560	10·81440	17
44	9·12892	9·99603	9·13289	10·86711	9·18137	9·99494	9·18644	10·81356	16
45	9·12985	9·99601	9·13384	10·86616	9·18220	9·99492	9·18728	10·81272	15
46	9·13078	9·99600	9·13478	10·86522	9·18302	9·99490	9·18812	10·81188	14
47	9·13171	9·99598	9·13573	10·86427	9·18383	9·99488	9·18896	10·81104	13
48	9·13263	9·99596	9·13667	10·86333	9·18465	9·99486	9·18979	10·81021	12
49	9·13355	9·99595	9·13761	10·86239	9·18547	9·99484	9·19063	10·80937	11
50	9·13447	9·99593	9·13854	10·86146	9·18628	9·99482	9·19146	10·80854	10
51	9·13539	9·99591	9·13948	10·86052	9·18709	9·99480	9·19229	10·80771	9
52	9·13630	9·99589	9·14041	10·85959	9·18790	9·99478	9·19312	10·80688	8
53	9·13722	9·99588	9·14134	10·85866	9·18871	9·99476	9·19395	10·80605	7
54	9·13813	9·99586	9·14227	10·85773	9·18952	9·99474	9·19478	10·80522	6
55	9·13904	9·99584	9·14320	10·85680	9·19033	9·99472	9·19561	10·80439	5
56	9·13994	9·99582	9·14412	10·85588	9·19113	9·99470	9·19643	10·80357	4
57	9·14085	9·99581	9·14504	10·85496	9·19193	9·99468	9·19725	10·80275	3
58	9·14175	9·99579	9·14597	10·85403	9·19273	9·99466	9·19807	10·80193	2
59	9·14266	9·99577	9·14688	10·85312	9·19353	9·99464	9·19889	10·80111	1
60	9·14356	9·99575	9·14780	10·85220	9·19433	9·99462	9·19971	10·80029	0
′	Cosine	Sine	Cotang.	Tang.	Cosine	Sine	Cotang.	Tang.	′
	82°				81°				

	9°				10°				
′	Sine	Cosine.	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	′
0	9·19433	9·99462	9·19971	10·80029	9·23967	9·99335	9·24632	10·75368	60
1	9·19513	9·99460	9·20053	10·79947	9·24039	9·99333	9·24706	10·75294	59
2	9·19592	9·99458	9·20134	10·79866	9·24110	9·99331	9·24779	10·75221	58
3	9·19672	9·99456	9·20216	10·79784	9·24181	9·99328	9·24853	10·75147	57
4	9·19751	9·99454	9·20297	10·79703	9·24253	9·99326	9·24926	10·75074	56
5	9·19830	9·99452	9·20378	10·79622	9·24324	9·99324	9·25000	10·75000	55
6	9·19909	9·99450	9·20459	10·79541	9·24395	9·99322	9·25073	10·74927	54
7	9·19988	9·99448	9·20540	10·79460	9·24466	9·99319	9·25146	10·74854	53
8	9·20067	9·99446	9·20621	10·79379	9·24536	9·99317	9·25219	10·74781	52
9	9·20145	9·99444	9·20701	10·79299	9·24607	9·99315	9·25292	10·74708	51
10	9·20223	9·99442	9·20782	10·79218	9·24677	9·99313	9·25365	10·74635	50
11	9·20302	9·99440	9·20862	10·79138	9·24748	9·99310	9·25437	10·74563	49
12	9·20380	9·99438	9·20942	10·79058	9·24818	9·99308	9·25510	10·74490	48
13	9·20458	9·99436	9·21022	10·78978	9·24888	9·99306	9·25582	10·74418	47
14	9·20535	9·99434	9·21102	10·78898	9·24958	9·99304	9·25655	10·74345	46
15	9·20613	9·99432	9·21182	10·78818	9·25028	9·99301	9·25727	10·74273	45
16	9·20691	9·99429	9·21261	10·78739	9·25098	9·99299	9·25799	10·74201	44
17	9·20768	9·99427	9·21341	10·78659	9·25168	9·99297	9·25871	10·74129	43
18	9·20845	9·99425	9·21420	10·78580	9·25237	9·99294	9·25943	10·74057	42
19	9·20922	9·99423	9·21499	10·78501	9·25307	9·99292	9·26015	10·73985	41
20	9·20999	9·99421	9·21578	10·78422	9·25376	9·99290	9·26086	10·73914	40
21	9·21076	9·99419	9·21657	10·78343	9·25445	9·99288	9·26158	10·73842	39
22	9·21153	9·99417	9·21736	10·78264	9·25514	9·99285	9·26229	10·73771	38
23	9·21229	9·99415	9·21814	10·78186	9·25583	9·99283	9·26301	10·73699	37
24	9·21306	9·99413	9·21893	10·78107	9·25652	9·99281	9·26372	10·73628	36
25	9·21382	9·99411	9·21971	10·78029	9·25721	9·99278	9·26443	10·73557	35
26	9·21458	9·99409	9·22049	10·77951	9·25790	9·99276	9·26514	10·73486	34
27	9·21534	9·99407	9·22127	10·77873	9·25858	9·99274	9·26585	10·73415	33
28	9·21610	9·99404	9·22205	10·77795	9·25927	9·99271	9·26655	10·73345	32
29	9·21685	9·99402	9·22283	10·77717	9·25995	9·99269	9·26726	10·73274	31
30	9·21761	9·99400	9·22361	10·77639	9·26063	9·99267	9·26797	10·73203	30
31	9·21836	9·99398	9·22438	10·77562	9·26131	9·99264	9·26867	10·73133	29
32	9·21912	9·99396	9·22516	10·77484	9·26199	9·99262	9·26937	10·73063	28
33	9·21987	9·99394	9·22593	10·77407	9·26267	9·99260	9·27008	10·72992	27
34	9·22062	9·99392	9·22670	10·77330	9·26335	9·99257	9·27078	10·72922	26
35	9·22137	9·99390	9·22747	10·77253	9·26403	9·99255	9·27148	10·72852	25
36	9·22211	9·99388	9·22824	10·77176	9·26470	9·99252	9·27218	10·72782	24
37	9·22286	9·99385	9·22901	10·77099	9·26538	9·99250	9·27288	10·72712	23
38	9·22361	9·99383	9·22977	10·77023	9·26605	9·99248	9·27357	10·72643	22
39	9·22435	9·99381	9·23054	10·76946	9·26672	9·99245	9·27427	10·72573	21
40	9·22509	9·99379	9·23130	10·76870	9·26739	9·99243	9·27496	10·72504	20
41	9·22583	9·99377	9·23206	10·76794	9·26806	9·99241	9·27566	10·72434	19
42	9·22657	9·99375	9·23283	10·76717	9·26873	9·99238	9·27635	10·72365	18
43	9·22731	9·99372	9·23359	10·76641	9·26940	9·99236	9·27704	10·72296	17
44	9·22805	9·99370	9·23435	10·76565	9·27007	9·99233	9·27773	10·72227	16
45	9·22878	9·99368	9·23510	10·76490	9·27073	9·99231	9·27842	10·72158	15
46	9·22952	9·99366	9·23586	10·76414	9·27140	9·99229	9·27911	10·72089	14
47	9·23025	9·99364	9·23661	10·76339	9·27206	9·99226	9·27980	10·72020	13
48	9·23098	9·99362	9·23737	10·76263	9·27273	9·99224	9·28049	10·71951	12
49	9·23171	9·99359	9·23812	10·76188	9·27339	9·99221	9·28117	10·71883	11
50	9·23244	9·99357	9·23887	10·76113	9·27405	9·99219	9·28186	10·71814	10
51	9·23317	9·99355	9·23962	10·76038	9·27471	9·99217	9·28254	10·71746	9
52	9·23390	9·99353	9·24037	10·75963	9·27537	9·99214	9·28323	10·71677	8
53	9·23462	9·99351	9·24112	10·75888	9·27602	9·99212	9·28391	10·71609	7
54	9·23535	9·99348	9·24186	10·75814	9·27668	9·99209	9·28459	10·71541	6
55	9·23607	9·99346	9·24261	10·75739	9·27734	9·98207	9·28527	10·71473	5
56	9·23679	9·99344	9·24335	10·75665	9·27799	9·99204	9·28595	10·71405	4
57	9·23752	9·99342	9·24410	10·75590	9·27864	9·99202	9·28662	10·71338	3
58	9·23823	9·99340	9·24484	10·75516	9 27930	9·99200	9·28730	10·71270	2
59	9·23895	9·99337	9·24558	10·75442	9·27995	9·99197	9·28798	10·71202	1
60	9·23967	9·99335	9·24632	10·75368	9·28060	9·99195	9·28865	10·71135	0
′	Cosine	Sine	Cotang.	Tang.	Cosine	Sine	Cotang.	Tang.	′
	80°				79°				

	11°				12°				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	′
0	9·28060	9·99195	9·28865	10·71135	9·31788	9·99040	9·32747	10·67253	60
1	9·28125	9·99192	9·28933	10·71067	9·31847	9·99038	9·32810	10·67190	59
2	9·28190	9·99190	9·29000	10·71000	9·31907	9·99035	9·32872	10·67128	58
3	9·28254	9·99187	9·29067	10·70933	9·31966	9·99032	9·32933	10·67067	57
4	9·28319	9·99185	9·29134	10·70866	9·32025	9·99030	9·32995	10·67005	56
5	9·28384	9·99182	9·29201	10·70799	9·32084	9·99027	9·33057	10·66943	55
6	9·28448	9·99180	9·29268	10·70732	9·32143	9·99024	9·33119	10·66881	54
7	9·28512	9·99177	9·29335	10·70665	9·32202	9·99022	9·33180	10·66820	53
8	9·28577	9·99175	9·29402	10·70598	9·32261	9·99019	9·33242	10·66758	52
9	9·28641	9·99172	9·29468	10·70532	9·32319	9·99016	9·33303	10·66697	51
10	9·28705	9·99170	9·29535	10·70465	9·32378	9·99013	9·33365	10·66635	50
11	9·28769	9·99167	9·29601	10·70399	9·32437	9·99011	9·33426	10·66574	49
12	9·28833	9·99165	9·29668	10·70332	9·32495	9·99008	9·33487	10·66513	48
13	9·28896	9·99162	9·29734	10·70266	9·32553	9·99005	9·33548	10·66452	47
14	9·28960	9·99160	9·29800	10·70200	9·32612	9·99002	9·33609	10·66391	46
15	9·29024	9·99157	9·29866	10·70134	9·32670	9·99000	9·33670	10·66330	45
16	9·29087	9·99155	9·29932	10·70068	9·32728	9·98997	9·33731	10·66269	44
17	9·29150	9·99152	9·29998	10·70002	9·32786	9·98994	9·33792	10·66208	43
18	9·29214	9·99150	9·30064	10·69936	9·32844	9·98991	9·33853	10·66147	42
19	9·29277	9·99147	9·30130	10·69870	9·32902	9·98989	9·33913	10·66087	41
20	9·29340	9·99145	9·30195	10·69805	9·32960	9·98986	9·33974	10·66026	40
21	9·29403	9·99142	9·30261	10·69739	9·33018	9·98983	9·34034	10·65966	39
22	9·29466	9·99140	9·30326	10·69674	9·33075	9·98980	9·34095	10·65905	38
23	9·29529	9·99137	9·30391	10·69609	9·33133	9·98978	9·34155	10·65845	37
24	9·29591	9·99135	9·30457	10·69543	9·33190	9·98975	9·34215	10·65785	36
25	9·29654	9·99132	9·30522	10·69478	9·33248	9·98972	9·34276	10·65724	35
26	9·29716	9·99130	9·30587	10·69413	9·33305	9·98969	9·34336	10·65664	34
27	9·29779	9·99127	9·30652	10·69348	9·33362	9·98967	9·34396	10·65604	33
28	9·29841	9·99124	9·30717	10·69283	9·33420	9·98964	9·34456	10·65544	32
29	9·29903	9·99122	9·30782	10·69218	9·33477	9·98961	9·34516	10·65484	31
30	9·29966	9·99119	9·30846	10·69154	9·33534	9·98958	9·34576	10·65424	30
31	9·30028	9·99117	9·30911	10·69089	9·33591	9·98955	9·34635	10·65365	29
32	9·30090	9·99114	9·30975	10·69025	9·33647	9·98953	9·34695	10·65305	28
33	9·30151	9·99112	9·31040	10·68960	9·33704	9·98950	9·34755	10·65245	27
34	9·30213	9·99109	9·31104	10·68896	9·33761	9·98947	9·34814	10·65186	26
35	9·30275	9·99106	9·31168	10·68832	9·33818	9·98944	9·34874	10·65126	25
36	9·30336	9·99104	9·31233	10·68767	9·33874	9·98941	9·34933	10·65067	24
37	9·30398	9·99101	9·31297	10·68703	9·33931	9·98938	9·34992	10·65008	23
38	9·30459	9·99099	9·31361	10·68639	9·33987	9·98936	9·35051	10·64949	22
39	9·30521	9·99096	9·31425	10·68575	9·34043	9·98933	9·35111	10·64889	21
40	9·30582	9·99093	9·31489	10·68511	9·34100	9·98930	9·35170	10·64830	20
41	9·30643	9·99091	9·31552	10·68448	9·34156	9·98927	9·35229	10·64771	19
42	9·30704	9·99088	9·31616	10·68384	9·34212	9·98924	9·35288	10·64712	18
43	9·30765	9·99086	9·31679	10·68321	9·34268	9·98921	9·35347	10·64653	17
44	9·30826	9·99083	9·31743	10·68257	9·34324	9·98919	9·35405	10·64595	16
45	9·30887	9·99080	9·31806	10·68194	9·34380	9·98916	9·35464	10·64536	15
46	9·30947	9·99078	9·31870	10·68130	9·34436	9·98913	9·35523	10·64477	14
47	9·31008	9·99075	9·31933	10·68067	9·34491	9·98910	9·35581	10·64419	13
48	9·31068	9·99072	9·31996	10·68004	9·34547	9·98907	9·35640	10·64360	12
49	9·31129	9·99070	9·32059	10·67941	9·34602	9·98904	9·35698	10·64302	11
50	9·31189	9·99067	9·32122	10·67878	9·34658	9·98901	9·35757	10·64243	10
51	9·31250	9·99064	9·32185	10·67815	9·34713	9·98898	9·35815	10·64185	9
52	9·31310	9·99062	9·32248	10·67752	9·34769	9·98896	9·35873	10·64127	8
53	9·31370	9·99059	9·32311	10·67689	9·34824	9·98893	9·35931	10·64069	7
54	9·31430	9·99056	9·32373	10·67627	9·34879	9·98890	9·35989	10·64011	6
55	9·31490	9·99054	9·32436	10·67564	9·34934	9·98887	9·36047	10·63953	5
56	9·31549	9·99051	9·32498	10·67502	9·34989	9·98884	9·36105	10·63895	4
57	9·31609	9·99048	9·32561	10·67439	9·35044	9·98881	9·36163	10·63837	3
58	9·31669	9·99046	9·32623	10·67377	9·35099	9·98878	9·36221	10·63779	2
59	9·31728	9·99043	9·32685	10·67315	9·35154	9·98875	9·36279	10·63721	1
60	9·31788	9·99040	9·32747	10·67253	9·35209	9·98872	9·36336	10·63664	0
′	Cosine	Sine	Cotang.	Tang.	Cosine	Sine	Cotang.	Tang.	′
	78°				77°				

	13°				14°				
′	Sine	Cosine	Tang.	Cotang	Sine	Cosine	Tang.	Cotang	′
0	9·35209	9·98872	9·36336	10·63664	9·38368	9·98690	9·39677	10·60323	60
1	9·35263	9·98869	9·36394	10·63606	9·38418	9·98687	9·39731	10·60269	59
2	9·35318	9·98867	9·36452	10·63548	9·38469	9·98684	9·39785	10·60215	58
3	9·35373	9·98864	9·36509	10·63491	9·38519	9·98681	9·39838	10·60162	57
4	9·35427	9·98861	9·36566	10·63434	9·38570	9·98678	9·39892	10·60108	56
5	9·35481	9·98858	9·36624	10·63376	9·38620	9·98675	9·39945	10·60055	55
6	9·35536	9·98855	9·36681	10·63319	9·38670	9·98671	9·39999	10·60001	54
7	9·35590	9·98852	9·36738	10·63262	9·38721	9·98668	9·40052	10·59948	53
8	9·35644	9·98849	9·36795	10·63205	9·38771	9·98665	9·40106	10·59894	52
9	9·35698	9·98846	9·36852	10·63148	9·38821	9·98662	9·40159	10·59841	51
10	9·35752	9·98843	9·36909	10·63091	9·38871	9·98659	9·40212	10·59788	50
11	9·35806	9·98840	9·36966	10·63034	9·38921	9·98656	9·40266	10·59734	49
12	9·35860	9·98837	9·37023	10·62977	9·38971	9·98652	9·40319	10·59681	48
13	9·35914	9·98834	9·37080	10·62920	9·39021	9·98649	9·40372	10·59628	47
14	9·35968	9·98831	9·37137	10·62863	9·39071	9·98646	9·40425	10·59575	46
15	9·36022	9·98828	9·37193	10·62807	9·39121	9·98643	9·40478	10·59522	45
16	9·36075	9·98825	9·37250	10·62750	9·39170	9·98640	9·40531	10·59469	44
17	9·36129	9·98822	9·37306	10·62694	9·39220	9·98636	9·40584	10·59416	43
18	9·36182	9·98819	9·37363	10·62637	9·39270	9·98633	9·40636	10·59364	42
19	9·36236	9·98816	9·37419	10·62581	9·39319	9·98630	9·40689	10·59311	41
20	9·36289	9·98813	9·37476	10·62524	9·39369	9·98627	9·40742	10·59258	40
21	9·36342	9·98810	9·37532	10·62468	9·39418	9·98623	9·40795	10·59205	39
22	9·36395	9·98807	9·37588	10·62412	9·39467	9·98620	9·40847	10·59153	38
23	9·36449	9·98804	9·37644	10·62356	9·39517	9·98617	9·40900	10·59100	37
24	9·36502	9·98801	9·37700	10·62300	9·39566	9·98614	9·40952	10·59048	36
25	9·36555	9·98798	9·37756	10·62244	9·39615	9·98610	9·41005	10·58995	35
26	9·36608	9·98795	9·37812	10·62188	9·39664	9·98607	9·41057	10·58943	34
27	9·36660	9·98792	9·37868	10·62132	9·39713	9·98604	9·41109	10·58891	33
28	9·36713	9·98789	9·37924	10·62076	9·39762	9·98601	9·41161	10·58839	32
29	9·36766	9·98786	9·37980	10·62020	9·39811	9·98597	9·41214	10·58786	31
30	9·36819	9·98783	9·38035	10·61965	9·39860	9·98594	9·41266	10·58734	30
31	9·36871	9·98780	9·38091	10·61909	9·39909	9·98591	9·41318	10·58682	29
32	9·36924	9·98777	9·38147	10·61853	9·39958	9·98588	9·41370	10·58630	28
33	9·36976	9·98774	9·38202	10·61798	9·40006	9·98584	9·41422	10·58578	27
34	9·37028	9·98771	9·38257	10·61743	9·40055	9·98581	9·41474	10·58526	26
35	9·37081	9·98768	9·38313	10·61687	9·40103	9·98578	9·41526	10·58474	25
36	9·37133	9·98765	9·38368	10·61632	9·40152	9·98574	9·41578	10·58422	24
37	9·37185	9·98762	9·38423	10·61577	9·40200	9·98571	9·41629	10·58371	23
38	9·37237	9·98759	9·38479	10·61521	9·40249	9·98568	9·41681	10·58319	22
39	9·37289	9·98756	9·38534	10·61466	9·40297	9·98565	9·41733	10·58267	21
40	9·37341	9.98753	9·38589	10·61411	9·40346	9·98561	9·41784	10·58216	20
41	9·37393	9.98750	9·38644	10·61356	9·40394	9·98558	9·41836	10·58164	19
42	9·37445	9.98746	9·38699	10·61301	9·40442	9·98555	9·41887	10·58113	18
43	9·37497	9.98743	9·38754	10·61246	9·40490	9·98551	9·41939	10·58061	17
44	9·37549	9.98740	9·38808	10·61192	9·40538	9·98548	9·41990	10·58010	16
45	9·37600	9.98737	9·38863	10·61137	9·40586	9·98545	9·42041	10·57959	15
46	9·37652	9.98734	9·38918	10·61082	9·40634	9·98541	9·42093	10·57907	14
47	9·37703	9.98731	9·38972	10·61028	9·40682	9·98538	9·42144	10·57856	13
48	9·37755	9.98728	9·39027	10·60973	9·40730	9·98535	9·42195	10·57805	12
49	9·37806	9.98725	9·39082	10·60918	9·40778	9·98531	9·42246	10·57754	11
50	9·37858	9·98722	9·39136	10·60864	9·40825	9·98528	9·42297	10·57703	10
51	9·37909	9·98719	9·39190	10·60810	9·40873	9·98525	9·42348	10·57652	9
52	9·37960	9·98715	9·39245	10·60755	9·40921	9·98521	9·42399	10·57601	8
53	9·38011	9·98712	9·39299	10·60701	9·40968	9·98518	9·42450	10·57550	7
54	9·38062	9·98709	9·39353	10·60647	9·41016	9·98515	9·42501	10·57499	6
55	9·38113	9·98706	9·39407	10·60593	9·41063	9·98511	9·42552	10·57448	5
56	9·38164	9·98703	9·39461	10·60539	9·41111	9·98508	9·42603	10·57397	4
57	9·38215	9·98700	9·39515	10·60485	9·41158	9·98505	9·42653	10·57347	3
58	9·38266	9·98697	9·39569	10·60431	9·41205	9·98501	9·42704	10·57296	2
59	9·38317	9·98694	9·39623	10·60377	9·41252	9·98498	9·42755	10·57245	1
60	9·38368	9·98690	9·39677	10·60323	9·41300	9·98494	9·42805	10·57195	0
′	Cosine	Sine	Cotang	Tang.	Cosine	Sine	Cotang	Tang.	′
	76°				75°				

	15°				16°				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	′
0	9·41300	9·98494	9·42805	10·57195	9·44034	9·98284	9·45750	10·54250	60
1	9·41347	9·98491	9·42856	10·57144	9·44078	9·98281	9·45797	10·54203	59
2	9·41394	9·98488	9·42906	10·57094	9·44122	9·98277	9·45845	10·54155	58
3	9·41441	9·98484	9·42957	10·57043	9·44166	9·98273	9·45892	10·54108	57
4	9·41488	9·98481	9·43007	10·56993	9·44210	9·98270	9·45940	10·54060	56
5	9·41535	9·98477	9·43057	10·56943	9·44253	9·98266	9·45987	10·54013	55
6	9·41582	9·98474	9·43108	10·56892	9·44297	9·98262	9·46035	10·53965	54
7	9·41628	9·98471	9·43158	10·56842	9·44341	9·98259	9·46082	10·53918	53
8	9·41675	9·98467	9·43208	10·56792	9·44385	9·98255	9·46130	10·53870	52
9	9·41722	9·98464	9·43258	10·56742	9·44428	9·98251	9·46177	10·53823	51
10	9·41768	9·98460	9·43308	10·56692	9·44472	9·98248	9·46224	10·53776	50
11	9·41815	9·98457	9·43358	10·56642	9·44516	9·98244	9·46271	10·53729	49
12	9·41861	9·98453	9·43408	10·56592	9·44559	9·98240	9·46319	10·53681	48
13	9·41908	9·98450	9·43458	10·56542	9·44602	9·98237	9·46366	10·53634	47
14	9·41954	9·98447	9·43508	10·56492	9·44646	9·98233	9·46413	10·53587	46
15	9·42001	9·98443	9·43558	10·56442	9·44689	9·98229	9·46460	10·53540	45
16	9·42047	9·98440	9·43607	10·56393	9·44733	9·98226	9·46507	10·53493	44
17	9·42093	9·98436	9·43657	10·56343	9·44776	9·98222	9·46554	10·53446	43
18	9·42140	9·98433	9·43707	10·56293	9·44819	9·98218	9·46601	10·53399	42
19	9·42186	9·98429	9·43756	10·56244	9·44862	9·98215	9·46648	10·53352	41
20	9·42232	9·98426	9·43806	10·56194	9·44905	9·98211	9·46694	10·53306	40
21	9·42278	9·98422	9·43855	10·56145	9·44948	9·98207	9·46741	10·53259	39
22	9·42324	9·98419	9·43905	10·56095	9·44992	9·98204	9·46788	10·53212	38
23	9·42370	9·98415	9·43954	10·56046	9·45035	9·98200	9·46835	10·53165	37
24	9·42416	9·98412	9·44004	10·55996	9·45077	9·98196	9·46881	10·53119	36
25	9·42461	9·98409	9·44053	10·55947	9·45120	9·98192	9·46928	10·53072	35
26	9·42507	9·98405	9·44102	10·55898	9·45163	9·98189	9·46975	10·53025	34
27	9·42553	9·98402	9·44151	10·55849	9·45206	9·98185	9·47021	10·52979	33
28	9·42599	9·98398	9·44201	10·55799	9·45249	9·98181	9·47068	10·52932	32
29	9·42644	9·98395	9·44250	10·55750	9·45292	9·98177	9·47114	10·52886	31
30	9·42690	9·98391	9·44299	10·55701	9·45334	9·98174	9·47160	10·52840	30
31	9·42735	9·98388	9·44348	10·55652	9·45377	9·98170	9·47207	10·52793	29
32	9·42781	9·98384	9·44397	10·55603	9·45419	9·98166	9·47253	10·52747	28
33	9·42826	9·98381	9·44446	10·55554	9·45462	9·98162	9·47299	10·52701	27
34	9·42872	9·98377	9·44495	10·55505	9·45504	9·98159	9·47346	10·52654	26
35	9·42917	9·98373	9·44544	10·55456	9·45547	9·98155	9·47392	10·52608	25
36	9·42962	9·98370	9·44592	10·55408	9·45589	9·98151	9·47438	10·52562	24
37	9·43008	9·98366	9·44641	10·55359	9·45632	9·98147	9·47484	10·52516	23
38	9·43053	9·98363	9·44690	10·55310	5·45674	9·98144	9·47530	10·52470	22
39	9·43098	9·98359	9·44738	10·55262	9·45716	9·98140	9·47576	10·52424	21
40	9·43143	9·98356	9·44787	10·55213	9·45758	9·98136	9·47622	10·52378	20
41	9·43188	9·98352	9·44836	10·55164	9·45801	9·98132	9·47668	10·52332	19
42	9·43233	9·98349	9·44884	10·55116	9·45843	9·98129	9·47714	10·52286	18
43	9·43278	9·98345	9·44933	10·55067	9·45885	9·98125	9·47760	10·52240	17
44	9·43323	9·98342	9·44981	10·55019	9·45927	9·98121	9·47806	10·52194	16
45	9·43367	9·98338	9·45029	10·54971	9·45969	9·98117	9·47852	10·52148	15
46	9·43412	9·98334	9·45078	10·54922	9·46011	9·98113	9·47897	10·52103	14
47	9·43457	9·98331	9·45126	10·54874	9·46053	9·98110	9·47943	10·52057	13
48	9·43502	9·98327	9·45174	10·54826	9·46095	9·98106	9·47989	10·52011	12
49	9·43546	9·98324	9·45222	10·54778	9·46136	9·98102	9·48035	10·51965	11
50	9·43591	9·98320	9·45271	10·54729	9·46178	9·98098	9·48080	10·51920	10
51	9·43635	9·98317	9·45319	10·54681	9·46220	9·98094	9·48126	10·51874	9
52	9·43680	9·98313	9·45367	10·54633	9·46262	9·98090	9·48171	10·51829	8
53	9·43724	9·98309	9·45415	10·54585	9·46303	9·98087	9·48217	10·51783	7
54	9·43769	9·98306	9·45463	10·54537	9·46345	9·98083	9·48262	10·51738	6
55	9·43813	9·98302	9·45511	10·54489	9·46386	9·98079	9·48307	10·51693	5
56	9·43857	9·98299	9·45559	10·54441	9·46428	9·98075	9·48353	10·51647	4
57	9·43901	9·98295	9·45606	10·54394	9·46469	9·98071	9·48398	10·51602	3
58	9·43946	9·98291	9·45654	10·54346	9·46511	9·98067	9·48443	10·51557	2
59	9·43990	9·98288	9·45702	10·54298	9·46552	9·98063	9·48489	10·51511	1
60	9·44034	9·98284	9·45750	10·54250	9·46594	9·98060	9·48534	10·51466	0
′	Cosine	Sine	Cotang.	Tang.	Cosine	Sine	Cotang.	Tang.	′
	74°				73°				

	17°				18°				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	′
0	9·46594	9·98060	9·48534	10·51466	9·48998	9·97821	9·51178	10·48822	60
1	9·46635	9·98056	9·48579	10·51421	9·49037	9·97817	9·51221	10·48779	59
2	9·46676	9·98052	9·48624	10·51376	9·49076	9·97812	9·51264	10·48736	58
3	9·46717	9·98048	9·48669	10·51331	9·49115	9·97808	9·51306	10·48694	57
4	9·46758	9·98044	9·48714	10·51286	9·49153	9·97804	9·51349	10·48651	56
5	9·46800	9·98040	9·48759	10·51241	9·49192	9·97800	9·51392	10·48608	55
6	9·46841	9·98036	9·48804	10·51196	9·49231	9·97796	9·51435	10·48565	54
7	9·46882	9·98032	9·48849	10·51151	9·49269	9·97792	9·51478	10·48522	53
8	9·46923	9·98029	9·48894	10·51106	9·49308	9·97788	9·51520	10·48480	52
9	9·46964	9·98025	9·48939	10·51061	9·49347	9·97784	9·51563	10·48437	51
10	9·47005	9·98021	9·48984	10·51016	9·49385	9·97779	9·51606	10·48394	50
11	9·47045	9·98017	9·49029	10·50971	9·49424	9·97775	9·51648	10·48352	49
12	9·47086	9·98013	9·49073	10·50927	9·49462	9·97771	9·51691	10·48309	48
13	9·47127	9·98009	9·49118	10·50882	9·49500	9·97767	9·51734	10·48266	47
14	9·47168	9·98005	9·49163	10·50837	9·49539	9·97763	9·51776	10·48224	46
15	9·47209	9·98001	9·49207	10·50793	9·49577	9·97759	9·51819	10·48181	45
16	9·47249	9·97997	9·49252	10·50748	9·49615	9·97754	9·51861	10·48139	44
17	9·47290	9·97993	9·49296	10·50704	9·49654	9·97750	9·51903	10·48097	43
18	9·47330	9·97989	9·49341	10·50659	9·49692	9·97746	9·51946	10·48054	42
19	9·47371	9·97986	9·49385	10·50615	9·49730	9·97742	9·51988	10·48012	41
20	9·47411	9·97982	9·49430	10·50570	9·49768	9·97738	9·52031	10·47969	40
21	9·47452	9·97978	9·49474	10·50526	9·49806	9·97734	9·52073	10·47927	39
22	9·47492	9·97974	9·49519	10·50481	9·49844	9·97729	9·52115	10·47885	38
23	9·47533	9·97970	9·49563	10·50437	9·49882	9·97725	9·52157	10·47843	37
24	9·47573	9·97966	9·49607	10·50393	9·49920	9·97721	9·52200	10·47800	36
25	9·47613	9·97962	9·49652	10·50348	9·49958	9·97717	9·52242	10·47758	35
26	9·47654	9·97958	9·49696	10·50304	9·49996	9·97713	9·52284	10·47716	34
27	9·47694	9·97954	9·49740	10·50260	9·50034	9·97708	9·52326	10·47674	33
28	9·47734	9·97950	9·49784	10·50216	9·50072	9·97704	9·52368	10·47632	32
29	9·47774	9·97946	9·49828	10·50172	9·50110	9·97700	9·52410	10·47590	31
30	9·47814	9·97942	9·49872	10·50128	9·50148	9·97696	9·52452	10·47548	30
31	9·47854	9·97938	9·49916	10·50084	9·50185	9·97691	9·52494	10·47506	29
32	9·47894	9·97934	9·49960	10·50040	9·50223	9·97687	9·52536	10·47464	28
33	9·47934	9·97930	9·50004	10·49996	9·50261	9·97683	9·52578	10·47422	27
34	9·47974	9·97926	9·50048	10·49952	9·50298	9·97679	9·52620	10·47380	26
35	9·48014	9·97922	9·50092	10·49908	9·50336	9·97674	9·52661	10·47339	25
36	9·48054	9·97918	9·50136	10·49864	9·50374	9·97670	9·52703	10·47297	24
37	9·48094	9·97914	9·50180	10·49820	9·50411	9·97666	9·52745	10·47255	23
38	9·48133	9·97910	9·50223	10·49777	9·50449	9·97662	9·52787	10·47213	22
39	9·48173	9·97906	9·50267	10·49733	9·50486	9·97657	9·52829	10·47171	21
40	9·48213	9·97902	9·50311	10·49689	9·50523	9·97653	9·52870	10·47130	20
41	9·48252	9·97898	9·50355	10·49645	9·50561	9·97649	9·52912	10·47088	19
42	9·48292	9·97894	9·50398	10·49602	9·50598	9·97645	9·52953	10·47047	18
43	9·48332	9·97890	9·50442	10·49558	9·50635	9·97640	9·52995	10·47005	17
44	9·48371	9·97886	9·50485	10·49515	9·50673	9·97636	9·53037	10·46963	16
45	9·48411	9·97882	9·50529	10·49471	9·50710	9·97632	9·53078	10·46922	15
46	9·48450	9·97878	9·50572	10·49428	9·50747	9·97628	9·53120	10·46880	14
47	9·48490	9·97874	9·50616	10·49384	9·50784	9·97623	9·53161	10·46839	13
48	9·48529	9·97870	9·50659	10·49341	9·50821	9·97619	9·53202	10·46798	12
49	9·48568	9·97866	9·50703	10·49297	9·50858	9·97615	9·53244	10·46756	11
50	9·48607	9·97861	9·50746	10·49254	9·50896	9·97610	9·53285	10·46715	10
51	9·48647	9·97857	9·50789	10·49211	9·50933	9·97606	9·53327	10·46673	9
52	9·48686	9·97853	9·50833	10·49167	9·50970	9·97602	9·53368	10·46632	8
53	9·48725	9·97849	9·50876	10·49124	9·51007	9·97597	9·53409	10·46591	7
54	9·48764	9·97845	9·50919	10·49081	9·51043	9·97593	9·53450	10·46550	6
55	9·48803	9·97841	9·50962	10·49038	9·51080	9·97589	9·53492	10·46508	5
56	9·48842	9·97837	9·51005	10·48995	9·51117	9·97584	9·53533	10·46467	4
57	9·48881	9·97833	9·51048	10·48952	9·51154	9·97580	9·53574	10·46426	3
58	9·48920	9·97829	9·51092	10·48908	9·51191	9·97576	9·53615	10·46385	2
59	9·48959	9·97825	9·51135	10·48865	9·51227	9·97571	9·53656	10·46344	1
60	9·48998	9·97821	9·51178	10·48822	9·51264	9·97567	9·53697	10·46303	0
′	Cosine	Sine	Cotang.	Tang.	Cosine	Sine	Cotang.	Tang.	′
	72°				71°				

	19°				20°				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	′
0	9·51264	9·97567	9·53697	10·46303	9·53405	9·97299	9·56107	10·43893	60
1	9·51301	9·97563	9·53738	10·46262	9·53440	9·97294	9·56146	10·43854	59
2	9·51338	9·97558	9·53779	10·46221	9·53475	9·97289	9·56185	10·43815	58
3	9·51374	9·97554	9·53820	10·46180	9·53509	9·97285	9·56224	10·43776	57
4	9·51411	9·97550	9·53861	10·46139	9·53544	9·97280	9·56264	10·43736	56
5	9·51447	9·97545	9·53902	10·46098	9·53578	9·97276	9·56303	10·43697	55
6	9·51484	9·97541	9·53943	10·46057	9·53613	9·97271	9·56342	10·43658	54
7	9·51520	9·97536	9·53984	10·46016	9·53647	9·97266	9·56381	10·43619	53
8	9·51557	9·97532	9·54025	10·45975	9·53682	9·97262	9·56420	10·43580	52
9	9·51593	9·97528	9·54065	10·45935	9·53716	9·97257	9·56459	10·43541	51
10	9·51629	9·97523	9·54106	10·45894	9·53751	9·97252	9·56498	10·43502	50
11	9·51666	9·97519	9·54147	10·45853	9·53785	9·97248	9·56537	10·43463	49
12	9·51702	9·97515	9·54187	10·45813	9·53819	9·97243	9·56576	10·43424	48
13	9·51738	9·97510	9·54228	10·45772	9·53854	9·97238	9·56615	10·43385	47
14	9·51774	9·97506	9·54269	10·45731	9·53888	9·97234	9·56654	10·43346	46
15	9·51811	9·97501	9·54309	10·45691	9·53922	9·97229	9·56693	10·43307	45
16	9·51847	9·97497	9·54350	10·45650	9·53957	9·97224	9·56732	10·43268	44
17	9·51883	9·97492	9·54390	10·45610	9·53991	9·97220	9·56771	10·43229	43
18	9·51919	9·97488	9·54431	10·45569	9·54025	9·97215	9·56810	10·43190	42
19	9·51955	9·97484	9·54471	10·45529	9·54059	9·97210	9·56849	10·43151	41
20	9·51991	9·97479	9·54512	10·45488	9·54093	9·97206	9·56887	10·43113	40
21	9·52027	9·97475	9·54552	10·45448	9·54127	9·97201	9·56926	10·43074	39
22	9·52063	9·97470	9·54593	10·45407	9·54161	9·97196	9·56965	10·43035	38
23	9·52099	9·97466	9·54633	10·45367	9·54195	9·97192	9·57004	10·42996	37
24	9·52135	9·97461	9·54673	10·45327	9·54229	9·97187	9·57042	10·42958	36
25	9·52171	9·97457	9·54714	10·45286	9·54263	9 97182	9·57081	10·42919	35
26	9·52207	9·97453	9·54754	10·45246	9·54297	9·97178	9·57120	10·42880	34
27	9·52242	9·97448	9·54794	10·45206	9·54331	9·97173	9·57158	10·42842	33
28	9·52278	9·97444	9·54835	10·45165	9·54365	9·97168	9·57197	10·42803	32
29	9·52314	9·97439	9·54875	10·45125	9·54399	9·97163	9·57235	10·42765	31
30	9·52350	9·97435	9·54915	10·45085	9·54433	9·97159	9·57274	10·42726	30
31	9·52385	9·97430	9·54955	10·45045	9·54466	9·97154	9·57312	10·42688	29
32	9·52421	9·97426	9·54995	10·45005	9·54500	9·97149	9·57351	10·42649	28
33	9·52456	9·97421	9·55035	10·44965	9·54534	9·97145	9·57389	10·42611	27
34	9·52492	9·97417	9·55075	10·44925	9·54567	9·97140	9·57428	10·42572	26
35	9·52527	9·97412	9·55115	10·44885	9·54601	9·97135	9·57466	10 42534	25
36	9·52563	9·97408	9·55155	10·44845	9·54635	9·97130	9·57504	10·42496	24
37	9·52598	9·97403	9·55195	10·44805	9·54668	9·97126	9·57543	10·42457	23
38	9·52634	9·97399	9·55235	10·44765	9·54702	9·97121	9·57581	10·42419	22
39	9·52669	9·97394	9·55275	10·44725	9·54735	9·97116	9·57619	10·42381	21
40	9·52705	9·97390	9·55315	10·44685	9·54769	9·97111	9·57658	10·42342	20
41	9·52740	9·97385	9·55355	10·44645	9·54802	9·97107	9·57696	10·42304	19
42	9·52775	9·97381	9·55395	10·44605	9·54836	9·97102	9·57734	10·42266	18
43	9·52811	9·97376	9·55434	10·44566	9·54869	9·97097	9·57772	10·42228	17
44	9·52846	9·97372	9·55474	10·44526	9·54903	9·97092	9·57810	10·42190	16
45	9·52881	9·97367	9·55514	10·44486	9·54936	9·97087	9·57849	10·42151	15
46	9·52916	9·97363	9·55554	10·44446	9·54969	9·97083	9·57887	10·42113	14
47	9·52951	9·97358	9·55593	10·44407	9·55003	9·97078	9·57925	10·42075	13
48	9·52986	9·97353	9·55633	10·44367	9·55036	9·97073	9·57963	10·42037	12
49	9·53021	9·97349	9·55673	10·44327	9·55069	9·97068	9·58001	10·41999	11
50	9·53056	9·97344	9·55712	10·44288	9·55102	9·97063	9·58039	10·41961	10
51	9·53092	9·97340	9·55752	10·44248	9·55136	9·97059	9·58077	10·41923	9
52	9·53126	9·97335	9·55791	10·44209	9·55169	9·97054	9·58115	10·41885	8
53	9·53161	9·97331	9·55831	10·44169	9·55202	9·97049	9·58153	10·41847	7
54	9·53196	9·97326	9·55870	10·44130	9·55235	9·97044	9·58191	10·41809	6
55	9·53231	9·97322	9·55910	10·44090	9·55268	9·97039	9·58229	10·41771	5
56	9·53266	9·97317	9·55949	10·44051	9·55301	9·97035	9·58267	10·41733	4
57	9·53301	9·97312	9·55989	10·44011	9·55334	9·97030	9·58304	10·41696	3
58	9·53336	9·97308	9·56028	10·43972	9·55367	9·97025	9·58342	10·41658	2
59	9·53370	9·97303	9·56067	10·43933	9·55400	9·97020	9·58380	10·41620	1
60	9·53405	9·97299	9·56107	10·43893	9·55433	9·97015	9·58418	10·41582	0
′	Cosine	Sine	Cotang.	Tang.	Cosine	Sine	Cotang.	Tang.	′
	70°				69°				

	21°				22°				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	′
0	9·55433	9·97015	9·58418	10·41582	9·57358	9·96717	9·60641	10·39359	60
1	9·55466	9·97010	9·58455	10·41545	9·57389	9·96711	9·60677	10·39323	59
2	9·55499	9·97005	9·58493	10·41507	9·57420	9·96706	9·60714	10·39286	58
3	9·55532	9·97001	9·58531	10·41469	9·57451	9·96701	9·60750	10·39250	57
4	9·55564	9·96996	9·58569	10·41431	9·57482	9·96696	9·60786	10·39214	56
5	9·55597	9·96991	9·58606	10·41394	9·57514	9·96691	9·60823	10·39177	55
6	9·55630	9·96986	9·58644	10·41356	9·57545	9·96686	9·60859	10·39141	54
7	9·55663	9·96981	9·58681	10·41319	9·57576	9·96681	9·60895	10·39105	53
8	9·55695	9·96976	9·58719	10·41281	9·57607	9·96676	9·60931	10·39069	52
9	9·55728	9·96971	9·58757	10·41243	9·57638	9·96670	9·60967	10·39033	51
10	9·55761	9·96966	9·58794	10·41206	9·57669	9·96665	9·61004	10·38996	50
11	9·55793	9·96962	9·58832	10·41168	9·57700	9·96660	9·61040	10·38960	49
12	9·55826	9·96957	9·58869	10·41131	9·57731	9·96655	9·61076	10·38924	48
13	9·55858	9·96952	9·58907	10·41093	9·57762	9·96650	9·61112	10·38888	47
14	9·55891	9·96947	9·58944	10·41056	9·57793	9·96645	9·61148	10·38852	46
15	9·55923	9·96942	9·58981	10·41019	9·57824	9·96640	9·61184	10·38816	45
16	9·55956	9·96937	9·59019	10·40981	9·57855	9·96634	9·61220	10·38780	44
17	9·55988	9·96932	9·59056	10·40944	9·57885	9·96629	9·61256	10·38744	43
18	9·56021	9·96927	9·59094	10·40906	9·57916	9·96624	9·61292	10·38708	42
19	9·56053	9·96922	9·59131	10·40869	9·57947	9·96619	9·61328	10·38672	41
20	9·56085	9·96917	9·59168	10·40832	9·57978	9·96614	9·61364	10·38636	40
21	9·56118	9·96912	9·59205	10·40795	9·58008	9·96608	9·61400	10·38600	39
22	9·56150	9·96907	9·59243	10·40757	9·58039	9·96603	9·61436	10·38564	38
23	9·56182	9·96903	9·59280	10·40720	9·58070	9·96598	9·61472	10·38528	37
24	9·56215	9·96898	9·59317	10·40683	9·58101	9·96593	9·61508	10·38492	36
25	9·56247	9·96893	9·59354	10·40646	9·58131	9·96588	9·61544	10·38456	35
26	9·56279	9·96888	9·59391	10·40609	9·58162	9·96582	9·61579	10·38421	34
27	9·56311	9·96883	9·59429	10·40571	9·58192	9·96577	9·61615	10·38385	33
28	9·56343	9·96878	9·59466	10·40534	9·58223	9·96572	9·61651	10·38349	32
29	9·56375	9·96873	9·59503	10·40497	9·58253	9·96567	9·61687	10·38313	31
30	9·56408	9·96868	9·59540	10·40460	9·58284	9·96562	9·61722	10·38278	30
31	9·56440	9·96863	9·59577	10·40423	9·58314	9·96556	9·61758	10·38242	29
32	9·56472	9·96858	9·59614	10·40386	9·58345	9·96551	9 61794	10·38206	28
33	9·56504	9·96853	9·59651	10·40349	9·58375	9·96546	9·61830	10·38170	27
34	9·56536	9·96848	9·59688	10·40312	9·58406	9·96541	9·61865	10·38135	26
35	9·56568	9·96843	9·59725	10·40275	9·58436	9·96535	9·61901	10·38099	25
36	9·56599	9·96838	9·59762	10·40238	9.58467	9·96530	9·61936	10·38064	24
37	9·56631	9·96833	9·59799	10·40201	9·58497	9·96525	9·61972	10·38028	23
38	9·56663	9·96828	9·59835	10·40165	9·58527	9·96520	9·62008	10·37992	22
39	9·56695	9·96823	9·59872	10·40128	9·58557	9·96514	9·62043	10·37957	21
40	9·56727	9·96818	9·59909	10·40091	9·58588	9·96509	9·62079	10·37921	20
41	9·56759	9·96813	9·59946	10·40054	9·58618	9·96504	9·62114	10·37886	19
42	9·56790	9·96808	9·59983	10·40017	9·58648	9·96498	9·62150	10·37850	18
43	9·56822	9·96803	9·60019	10·39981	9·58678	9·96493	9·62185	10·37815	17
44	9·56854	9·96798	9·60056	10·39944	9·58709	9·96488	9·62221	10·37779	16
45	9·56886	9·96793	9·60093	10·39907	9·58739	9·96483	9·62256	10·37744	15
46	9·56917	9·96788	9·60130	10·39870	9·58769	9·96477	9·62292	10·37708	14
47	9·56949	9·96783	9·60166	10·39834	9·58799	9·96472	9·62327	10·37673	13
48	9·56980	9·96778	9·60203	10·39797	9·58829	9·96467	9·62362	10·37638	12
49	9·57012	9·96772	9·60240	10·39760	9·58859	9·96461	9·62398	10·37602	11
50	9·57044	9·96767	9·60276	10·39724	9·58889	9·96456	9·62433	10·37567	10
51	9·57075	9·96762	9·60313	10·39687	9·58919	9·96451	9·62468	10·37532	9
52	9·57107	9·96757	9·60349	10·39651	9·58949	9·96445	9·62504	10·37496	8
53	9·57138	9·96752	9·60386	10·39614	9·58979	9·96440	9·62539	10·37461	7
54	9·57169	9·96747	9·60422	10·39578	9·59009	9·96435	9·62574	10·37426	6
55	9·57201	9·96742	9·60459	10·39541	9·59039	9·96429	9·62609	10·37391	5
56	9·57232	9·96737	9·60495	10·39505	9·59069	9·96424	9·62645	10·37355	4
57	9·57264	9·96732	9·60532	10·39468	9·59098	9·96419	9·62680	10·37320	3
58	9·57295	9·96727	9·60568	10·39432	9·59128	9·96413	9·62715	10·37285	2
59	9·57326	9·96722	9·60605	10·39395	9·59158	9·96408	9·62750	10·37250	1
60	9·57358	9·96717	9·60641	10·39359	9·59188	9·96403	9·62785	10·37215	0
′	Cosine	Sine	Cotang.	Tang.	Cosine	Sine	Cotang.	Tang.	′
	68°				67°				

	23°				24°				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	′
0	9·59188	9·96403	9·62785	10·37215	9·60931	9·96073	9·64858	10·35142	60
1	9·59218	9·96397	9·62820	10·37180	9·60960	9·96067	9·64892	10·35108	59
2	9·59247	9·96392	9·62855	10·37145	9·60988	9·96062	9·64926	10·35074	58
3	9·59277	9·96387	9·62890	10·37110	9·61016	9·96056	9·64960	10·35040	57
4	9·59307	9·96381	9·62926	10·37074	9·61045	9·96050	9·64994	10·35006	56
5	9·59336	9·96376	9·62961	10·37039	9·61073	9·96045	9·65028	10·34972	55
6	9·59366	9·96370	9·62996	10·37004	9·61101	9·96039	9·65062	10·34938	54
7	9·59396	9·96365	9·63031	10·36969	9·61129	9·96034	9·65096	10·34904	53
8	9·59425	9·96360	9·63066	10·36934	9·61158	9·96028	9·65130	10·34870	52
9	9·59455	9·96354	9·63101	10·36899	9·61186	9·96022	9·65164	10·34836	51
10	9·59484	9·96349	9·63135	10·36865	9·61214	9·96017	9·65197	10·34803	50
11	9·59514	9·96343	9·63170	10·36830	9·61242	9·96011	9·65231	10·34769	49
12	9·59543	9·96338	9·63205	10·36795	9·61270	9·96005	9·65265	10·34735	48
13	9·59573	9·96333	9·63240	10·36760	9·61298	9·96000	9·65299	10·34701	47
14	9·59602	9·96327	9·63275	10·36725	9·61326	9·95994	9·65333	10·34667	46
15	9·59632	9·96322	9·63310	10·36690	9·61354	9·95988	9·65366	10·34634	45
16	9·59661	9·96316	9·63345	10·36655	9·61382	9·95982	9·65400	10·34600	44
17	9·59690	9·96311	9·63379	10·36621	9·61411	9·95977	9·65434	10·34566	43
18	9·59720	9·96305	9·63414	10·36586	9·61438	9·95971	9·65467	10·34533	42
19	9·59749	9·96300	9·63449	10·36551	9·61466	9·95965	9·65501	10·34499	41
20	9·59778	9·96294	9·63484	10·36516	9·61494	9·95960	9·65535	10·34465	40
21	9·59808	9·96289	9·63519	10·36481	9·61522	9·95954	9·65568	10·34432	39
22	9·59837	9·96284	9·63553	10·36447	9·61550	9·95948	9·65602	10·34398	38
23	9·59866	9·96278	9·63588	10·36412	9·61578	9·95942	9·65636	10·34364	37
24	9·59895	9·96273	9·63623	10·36377	9·61606	9·95937	9·65669	10·34331	36
25	9·59924	9·96267	9·63657	10·36343	9·61634	9·95931	9·65703	10·34297	35
26	9·59954	9·96262	9·63692	10·36308	9·61662	9·95925	9·65736	10·34264	34
27	9·59983	9·96256	9·63726	10·36274	9·61689	9·95920	9·65770	10·34230	33
28	9·60012	9·96251	9·63761	10·36239	9·61717	9·95914	9·65803	10·34197	32
29	9·60041	9·96245	9·63796	10·36204	9·61745	9·95908	9·65837	10·34163	31
30	9·60070	9·96240	9·63830	10·36170	9·61773	9·95902	9·65870	10·34130	30
31	9·60099	9·96234	9·63865	10·36135	9·61800	9·95897	9·65904	10·34096	29
32	9·60128	9·96229	9·63899	10·36101	9·61828	9·95891	9·65937	10·34063	28
33	9·60157	9·96223	9·63934	10·36066	9·61856	9·95885	9·65971	10·34029	27
34	9·60186	9·96218	9·63968	10·36032	9·61883	9·95879	9·66004	10·33996	26
35	9·60215	9·96212	9·64003	10·35997	9·61911	9·95873	9·66038	10·33962	25
36	9·60244	9·96207	9·64037	10·35963	9·61939	9·95868	9·66071	10·33929	24
37	9·60273	9·96201	9·64072	10·35928	9·61966	9·95862	9·66104	10·33896	23
38	9·60302	9·96196	9·64106	10·35894	9·61994	9·95856	9·66138	10·33862	22
39	9·60331	9·96190	9·64140	10·35860	9·62021	9·95850	9·66171	10·33829	21
40	9·60359	9·96185	9·64175	10·35825	9·62049	9·95844	9·66204	10·33796	20
41	9·60388	9·96179	9·64209	10·35791	9·62076	9·95839	9·66238	10·33762	19
42	9·60417	9·96174	9·64243	10·35757	9·62104	9·95833	9·66271	10·33729	18
43	9·60446	9·96168	9·64278	10·35722	9·62131	9·95827	9·66304	10·33696	17
44	9·60474	9·96162	9·64312	10·35688	9·62159	9·95821	9·66337	10·33663	16
45	9·60503	9·96157	9·64346	10·35654	9·62186	9·95815	9·66371	10·33629	15
46	9·60532	9·96151	9·64381	10·35619	9·62214	9·95810	9·66404	10·33596	14
47	9·60561	9·96146	9·64415	10·35585	9·62241	9·95804	9·66437	10·33563	13
48	9·60589	9·96140	9·64449	10·35551	9·62268	9·95798	9·66470	10·33530	12
49	9·60618	9·96135	9·64483	10·35517	9·62296	9·95792	9·66503	10·33497	11
50	9·60646	9·96129	9·64517	10·35483	9·62323	9·95786	9·66537	10·33463	10
51	9·60675	9·96123	9·64552	10·35448	9·62350	9·95780	9·66570	10·33430	9
52	9·60704	9·96118	9·64586	10·35414	9·62377	9·95775	9·66603	10·33397	8
53	9·60732	9·96112	9·64620	10·35380	9·62405	9·95769	9·66636	10·33364	7
54	9·60761	9·96107	9·64654	10·35346	9·62432	9·95763	9·66669	10·33331	6
55	9·60789	9·96101	9·64688	10·35312	9·62459	9·95757	9·66702	10·33298	5
56	9·60818	9·96095	9·64722	10·35278	9·62486	9·95751	9·66735	10·33265	4
57	9·60846	9·96090	9·64756	10·35244	9·62513	9·95745	9·66768	10·33232	3
58	9·60875	9·96084	9·64790	10·35210	9·62541	9·95739	9·66801	10·33199	2
59	9·60903	9·96079	9·64824	10·35176	9·62568	9·95733	9·66834	10·33166	1
60	9·60931	9·96073	9·64858	10·35142	9·62595	9·95728	9·66867	10·33133	0
′	Cosine	Sine	Cotang.	Tang.	Cosine	Sine	Cotang.	Tang.	′
	66°				65°				

	25°				26°				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	′
0	9·62595	9·95728	9·66867	10·33133	9·64184	9·95366	9·68818	10·31182	60
1	9·62622	9·95722	9·66900	10·33100	9·64210	9·95360	9·68850	10·31150	59
2	9·62649	9·95716	9·66933	10·33067	9·64236	9·95354	9·68882	10·31118	58
3	9·62676	9·95710	9·66966	10·33034	9·64262	9·95348	9·68914	10·31086	57
4	9·62703	9·95704	9·66999	10·33001	9·64288	9·95341	9·68946	10·31054	56
5	9·62730	9·95698	9·67032	10·32968	9·64313	9·95335	9·68978	10·31022	55
6	9·62757	9·95692	9·67065	10·32935	9·64339	9·95329	9·69010	10·30990	54
7	9·62784	9·95686	9·67098	10·32902	9·64365	9·95323	9·69042	10·30958	53
8	9·62811	9·95680	9·67131	10·32869	9·64391	9·95317	9·69074	10·30926	52
9	9·62838	9·95674	9·67163	10·32837	9·64417	9·95310	9·69106	10·30894	51
10	9·62865	9·95668	9·67196	10·32804	9·64442	9·95304	9·69138	10·30862	50
11	9·62892	9·95663	9·67229	10·32771	9·64468	9·95298	9·69170	10·30830	49
12	9·62918	9·95657	9·67262	10·32738	9·64494	9·95292	9·69202	10·30798	48
13	9·62945	9·95651	9·67295	10·32705	9·64519	9·95286	9·69234	10·30766	47
14	9·62972	9·95645	9·67327	10·32673	9·64545	9·95279	9·69266	10·30734	46
15	9·62999	9·95639	9·67360	10·32640	9·64571	9·95273	9·69298	10·30703	45
16	9·63026	9·95633	9·67393	10·32607	9·64596	9·95267	9·69329	10·30671	44
17	9·63052	9·95627	9·67426	10·32574	9·64622	9·95261	9·69361	10·30639	43
18	9·63079	9·95621	9·67458	10·32542	9·64647	9·95254	9·69393	10·30607	42
19	9·63106	9·95615	9·67491	10·32509	9·64673	9·95248	9·69425	10·30575	41
20	9·63133	9·95609	9·67524	10·32476	9·64698	9·95242	9·69457	10·30543	40
21	9·63159	9·95603	9·67556	10·32444	9·64724	9·95236	9·69488	10·30512	39
22	9·63186	9·95597	9·67589	10·32411	9·64749	9·95229	9·69520	10·30480	38
23	9·63213	9·95591	9·67622	10·32378	9·64775	9·95223	9·69552	10·30448	37
24	9·63239	9·95585	9·67654	10·32346	9·64800	9·95217	9·69584	10·30416	36
25	9·63266	9·95579	9·67687	10·32313	9·64826	9·95211	9·69615	10·30385	35
26	9·63292	9·95573	9·67719	10·32281	9·64851	9·95204	9·69647	10·30353	34
27	9·63319	9·95567	9·67752	10·32248	9·64877	9·95198	9·69679	10·30321	33
28	9·63345	9·95561	9·67785	10·32215	9·64902	9·95192	9·69710	10·30290	32
29	9·63372	9·95555	9·67817	10·32183	9·64927	9·95185	9·69742	10·30258	31
30	9·63398	9·95549	9·67850	10·32150	9·64953	9·95179	9·69774	10·30226	30
31	9·63425	9·95543	9·67882	10·32118	9·64978	9·95173	9·69805	10·30195	29
32	9·63451	9·95537	9·67915	10·32085	9·65003	9·95167	9·69837	10·30163	28
33	9·63478	9·95531	9·67947	10·32053	9·65029	9·95160	9·69868	10·30132	27
34	9·63504	9·95525	9·67980	10·32020	9·65054	9·95154	9·69900	10·30100	26
35	9·63531	9·95519	9·68012	10·31988	9·65079	9·95148	9·69932	10·30068	25
36	9·63557	9·95513	9·68044	10·31956	9·65104	9·95141	9·69963	10·30037	24
37	9·63583	9·95507	9·68077	10·31923	9·65130	9·95135	9·69995	10·30005	23
38	9·63610	9·95500	9·68109	10·31891	9·65155	9·95129	9·70026	10·29974	22
39	9·63636	9·95494	9·68142	10·31858	9·65180	9·95122	9·70058	10·29942	21
40	9·63662	9·95488	9·68174	10·31826	9·65205	9·95116	9·70089	10·29911	20
41	9·63689	9·95482	9·68206	10·31794	9·65230	9·95110	9·70121	10·29879	19
42	9·63715	9·95476	9·68239	10·31761	9·65255	9·95103	9·70152	10·29848	18
43	9·63741	9·95470	9·68271	10·31729	9·65281	9·95097	9·70184	10·29816	17
44	9·63767	9·95464	9·68303	10·31697	9·65306	9·95090	9·70215	10·29785	16
45	9·63794	9·95458	9·68336	10·31664	9·65331	9·95084	9·70247	10·29753	15
46	9·63820	9·95452	9·68368	10·31632	9·65356	9·95078	9·70278	10·29722	14
47	9·63846	9·95446	9·68400	10·31600	9·65381	9·95071	9·70309	10·29691	13
48	9·63872	9·95440	9·68432	10·31568	9·65406	9·95065	9·70341	10·29659	12
49	9·63898	9·95434	9·68465	10·31535	9·65431	9·95059	9·70372	10·29628	11
50	9·63924	9·95427	9·68497	10·31503	9·65456	9·95052	9·70404	10·29596	10
51	9·63950	9·95421	9·68529	10·31471	9·65481	9·95046	9·70435	10·29565	9
52	9·63976	9·95415	9·68561	10·31439	9·65506	9·95039	9·70466	10·29534	8
53	9·64002	9·95409	9·68593	10·31407	9·65531	9·95033	9·70498	10·29502	7
54	9·64028	9·95403	9·68626	10·31374	9·65556	9·95027	9·70529	10·29471	6
55	9·64054	9·95397	9·68658	10·31342	9·65580	9·95020	9·70560	10·29440	5
56	9·64080	9·95391	9·68690	10·31310	9·65605	9·95014	9·70592	10·29408	4
57	9·64106	9·95384	9·68722	10·31278	9·65630	9·95007	9·70623	10·29377	3
58	9·64132	9·95378	9·68754	10·31246	9·65655	9·95001	9·70654	10·29346	2
59	9·64158	9·95372	9·68786	10·31214	9·65680	9·94995	9·70685	10·29315	1
60	9·64184	9·95366	9·68818	10·31182	9·65705	9·94988	9·70717	10·29283	0
	Cosine	Sine	Cotang.	Tang.	Cosine	Sine	Cotang.	Tang.	′
	64°				63°				

	27°				28°				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	′
0	9·65705	9·94988	9·70717	10·29283	9·67161	9·94593	9·72567	10·27433	60
1	9·65729	9·94982	9·70748	10·29252	9·67185	9·94587	9·72598	10·27402	59
2	9·65754	9·94975	9·70779	10·29221	9·67208	9·94580	9·72628	10·27372	58
3	9·65779	9·94969	9·70810	10·29190	9·67232	9·94573	9·72659	10·27341	57
4	9·65804	9·94962	9·70841	10·29159	9·67256	9·94567	9·72689	10·27311	56
5	9·65828	9·94956	9·70873	10·29127	9·67280	9·94560	9·72720	10·27280	55
6	9·65853	9·94949	9·70904	10·29096	9·67303	9·94553	9·72750	10·27250	54
7	9·65878	9·94943	9·70935	10·29065	9·67327	9·94546	9·72780	10·27220	53
8	9·65902	9·94936	9·70966	10·29034	9·67350	9·94540	9·72811	10·27189	52
9	9·65927	9·94930	9·70997	10·29003	9·67374	9·94533	9·72841	10·27159	51
10	9·65952	9·94923	9·71028	10·28972	9·67398	9·94526	9·72872	10·27128	50
11	9·65976	9·94917	9·71059	10·28941	9·67421	9·94519	9·72902	10·27098	49
12	9·66001	9·94911	9·71090	10·28910	9·67445	9·94513	9·72932	10·27068	48
13	9·66025	9·94904	9·71121	10·28879	9·67468	9·94506	9·72963	10·27037	47
14	9·66050	9·94898	9·71153	10·28847	9·67492	9·94499	9·72993	10·27007	46
15	9·66075	9·94891	9·71184	10·28816	9·67515	9·94492	9·73023	10·26977	45
16	9·66099	9·94885	9·71215	10·28785	9·67539	9·94485	9·73054	10·26946	44
17	9·66124	9·94878	9·71246	10·28754	9·67562	9·94479	9·73084	10·26916	43
18	9·66148	9·94871	9·71277	10·28723	9·67586	9·94472	9·73114	10·26886	42
19	9·66173	9·94865	9·71308	10·28692	9·67609	9·94465	9·73144	10·26856	41
20	9·66197	9·94858	9·71339	10·28661	9·67633	9·94458	9·73175	10·26825	40
21	9·66221	9·94852	9·71370	10·28630	9·67656	9·94451	9·73205	10·26795	39
22	9·66246	9·94845	9·71401	10·28599	9·67680	9·94445	9·73235	10·26765	38
23	9·66270	9·94839	9·71431	10·28569	9·67703	9·94438	9·73265	10·26735	37
24	9·66295	9·94832	9·71462	10·28538	9·67726	9·94431	9·73295	10·26705	36
25	9·66319	9·94826	9·71493	10·28507	9·67750	9·94424	9·73326	10·26674	35
26	9·66343	9·94819	9·71524	10·28476	9·67773	9·94417	9·73356	10·26644	34
27	9·66368	9·94813	9·71555	10·28445	9·67796	9·94410	9·73386	10·26614	33
28	9·66392	9·94806	9·71586	10·28414	9·67820	9·94404	9·73416	10·26584	32
29	9·66416	9·94799	9·71617	10·28383	9·67843	9·94397	9·73446	10·26554	31
30	9·66441	9·94793	9·71648	10·28352	9·67866	9·94390	9·73476	10·26524	30
31	9·66465	9·94786	9·71679	10·28321	9·67890	9·94383	9·73507	10·26493	29
32	9·66489	9·94780	9·71709	10·28291	9·67913	9·94376	9·73537	10·26463	28
33	9·66513	9·94773	9·71740	10·28260	9·67936	9·94369	9·73567	10·26433	27
34	9·66537	9·94767	9·71771	10·28229	9·67959	9·94362	9·73597	10·26403	26
35	9·66562	9·94760	9·71802	10·28198	9·67982	9·94355	9·73627	10·26373	25
36	9·66586	9·94753	9·71833	10·28167	9·68006	9·94349	9·73657	10·26343	24
37	9·66610	9·94747	9·71863	10·28137	9·68029	9·94342	9·73687	10·26313	23
38	9·66634	9·94740	9·71894	10·28106	9·68052	9·94335	9·73717	10·26283	22
39	9·66658	9·94734	9·71925	10·28075	9·68075	9·94328	9·73747	10·26253	21
40	9·66682	9·94727	9·71955	10·28045	9·68098	9·94321	9·73777	10·26223	20
41	9·66706	9·94720	9·71986	10·28014	9·68121	9·94314	9·73807	10·26193	19
42	9·66731	9·94714	9·72017	10·27983	9·68144	9·94307	9·73837	10·26163	18
43	9·66755	9·94707	9·72048	10·27952	9·68167	9·94300	9·73867	10·26133	17
44	9·66779	9·94700	9·72078	10·27922	9·68190	9·94293	9·73897	10·26103	16
45	9·66803	9·94694	9·72109	10·27891	9·68213	9·94286	9·73927	10·26073	15
46	9·66827	9·94687	9·72140	10·27860	9·68237	9·94279	9·73957	10·26043	14
47	9·66851	9·94680	9·72170	10·27830	9·68260	9·94273	9·73987	10·26013	13
48	9·66875	9·94674	9·72201	10·27799	9·68283	9·94266	9·74017	10·25983	12
49	9·66899	9·94667	9·72231	10·27769	9·68305	9·94259	9·74047	10·25953	11
50	9·66922	9·94660	9·72262	10·27738	9·68328	9·94252	9·74077	10·25923	10
51	9·66946	9·94654	9·72293	10·27707	9·68351	9·94245	9·74107	10·25893	9
52	9·66970	9·94647	9·72323	10·27677	9·68374	9·94238	9·74137	10·25863	8
53	9·66994	9·94640	9·72354	10·27646	9·68394	9·94231	9·74166	10·25834	7
54	9·67018	9·94634	9·72384	10·27616	9·68420	9·94224	9·74196	10·25804	6
55	9·67042	9·94627	9·72415	10·27585	9·68443	9·94217	9·74226	10·25774	5
56	9·67066	9·94620	9·72445	10·27555	9·68466	9·94210	9·74256	10·25744	4
57	9·67090	9·94614	9·72476	10·27524	9·68489	9·94203	9·74286	10·25714	3
58	9·67113	9·94607	9·72506	10·27494	9·68512	9·94196	9·74316	10·25684	2
59	9·67137	9·94600	9·72537	10·27463	9·68534	9·94189	9·74345	10·25655	1
60	9·67161	9·94593	9·72567	10·27433	9·68557	9·94182	9·74375	10·25625	0
′	Cosine	Sine	Cotang.	Tang.	Cosine	Sine	Cotang.	Tang.	′
	62°				61°				

	29°				30°				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	′
0	9·68557	9·94182	9·74375	10·25625	9·69897	9·93753	9·76144	10·23856	60
1	9·68580	9·94175	9·74405	10·25595	9·69919	9·93746	9·76173	10·23827	59
2	9·68603	9·94168	9·74435	10·25565	9·69941	9·93738	9·76202	10·23798	58
3	9·68625	9·94161	9·74465	10·25535	9·69963	9·93731	9·76231	10·23769	57
4	9·68648	9·94154	9·74494	10·25506	9·69984	9·93724	9·76261	10·23739	56
5	9·68671	9·94147	9·74524	10·25476	9·70006	9·93717	9·76290	10·23710	55
6	9·68694	9·94140	9·74554	10·25446	9·70028	9·93709	9·76319	10·23681	54
7	9·68716	9·94133	9·74583	10·25417	9·70050	9·93702	9·76348	10·23652	53
8	9·68739	9·94126	9·74613	10·25387	9·70072	9·93695	9·76377	10·23623	52
9	9·68762	9·94119	9·74643	10·25357	9·70093	9·93687	9·76406	10·23594	51
10	9·68784	9·94112	9·74673	10·25327	9·70115	9·93680	9·76435	10·23565	50
11	9·68807	9·94105	9·74702	10·25298	9·70137	9·93673	9·76464	10·23536	49
12	9·68829	9·94098	9·74732	10·25268	9·70159	9·93665	9·76493	10·23507	48
13	9·68852	9·94090	9·74762	10·25238	9·70180	9·93658	9·76522	10·23478	47
14	9·68875	9·94083	9·74791	10·25209	9·70202	9·93650	9·76551	10·23449	46
15	9·68897	9·94076	9·74821	10·25179	9·70224	9·93643	9·76580	10·23420	45
16	9·68920	9·94069	9·74851	10·25149	9·70245	9·93636	9·76609	10·23391	44
17	9·68942	9·94062	9·74880	10·25120	9·70267	9·93628	9·76639	10·23361	43
18	9·68965	9·94055	9·74910	10·25090	9·70288	9·93621	9·76668	10·23332	42
19	9·68987	9·94048	9·74939	10·25061	9·70310	9·93614	9·76697	10·23303	41
20	9·69010	9·94041	9·74969	10·25031	9·70332	9·93606	9·76725	10·23275	40
21	9·69032	9·94034	9·74998	10·25002	9·70353	9·93599	9·76754	10·23246	39
22	9·69055	9·94027	9·75028	10·24972	9·70375	9·93591	9·76783	10·23217	38
23	9·69077	9·94020	9·75058	10·24942	9·70396	9·93584	9·76812	10·23188	37
24	9·69100	9·94012	9·75087	10·24913	9·70418	9·93577	9·76841	10·23159	36
25	9·69122	9·94005	9·75117	10·24883	9·70439	9·93569	9·76870	10·23130	35
26	9·69144	9·93998	9·75146	10·24854	9·70461	9·93562	9·76899	10·23101	34
27	9·69167	9·93991	9·75176	10·24824	9·70482	9·93554	9·76928	10·23072	33
28	9·69189	9·93984	9·75205	10·24795	9·70504	9·93547	9·76957	10·23043	32
29	9·69212	9·93977	9·75235	10·24765	9·70525	9·93539	9·76986	10·23014	31
30	9·69234	9·93970	9·75264	10·24736	9·70547	9·93532	9·77015	10·22985	30
31	9·69256	9·93963	9·75294	10·24706	9·70568	9·93525	9·77044	10·22956	29
32	9·69279	9·93955	9·75323	10·24677	9·70590	9·93517	9·77073	10·22927	28
33	9·69301	9·93948	9·75353	10·24647	9·70611	9·93510	9·77101	10·22899	27
34	9·69323	9·93941	9·75382	10·24618	9·70633	9·93502	9·77130	10·22870	26
35	9·69345	9·93934	9·75411	10·24589	9·70654	9·93495	9·77159	10·22841	25
36	9·69368	9·93927	9·75441	10·24559	9·70675	9·93487	9·77188	10·22812	24
37	9 69390	9·93920	9·75470	10·24530	9·70697	9·93480	9·77217	10·22783	23
38	9·69412	9·93912	9·75500	10·24500	9·70718	9·93472	9·77246	10·22754	22
39	9·69434	9·93905	9·75529	10·24471	9·70739	9·93465	9·77274	10·22726	21
40	9·69456	9·93898	9·75558	10·24442	9·70761	9·93457	9·77303	10·22697	20
41	9·69479	9·93891	9·75588	10·24412	9·70782	9·93450	9·77332	10·22668	19
42	9·69501	9·93884	9·75617	10·24383	9·70803	9·93442	9·77361	10·22639	18
43	9·69523	9·93876	9·75647	10·24353	9·70824	9·93435	9·77390	10·22610	17
44	9·69545	9·93869	9·75676	10·24324	9·70846	9·93427	9·77418	10·22582	16
45	9·69567	9·93862	9·75705	10·24295	9·70867	9·93420	9·77447	10·22553	15
46	9·69589	9·93855	9·75735	10·24265	9·70888	9·93412	9·77476	10·22524	14
47	9·69611	9·93847	9·75764	10·24236	9·70909	9·93405	9·77505	10·22495	13
48	9·69633	9·93840	9·75793	10·24207	9·70931	9·93397	9·77533	10·22467	12
49	9·69655	9·93833	9·75822	10·24178	9·70952	9·93390	9·77562	10·22438	11
50	9·69677	9·93826	9·75852	10·24148	9·70973	9·93382	9·77591	10·22409	10
51	9·69699	9·93819	9·75881	10·24119	9·70994	9·93375	9·77619	10·22381	9
52	9·69721	9·93811	9·75910	10·24090	9·71015	9·93367	9·77648	10·22352	8
53	9·69743	9·93804	9·75939	10·24061	9·71036	9·93360	9·77677	10·22323	7
54	9·69765	9·93797	9·75969	10·24031	9·71058	9·93352	9·77706	10·22294	6
55	9·69787	9·93789	9·75998	10·24002	9·71079	9·93344	9·77734	10·22266	5
56	9·69809	9·93782	9·76027	10·23973	9·71100	9·93337	9·77763	10·22237	4
57	9·69831	9·93775	9·76056	10·23944	9·71121	9·93329	9·77791	10·22209	3
58	9·69853	9·93768	9·76086	10·23914	9·71142	9·93322	9·77820	10·22180	2
59	9·69875	9·93760	9·76115	10·23885	9·71163	9·93314	9·77849	10·22151	1
60	9·69897	9·93753	9·76144	10·23856	9·71184	9·93307	9·77877	10·22123	0
′	Cosine	Sine	Cotang.	Tang.	Cosine	Sine	Cotang.	Tang.	′
	60°				59°				

	31°				32°				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	
0	9·71184	9·93307	9·77877	10·22123	9·72421	9·92842	9·79579	10·20421	60
1	9·71205	9·93299	9·77906	10·22094	9·72441	9·92834	9·79607	10·20393	59
2	9·71226	9·93291	9·77935	10·22065	9·72461	9·92826	9·79635	10·20365	58
3	9·71247	9·93284	9·77963	10·22037	9·72482	9·92818	9·79663	10·20337	57
4	9·71268	9·93276	9·77992	10·22008	9·72502	9·92810	9·79691	10·20309	56
5	9·71289	9·93269	9·78020	10·21980	9·72522	9·92803	9·79719	10·20281	55
6	9·71310	9·93261	9·78049	10·21951	9·72542	9·92795	9·79747	10·20253	54
7	9·71331	9·93253	9·78077	10·21923	9·72562	9·92787	9·79776	10·20224	53
8	9·71352	9·93246	9·78106	10·21894	9·72582	9·92779	9·79804	10·20196	52
9	9·71373	9·93238	9·78135	10·21865	9·72602	9·92771	9·79832	10·20168	51
10	9·71393	9·93230	9·78163	10·21837	9·72622	9·92763	9·79860	10·20140	50
11	9·71414	9·93223	9·78192	10·21808	9·72643	9·92755	9·79888	10·20112	49
12	9·71435	9·93215	9·78220	10·21780	9·72663	9·92747	9·79916	10·20084	48
13	9·71456	9·93207	9·78249	10·21751	9·72683	9·92739	9·79944	10·20056	47
14	9·71477	9·93200	9·78277	10·21723	9·72703	9·92731	9·79972	10·20028	46
15	9·71498	9·93192	9·78306	10·21694	9·72723	9·92723	9·80000	10·20000	45
16	9·71519	9·93184	9·78334	10·21666	9·72743	9·92715	9·80028	10·19972	44
17	9·71539	9·93177	9·78363	10·21637	9·72763	9·92707	9·80056	10·19944	43
18	9·71560	9·93169	9·78391	10·21609	9·72783	9·92699	9·80084	10·19916	42
19	9·71581	9·93161	9·78419	10·21581	9·72803	9·92691	9·80112	10·19888	41
20	9·71602	9·93154	9·78448	10·21552	9·72823	9·92683	9·80140	10·19860	40
21	9·71622	9·93146	9·78476	10·21524	9·72843	9·92675	9·80168	10·19832	39
22	9·71643	9·93138	9·78505	10·21495	9·72863	9·92667	9·80195	10·19805	38
23	9·71664	9·93131	9·78533	10·21467	9·72883	9·92659	9·80223	10·19777	37
24	9·71685	9·93123	9·78562	10·21438	9·72902	9·92651	9·80251	10·19749	36
25	9·71705	9·93115	9·78590	10·21410	9·72922	9·92643	9·80279	10·19721	35
26	9·71726	9·93108	9·78618	10·21382	9·72942	9·92635	9·80307	10·19693	34
27	9·71747	9·93100	9·78647	10·21353	9·72962	9·92627	9·80335	10·19665	33
28	9·71767	9·93092	9·78675	10·21325	9·72982	9·92619	9·80363	10·19637	32
29	9·71788	9·93084	9·78704	10·21296	9·73002	9·92611	9·80391	10·19609	31
30	9·71809	9·93077	9·78732	10·21268	9·73022	9·92603	9·80419	10·19581	30
31	9·71829	9·93069	9·78760	10·21240	9·73041	9·92595	9·80447	10·19553	29
32	9·71850	9·93061	9·78789	10·21211	9·73061	9·92587	9·80474	10·19526	28
33	9·71870	9·93053	9·78817	10·21183	9·73081	9·92579	9·80502	10·19498	27
34	9·71891	9·93046	9·78845	10·21155	9·73101	9·92571	9·80530	10·19470	26
35	9·71911	9·93038	9·78874	10·21126	9·73121	9·92563	9·80558	10·19442	25
36	9·71932	9·93030	9·78902	10·21098	9·73140	9·92555	9·80586	10·19414	24
37	9·71952	9·93022	9·78930	10·21070	9·73160	9·92546	9·80614	10·19386	23
38	9·71973	9·93014	9·78959	10·21041	9·73180	9·92538	9·80642	10·19358	22
39	9·71994	9·93007	9·78987	10·21013	9·73200	9·92530	9·80669	10·19331	21
40	9·72014	9·92999	9·79015	10·20985	9·73219	9·92522	9·80697	10·19303	20
41	9·72034	9·92991	9·79043	10·20957	9·73239	9·92514	9·80725	10·19275	19
42	9·72055	9·92983	9·79072	10·20928	9·73259	9·92506	9·80753	10·19247	18
43	9·72075	9·92976	9·79100	10·20900	9·73278	9·92498	9·80781	10·19219	17
44	9·72096	9·92968	9·79128	10·20872	9·73298	9·92490	9·80808	10·19192	16
45	9·72116	9·92960	9·79156	10·20844	9·73318	9·92482	9·80836	10·19164	15
46	9·72137	9·92952	9·79185	10·20815	9·73337	9·92473	9·80864	10·19136	14
47	9·72157	9·92944	9·79213	10·20787	9·73357	9·92465	9·80892	10·19108	13
48	9·72177	9·92936	9·79241	10·20759	9·73377	9·92457	9·80919	10·19081	12
49	9·72198	9·92929	9·79269	10·20731	9·73396	9·92449	9·80947	10·19053	11
50	9·72218	9·92921	9·79297	10·20703	9·73416	9·92441	9·80975	10·19025	10
51	9·72238	9·92913	9·79326	10·20674	9·73435	9·92433	9·81003	10·18997	9
52	9·72259	9·92905	9·79354	10·20646	9·73455	9·92425	9·81030	10·18970	8
53	9·72279	9·92897	9·79382	10·20618	9·73474	9·92416	9·81058	10·18942	7
54	9·72299	9·92889	9·79410	10·20590	9·73494	9·92408	9·81086	10·18914	6
55	9·72320	9·92881	9·79438	10·20562	9·73513	9·92400	9·81113	10·18887	5
56	9·72340	9·92874	9·79466	10·20534	9·73533	9·92392	9·81141	10·18859	4
57	9·72360	9·92866	9·79495	10·20505	9·73552	9·92384	9·81169	10·18831	3
58	9·72381	9·92858	9·79523	10·20477	9·73572	9·92376	9·81196	10·18804	2
59	9·72401	9·92850	9·79551	10·20449	9·73591	9·92367	9·81224	10·18776	1
60	9·72421	9·92842	9·79579	10·20421	9·73611	9·92359	9·81252	10·18748	0
′	Cosine	Sine	Cotang.	Tang.	Cosine	Sine	Cotang.	Tang.	′
	58°				57°				

	33°				34°				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	′
0	9·73611	9·92359	9·81252	10·18748	9·74756	9·91857	9·82899	10·17101	60
1	9·73630	9·92351	9·81279	10·18721	9·74775	9·91849	9·82926	10·17074	59
2	9·73650	9·92343	9·81307	10·18693	9·74794	9·91840	9·82953	10·17047	58
3	9·73669	9·92334	9·81335	10·18665	9·74812	9·91832	9·82980	10·17020	57
4	9·73689	9·92326	9·81362	10·18638	9·74831	9·91823	9·83008	10·16992	56
5	9·73708	9·92318	9·81390	10·18610	9·74850	9·91815	9·83035	10·16965	55
6	9·73727	9·92310	9·81418	10·18582	9·74868	9·91806	9·83062	10·16938	54
7	9·73747	9·92302	9·81445	10·18555	9·74887	9·91798	9·83089	10·16911	53
8	9·73766	9·92293	9·81473	10·18527	9·74906	9·91789	9·83117	10·16883	52
9	9·73785	9·92285	9·81500	10·18500	9·74924	9·91781	9·83144	10·16856	51
10	9·73805	9·92277	9·81528	10·18472	9·74943	9·91772	9·83171	10·16829	50
11	9·73824	9·92269	9·81556	10·18444	9·74961	9·91763	9·83198	10·16802	49
12	9·73843	9·92260	9·81583	10·18417	9·74980	9·91755	9·83225	10·16775	48
13	9·73863	9·92252	9·81611	10·18389	9·74999	9·91746	9·83252	10·16748	47
14	9·73882	9·92244	9·81638	10·18362	9·75017	9·91738	9·83280	10·16720	46
15	9·73901	9·92235	9·81666	10·18334	9·75036	9·91729	9·83307	10·16693	45
16	9·73921	9·92227	9·81693	10·18307	9·75054	9·91720	9·83334	10·16666	44
17	9·73940	9·92219	9·81721	10·18279	9·75073	9·91712	9·83361	10·16639	43
18	9·73959	9·92211	9·81748	10·18252	9·75091	9·91703	9·83388	10·16612	42
19	9·73978	9·92202	9·81776	10·18224	9·75110	9·91695	9·83415	10·16585	41
20	9·73997	9·92194	9·81803	10·18197	9·75128	9·91686	9·83442	10·16558	40
21	9·74017	9·92186	9·81831	10·18169	9·75147	9·91677	9·83470	10·16530	39
22	9·74036	9·92177	9·81858	10·18142	9·75165	9·91669	9·83497	10·16503	38
23	9·74055	9·92169	9·81886	10·18114	9·75184	9·91660	9·83524	10·16476	37
24	9·74074	9·92161	9·81913	10·18087	9·75202	9·91651	9·83551	10·16449	36
25	9·74093	9·92152	9·81941	10·18059	9·75221	9·91643	9·83578	10·16422	35
26	9·74113	9·92144	9·81968	10·18032	9·75239	9·91634	9·83605	10·16395	34
27	9·74132	9·92136	9·81996	10·18004	9·75258	9·91625	9·83632	10·16368	33
28	9·74151	9·92127	9·82023	10·17977	9·75276	9·91617	9·83659	10·16341	32
29	9·74170	9·92119	9·82051	10·17949	9·75294	9·91608	9·83686	10·16314	31
30	9·74189	9·92111	9·82078	10·17922	9·75313	9·91599	9·83713	10·16287	30
31	9·74208	9·92102	9·82106	10·17894	9·75331	9·91591	9·83740	10·16260	29
32	9·74227	9·92094	9·82133	10·17867	9·75350	9·91582	9·83768	10·16232	28
33	9·74246	9·92086	9·82161	10·17839	9·75368	9·91573	9·83795	10·16205	27
34	9·74265	9·92077	9·82188	10·17812	9·75386	9·91565	9·83822	10·16178	26
35	9·74284	9·92069	9·82215	10·17785	9·75405	9·91556	9·83849	10·16151	25
36	9·74303	9·92060	9·82243	10·17757	9·75423	9·91547	9·83876	10·16124	24
37	9·74322	9·92052	9·82270	10·17730	9·75441	9·91538	9·83903	10·16097	23
38	9·74341	9·92044	9·82298	10·17702	9·75459	9·91530	9·83930	10·16070	22
39	9·74360	9·92035	9·82325	10·17675	9·75478	9·91521	9·83957	10·16043	21
40	9·74379	9·92027	9·82352	10·17648	9·75496	9·91512	9·83984	10·16016	20
41	9·74398	9·92018	9·82380	10·17620	9·75514	9·91504	9·84011	10·15989	19
42	9·74417	9·92010	9·82407	10·17593	9·75533	9·91495	9·84038	10·15962	18
43	9·74436	9·92002	9·82435	10·17565	9·75551	9·91486	9·84065	10·15935	17
44	9·74455	9·91993	9·82462	10·17538	9·75569	9·91477	9·84092	10·15908	16
45	9·74474	9·91985	9·82489	10·17511	9·75587	9·91469	9·84119	10·15881	15
46	9·74493	9·91976	9·82517	10·17483	9·75605	9·91460	9·84146	10·15854	14
47	9·74512	9·91968	9·82544	10·17456	9·75624	9·91451	9·84173	10·15827	13
48	9·74531	9·91959	9·82571	10·17429	9·75642	9·91442	9·84200	10·15800	12
49	9·74549	9·91951	9·82599	10·17401	9·75660	9·91433	9·84227	10·15773	11
50	9·74568	9·91942	9·82626	10·17374	9·75678	9·91425	9·84254	10·15746	10
51	9·74587	9·91934	9·82653	10·17347	9·75696	9·91416	9·84280	10·15720	9
52	9·74606	9·91925	9·82681	10·17319	9·75714	9·91407	9·84307	10·15693	8
53	9·74625	9·91917	9·82708	10·17292	9·75733	9·91398	9·84334	10·15666	7
54	9·74644	9·91908	9·82735	10·17265	9·75751	9·91389	9·84361	10·15639	6
55	9·74662	9·91900	9·82762	10·17238	9·75769	9·91381	9·84388	10·15612	5
56	9·74681	9·91891	9·82790	10·17210	9·75787	9·91372	9·84415	10·15585	4
57	9·74700	9·91883	9·82817	10·17183	9·75805	9·91363	9·84442	10·15558	3
58	9·74719	9·91874	9·82844	10·17156	9·75823	9·91354	9·84469	10·15531	2
59	9·74737	9·91866	9·82871	10·17129	9·75841	9·91345	9·84496	10·15504	1
60	9·74756	9·91857	9·82899	10·17101	9·75859	9·91336	9·84523	10·15477	0
′	Cosine	Sine	Cotang.	Tang.	Cosine	Sine	Cotang.	Tang.	′
	56°				55°				

	35°				36°				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	′
0	9·75859	9·91336	9·84523	10·15477	9·76922	9·90796	9·86126	10·13874	60
1	9·75877	9·91328	9·84550	10·15450	9·76939	9·90787	9·86153	10·13847	59
2	9·75895	9·91319	9·84576	10·15424	9·76957	9·90777	9·86179	10·13821	58
3	9·75913	9·91310	9·84603	10·15397	9·76974	9·90768	9·86206	10·13794	57
4	9·75931	9·91301	9·84630	10·15370	9·76991	9·90759	9·86232	10·13768	56
5	9·75949	9·91292	9·84657	10·15343	9·77009	9·90750	9·86259	10·13741	55
6	9·75967	9·91283	9·84684	10·15316	9·77026	9·90741	9·86285	10·13715	54
7	9·75985	9·91274	9·84711	10·15289	9·77043	9·90731	9·86312	10·13688	53
8	9·76003	9·91266	9·84738	10·15262	9·77061	9·90722	9·86338	10·13662	52
9	9·76021	9·91257	9·84764	10·15236	9·77078	9·90713	9·86365	10·13635	51
10	9·76039	9·91248	9·84791	10·15209	9·77095	9·90704	9·86392	10·13608	50
11	9·76057	9·91239	9·84818	10·15182	9·77112	9·90694	9·86418	10·13582	49
12	9·76075	9·91230	9·84845	10·15155	9·77130	9·90685	9·86445	10·13555	48
13	9·76093	9·91221	9·84872	10·15128	9·77147	9·90676	9·86471	10·13529	47
14	9·76111	9·91212	9·84899	10·15101	9·77164	9·90667	9·86498	10·13502	46
15	9 76129	9·91203	9·84925	10·15075	9·77181	9·90657	9·86524	10·13476	45
16	9·76146	9·91194	9·84952	10·15048	9·77199	9·90648	9·86551	10·13449	44
17	9·76164	9·91185	9·84979	10·15021	9·77216	9·90639	9·86577	10·13423	43
18	9·76182	9·91176	9·85006	10·14994	9·77233	9·90630	9·86603	10·13397	42
19	9·76200	9·91167	9·85033	10·14967	9·77250	9·90620	9·86630	10·13370	41
20	9·76218	9·91158	9·85059	10·14941	9·77268	9·90611	9·86656	10·13344	40
21	9·76236	9·91149	9·85086	10·14914	9·77285	9·90602	9·86683	10·13317	39
22	9·76253	9·91141	9·85113	10·14887	9·77302	9·90592	9·86709	10·13291	38
23	9·76271	9·91132	9·85140	10·14860	9·77319	9·90583	9·86736	10·13264	37
24	9·76289	9·91123	9·85166	10·14834	9·77336	9·90574	9·86762	10·13238	36
25	9·76307	9·91114	9·85193	10·14807	9·77353	9·90565	9·86789	10·13211	35
26	9·76324	9·91105	9·85220	10·14780	9·77370	9·90555	9·86815	10·13185	34
27	9·76342	9·91096	9·85247	10·14753	9·77387	9·90546	9·86842	10·13158	33
28	9·76360	9·91087	9·85273	10·14727	9·77405	9·90537	9·86868	10·13132	32
29	9·76378	9·91078	9·85300	10·14700	9·77422	9·90527	9·86894	10·13106	31
30	9·76395	9·91069	9·85327	10·14673	9·77439	9·90518	9·86921	10·13079	30
31	9·76413	9·91060	9·85354	10·14646	9·77456	9·90509	9·86947	10·13053	29
32	9·76431	9·91051	9·85380	10·14620	9·77473	9·90499	9·86974	10·13026	28
33	9·76448	9·91042	9·85407	10·14593	9·77490	9·90490	9·87000	10·13000	27
34	9·76466	9·91033	9·85434	10·14566	9·77507	9·90480	9·87027	10·12973	26
35	9·76484	9·91023	9·85460	10·14540	9·77524	9·90471	9·87053	10·12947	25
36	9·76501	9·91014	9·85487	10·14513	9·77541	9·90462	9·87079	10·12921	24
37	9·76519	9·91005	9·85514	10·14486	9·77558	9·90452	9·87106	10·12894	23
38	9·76537	9·90996	9·85540	10·14460	9·77575	9·90443	9·87132	10·12868	22
39	9·76554	9·90987	9·85567	10·14433	9·77592	9·90434	9·87158	10·12842	21
40	9·76572	9·90978	9·85594	10·14406	9·77609	9·90424	9·87185	10·12815	20
41	9·76590	9·90969	9·85620	10·14380	9·77626	9·90415	9·87211	10·12789	19
42	9·76607	9·90960	9·85647	10·14353	9·77643	9·90405	9·87238	10·12762	18
43	9·76625	9·90951	9·85674	10·14326	9·77660	9·90396	9·87264	10·12736	17
44	9·76642	9·90942	9·85700	10·14300	9·77677	9·90386	9·87290	10·12710	16
45	9·76660	9·90933	9·85727	10·14273	9·77694	9·90377	9·87317	10·12683	15
46	9·76677	9·90924	9·85754	10·14246	9·77711	9·90368	9·87343	10·12657	14
47	9·76695	9·90915	9·85780	10·14220	9·77728	9·90358	9·87369	10·12631	13
48	9·76712	9·90905	9·85807	10·14193	9·77744	9·90349	9·87396	10·12604	12
49	9·76730	9·90896	9·85834	10·14166	9·77761	9·90339	9·87422	10·12578	11
50	9·76747	9·90887	9·85860	10·14140	9·77778	9·90330	9·87448	10·12552	10
51	9·76765	9·90878	9·85887	10·14113	9·77795	9·90320	9·87475	10·12525	9
52	9·76782	9·90869	9·85913	10·14087	9·77812	9·90311	9·87501	10·12499	8
53	9·76800	9·90860	9·85940	10·14060	9·77829	9·90301	9·87527	10·12473	7
54	9·76817	9·90851	9·85967	10·14033	9·77846	9·90292	9·87554	10·12446	6
55	9·76835	9·90842	9·85993	10·14007	9·77862	9·90282	9·87580	10·12420	5
56	9·76852	9·90832	9·86020	10·13980	9·77879	9·90273	9·87606	10·12394	4
57	9·76870	9·90823	9·86046	10·13954	9·77896	9·90263	9·87633	10·12367	3
58	9·76887	9·90814	9·86073	10·13927	9·77913	9·90254	9·87659	10·12341	2
59	9·76904	9·90805	9·86100	10·13900	9·77930	9·90244	9·87685	10·12315	1
60	9·76922	9·90796	9·86126	10·13874	9·77946	9·90235	9·87711	10·12289	0
′	Cosine	Sine	Cotang.	Tang.	Cosine	Sine	Cotang.	Tang.	′
	54°				53°				

	37°				38°				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	′
0	9·77946	9·90235	9·87711	10·12289	9·78934	9·89653	9·89281	10·10719	60
1	9·77963	9·90225	9·87738	10·12262	9·78950	9·89643	9·89307	10·10693	59
2	9·77980	9·90216	9·87764	10·12236	9·78967	9·89633	9·89333	10·10667	58
3	9·77997	9·90206	9·87790	10·12210	9·78983	9·89624	9·89359	10·10641	57
4	9·78013	9·90197	9·87817	10·12183	9·78999	9·89614	9·89385	10·10615	56
5	9·78030	9·90187	9·87843	10·12157	9·79015	9·89604	9·89411	10·10589	55
6	9·78047	9·90178	9·87869	10·12131	9·79031	9·89594	9·89437	10·10563	54
7	9·78063	9·90168	9·87895	10·12105	9·79047	9·89584	9·89463	10·10537	53
8	9·78080	9·90159	9·87922	10·12078	9·79063	9·89574	9·89489	10·10511	52
9	9·78097	9·90149	9·87948	10·12052	9·79079	9·89564	9·89515	10·10485	51
10	9·78113	9·90139	9·87974	10·12026	9·79095	9·89554	9·89541	10·10459	50
11	9·78130	9·90130	9·88000	10·12000	9·79111	9·89544	9·89567	10·10433	49
12	9·78147	9·90120	9·88027	10·11973	9·79128	9·89534	9·89593	10·10407	48
13	9·78163	9·90111	9·88053	10·11947	9·79144	9·89524	9·89619	10·10381	47
14	9·78180	9·90101	9·88079	10·11921	9·79160	9·89514	9·89645	10·10355	46
15	9·78197	9·90091	9·88105	10·11895	9·79176	9·89504	9·89671	10·10329	45
16	9·78213	9·90082	9·88131	10·11869	9·79192	9·89495	9·89697	10·10303	44
17	9·78230	9·90072	9·88158	10·11842	9·79208	9·89485	9·89723	10·10277	43
18	9·78246	9·90063	9·88184	10·11816	9·79224	9·89475	9·89749	10·10251	42
19	9·78263	9·90053	9·88210	10·11790	9·79240	9·89465	9·89775	10·10225	41
20	9·78280	9·90043	9·88236	10·11764	9·79256	9·89455	9·89801	10·10199	40
21	9·78296	9·90034	9·88262	10·11738	9·79272	9·89445	9·89827	10·10173	39
22	9·78313	9·90024	9·88289	10·11711	9·79288	9·89435	9·89853	10·10147	38
23	9·78329	9·90014	9·88315	10·11685	9·79304	9·89425	9·89879	10·10121	37
24	9·78346	9·90005	9·88341	10·11659	9·79319	9·89415	9·89905	10·10095	36
25	9·78362	9·89995	9·88367	10·11633	9·79335	9·89405	9·89931	10·10069	35
26	9·78379	9·89985	9·88393	10·11607	9·79351	9·89395	9·89957	10·10043	34
27	9·78395	9·89976	9·88420	10·11580	9·79367	9·89385	9·89983	10·10017	33
28	9·78412	9·89966	9·88446	10·11554	9·79383	9·89375	9·90009	10·09991	32
29	9·78428	9·89956	9·88472	10·11528	9·79399	9·89364	9·90035	10·09965	31
30	9·78445	9·89947	9·88498	10·11502	9·79415	9·89354	9·90061	10·09939	30
31	9·78461	9·89937	9·88524	10·11476	9·79431	9·89344	9·90086	10·09914	29
32	9·78478	9·89927	9·88550	10·11450	9·79447	9·89334	9·90112	10·09888	28
33	9·78494	9·89918	9·88577	10·11424	9·79463	9·89324	9·90138	10·09862	27
34	9·78510	9·89908	9·88603	10·11397	9·79478	9·89314	9·90164	10·09836	26
35	9·78527	9·89898	9 88629	10·11371	9·79494	9·89304	9·90190	10·09810	25
36	9·78543	9·89888	9·88655	10·11345	9·79510	9·89294	9·90216	10·09784	24
37	9·78560	9·89879	9·88681	10·11319	9·79526	9·89284	9·90242	10·09758	23
38	9·78576	9·89869	9·88707	10·11293	9·79542	9·89274	9·90268	10·09732	22
39	9·78592	9·89859	9·88733	10·11267	9·79558	9·89264	9·90294	10·09706	21
40	9·78609	9·89849	9·88759	10·11241	9·79573	9·89254	9·90320	10·09680	20
41	9·78625	9·89840	9·88786	10·11214	9·79589	9·89244	9·90346	10·09654	19
42	9·78642	9·89830	9·88812	10·11188	9·79605	9·89233	9·90371	10·09629	18
43	9·78658	9·89820	9·88838	10·11162	9·79621	9·89223	9·90397	10·09603	17
44	9·78674	9·89810	9·88864	10·11136	9·79636	9·89213	9·90423	10·09577	16
45	9·78691	9·89801	9·88890	10·11110	9·79652	9·89203	9·90449	10·09551	15
46	9·78707	9·89791	9·88916	10·11084	9·79668	9·89193	9·90475	10·09525	14
47	9·78723	9·89781	9·88942	10·11058	9·79684	9·89183	9·90501	10·09499	13
48	9·78739	9·89771	9·88968	10·11032	9·79699	9·89173	9·90527	10·09473	12
49	9·78756	9·89761	9·88994	10·11006	9·79715	9·89162	9·90553	10·09447	11
50	9·78772	9·89752	9·89020	10·10980	9·79731	9·89152	9·90578	10·09422	10
51	9·78788	9·89742	9·89046	10·10954	9·79746	9·89142	9·90604	10·09396	9
52	9·78805	9·89732	9·89073	10·10927	9·79762	9·89132	9·90630	10·09370	8
53	9·78821	9·89722	9·89099	10·10901	9·79778	9·89122	9·90656	10·09344	7
54	9·78837	9·89712	9·89125	10·10875	9·79793	9·89112	9·90682	10·09318	6
55	9·78853	9·89702	9·89151	10·10849	9·79809	9·89101	9·90708	10·09292	5
56	9·78869	9·89693	9·89177	10·10823	9·79825	9 89091	9·90734	10·09266	4
57	9·78886	9·89683	9·89203	10·10797	9·79840	9·89081	9·90759	10·09241	3
58	9·78902	9·89673	9·89229	10·10771	9·79856	9·89071	9·90785	10·09215	2
59	9·78918	9·89663	9·89255	10·10745	9·79872	9·89060	9·90811	10·09189	1
60	9·78934	9·89653	9·89281	10·10719	9·79887	9·89050	9·90837	10·09163	0
′	Cosine	Sine	Cotang.	Tang.	Cosine	Sine	Cotang.	Tang.	′
	52°				51°				

	39°				40°				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	′
0	9·79887	9·83050	9·90837	10·09163	9·80807	9·88425	9·92381	10·07619	60
1	9·79903	9·89040	9·90868	10·09137	9·80822	9·88415	9·92407	10·07593	59
2	9·79918	9·89030	9·90889	10·09111	9·80837	9·88404	9·92433	10·07567	58
3	9·79934	9·89020	9·90914	10·09086	9·80852	9·88394	9·92458	10·07542	57
4	9·79950	9·89009	9·90940	10·09060	9·80867	9·88383	9·92484	10·07516	56
5	9·79965	9·88999	9·90966	10·09034	9·80882	9·88372	9·92510	10·07490	55
6	9·79981	9·88989	9·90992	10·09008	9·80897	9·88362	9·92535	10·07465	54
7	9·79996	9·88978	9·91018	10·08982	9·80912	9·88351	9·92561	10·07439	53
8	9·80012	9·88968	9·91043	10·08957	9·80927	9·88340	9·92587	10·07413	52
9	9·80027	9·88958	9·91069	10·08931	9·80942	9·88330	9·92612	10·07388	51
10	9·80043	9·88948	9·91095	10·08905	9·80957	9·88319	9·92638	10·07362	50
11	9·80058	9·88937	9·91121	10·08879	9·80972	9·88308	9·92663	10·07337	49
12	9·80074	9·88927	9·91147	10·08853	9·81987	9·88298	9·92689	10·07311	48
13	9·80089	9·88917	9·91172	10·08828	9·81002	9·88287	9·92715	10·07285	47
14	9·80105	9·88906	9·91198	10·08802	9·81017	9·88276	9·92740	10·07260	46
15	9·80120	9·88896	9·91224	10·08776	9·81032	9·88266	9·92766	10·07234	45
16	9·80136	9·88886	9·91250	10·08750	9·81047	9·88255	9·92792	10·07208	44
17	9·80151	9·88875	9·91276	10·08724	9·81061	9·88244	9·92817	10·07183	43
18	9·80166	9·88865	9·91301	10·08699	9·81076	9·88234	9·92843	10·07157	42
19	9·80182	9·88855	9·91327	10·08673	9·81091	9·88223	9·92868	10·07132	41
20	9·80197	9·88844	9·91353	10·08647	9·81106	9·88212	9·92894	10·07106	40
21	9·80213	9·88834	9·91379	10·08621	9·81121	9·88201	9·92920	10·07080	39
22	9·80228	9·88824	9·91404	10·08596	9·81136	9 88191	9·92945	10·07055	38
23	9·80244	9·88813	9·91430	10·08570	9·81151	9·88180	9·92971	10·07029	37
24	9·80259	9·88803	9·91456	10·08544	9·81166	9·88169	9·92996	10·07004	36
25	9·80274	9·88793	9·91482	10·08518	9·81180	9·88158	9·93022	10·06978	35
26	9·80290	9·88782	9·91507	10·08493	9·81195	9·88148	9·93048	10·06952	34
27	9·80305	9·88772	9·91533	10·08467	9·81210	9·88137	9·93073	10·06927	33
28	9·80320	9·88761	9·91559	10·08441	9·81225	9·88126	9·93099	10·06901	32
29	9·80336	9·88751	9·91585	10·08415	9·81240	9·88115	9·93124	10·06876	31
30	9·80351	9·88741	9·91610	10·08390	9·81254	9·88105	9·93150	10·06850	30
31	9·80366	9·88730	9·91636	10·08364	9·81269	9·88094	9·93175	10·06825	29
32	9·80382	9·88720	9·91662	10·08338	9·81284	9·88083	9·93201	10·06799	28
33	9·80397	9·88709	9·91688	10·08312	9·81299	9·88072	9·93227	10·06773	27
34	9·80412	9·88699	9·91713	10·08287	9·81314	9·88061	9·93252	10·06748	26
35	9·80428	9·88688	9·91739	10·08261	9·81328	9·88051	9·93278	10·06722	25
36	9·80443	9·88678	9·91765	10·08235	9·81343	9·88040	9·93303	10·06697	24
37	9·80458	9·88668	9·91791	10·08209	9·81358	9·88029	9·93329	10·06671	23
38	9·80473	9·88657	9·91816	10·08184	9·81372	9·88018	9·93354	10·06646	22
39	9·80489	9·88647	9·91842	10·08158	9·81387	9·88007	9·93380	10·06620	21
40	9·80504	9·88636	9·91868	10·08132	9·81402	9·87996	9·93406	10·06594	20
41	9·80519	9·88626	9·91893	10·08107	9·81417	9·87985	9·93431	10·06569	19
42	9·80534	9·88615	9·91919	10·08081	9·81431	9·87975	9·93457	10·06543	18
43	9·80550	9·88605	9·91945	10·08055	9·81446	9·87964	9·93482	10·06518	17
44	9·80565	9·88594	9·91971	10·08029	9·81461	9·87953	9·93508	10·06492	16
45	9·80580	9·88584	9·91996	10·08004	9·81475	9·87942	9·93533	10·06467	15
46	9·80595	9·88573	9·92022	10·07978	9·81490	9·87931	9·93559	10·06441	14
47	9·80610	9·88563	9·92048	10·07952	9·81505	9·87920	9·93584	10·06416	13
48	9·80625	9·88552	9·92073	10·07927	9·81519	9·87909	9·93610	10·06390	12
49	9·80641	9·88542	9·92099	10·07901	9·81534	9·87898	9·93636	10·06364	11
50	9·80656	9·88531	9·92125	10·07875	9·81549	9·87887	9·93661	10·06339	10
51	9·80671	9·88521	9·92150	10·07850	9·81563	9·87877	9·93687	10·06313	9
52	9·80686	9·88510	9·92176	10·07824	9·81578	9·87866	9·93712	10·06288	8
53	9·80701	9·88499	9·92202	10·07798	9·81592	9·87855	9·93738	10·06262	7
54	9·80716	9·88489	9·92227	10·07773	9·81607	9·87844	9·93763	10·06237	6
55	9·80731	9·88478	9·92253	10·07747	9·81622	9·87833	9·93789	10·06211	5
56	9·80746	9·88468	9·92279	10·07721	9·81636	9·87822	9·93814	10·06186	4
57	9·80762	9·88457	9·92304	10·07696	9·81651	9·87811	9·93840	10·06160	3
58	9·80777	9·88447	9·92330	10·07670	9·81665	9·87800	9·93865	10·06135	2
59	9·80792	9·88436	9·92356	10·07644	9·81680	9·87789	9·93891	10·06109	1
60	9·80807	9·88425	9·92381	10·07619	9·81694	9·87778	9·93916	10·06084	0
′	Cosine	Sine	Cotang.	Tang.	Cosine	Sine	Cotang.	Tang.	′
	50°				49°				

	41°				42°				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	′
0	9·81694	9·87778	9·93916	10·06084	9·82551	9·87107	9·95444	10·04556	60
1	9·81709	9·87767	9·93942	10·06058	9·82565	9·87096	9·95469	10·04531	59
2	9·81723	9·87756	9·93967	10·06033	9·82579	9·87085	9·95495	10·04505	58
3	9·81738	9·87745	9·93993	10·06007	9·82593	9·87073	9·95520	10·04480	57
4	9·81752	9·87734	9·94018	10·05982	9·82607	9·87062	9·95545	10·04455	56
5	9·81767	9·87723	9·94044	10·05956	9·82621	9·87050	9·95571	10·04429	55
6	9·81781	9·87712	9·94069	10·05931	9·82635	9·87039	9·95596	10·04404	54
7	9·81796	9·87701	9·94095	10·05905	9·82649	9·87028	9·95622	10·04378	53
8	9·81810	9·87690	9·94120	10·05880	9·82663	9·87016	9·95647	10·04353	52
9	9·81825	9·87679	9·94146	10·05854	9·82677	9·87005	9·95672	10·04328	51
10	9·81839	9·87668	9·94171	10·05829	9·82691	9·86993	9·95698	10·04302	50
11	9·81854	9·87657	9·94197	10·05803	9·82705	9·86982	9·95723	10·04277	49
12	9·81868	9·87646	9·94222	10·05778	9·82719	9·86970	9·95748	10·04252	48
13	9·81882	9·87635	9·94248	10·05752	9·82733	9·86959	9·95774	10·04226	47
14	9·81897	9·87624	9·94273	10·05727	9·82747	9·86947	9·95799	10·04201	46
15	9·81911	9·87613	9·94299	10·05701	9·82761	9·86936	9·95825	10·04175	45
16	9·81926	9·87601	9·94324	10·05676	9·82775	9·86924	9·95850	10·04150	44
17	9·81940	9·87590	9·94350	10·05650	9·82788	9·86913	9·95875	10·04125	43
18	9·81955	9·87579	9·94375	10·05625	9·82802	9·86902	9·95901	10·04099	42
19	9·81969	9·87568	9·94401	10·05599	9·82816	9·86890	9·95926	10·04074	41
20	9·81983	9·87557	9·94426	10·05574	9·82830	9·86879	9·95952	10·04048	40
21	9·81998	9·87546	9·94452	10·05548	9·82844	9·86867	9·95977	10·04023	39
22	9·82012	9·87535	9·94477	10·05523	9·82858	9·86855	9·96002	10·03998	38
23	9·82026	9·87524	9·94503	10·05497	9·82872	9·86844	9·96028	10·03972	37
24	9·82041	9·87513	9·94528	10·05472	9·82885	9·86832	9·96053	10·03947	36
25	9·82055	9·87501	9·94554	10·05446	9·82899	9·86821	9·96078	10·03922	35
26	9·82069	9·87490	9·94579	10·05421	9·82913	9·86809	9·96104	10·03896	34
27	9·82084	9·87479	9·94604	10·05396	9·82927	9·86798	9·96129	10·03871	33
28	9·82098	9·87468	9·94630	10·05370	9·82941	9·86786	9·96155	10·03845	32
29	9·82112	9·87457	9·94655	10·05345	9·82955	9·86775	9·96180	10·03820	31
30	9·82126	9·87446	9·94681	10·05319	9·82968	9·86763	9·96205	10·03795	30
31	9·82141	9·87434	9·94706	10·05294	9·82982	9·86752	9·96231	10·03769	29
32	9·82155	9·87423	9·94732	10·05268	9·82996	9·86740	9·96256	10·03744	28
33	9·82169	9·87412	9·94757	10·05243	9·83010	9·86728	9·96281	10·03719	27
34	9·82184	9·87401	9·94783	10·05217	9·83023	9·86717	9·96307	10·03693	26
35	9·82198	9·87390	9·94808	10·05192	9·83037	9·86705	9·96332	10·03668	25
36	9·82212	9·87378	9·94834	10·05166	9·83051	9·86694	9·96357	10·03643	24
37	9·82226	9·87367	9·94859	10·05141	9·83065	9·86682	9·96383	10·03617	23
38	9·82240	9·87356	9·94884	10·05116	9·83078	9·86670	9·96408	10·03592	22
39	9·82255	9·87345	9·94910	10·05090	9·83092	9·86659	9·96433	10·03567	21
40	9·82269	9·87334	9·94935	10·05065	9·83106	9·86647	9·96459	10·03541	20
41	9·82283	9·87322	9·94961	10·05039	9·83120	9·86635	9·96484	10·03516	19
42	9·82297	9·87311	9·94986	10·05014	9·83133	9·86624	9·96510	10·03490	18
43	9·82311	9·87300	9·95012	10·04988	9·83147	9·86612	9·96535	10·03465	17
44	9·82326	9·87288	9·95037	10·04963	9·83161	9·86600	9·96560	10·03440	16
45	9·82340	9·87277	9·95062	10·04938	9·83174	9·86589	9·96586	10·03414	15
46	9·82354	9·87266	9·95088	10·04912	9·83188	9·86577	9·96611	10·03389	14
47	9·82368	9·87255	9·95113	10·04887	9·83202	9·86565	9·96636	10·03364	13
48	9·82382	9·87243	9·95139	10·04861	9·83215	9·86554	9·96662	10·03338	12
49	9·82396	9·87232	9·95164	10·04836	9·83229	9·86542	9·96687	10·03313	11
50	9·82410	9·87221	9·95190	10·04810	9·83242	9·86530	9·96712	10·03288	10
51	9·82424	9·87209	9·95215	10·04785	9·83256	9·86518	9·96738	10·03262	9
52	9·82439	9·87198	9·95240	10·04760	9·83270	9·86507	9·96763	10·03237	8
53	9·82453	9·87187	9·95266	10·04734	9·83283	9·86495	9·96788	10·03212	7
54	9·82467	9·87175	9·95291	10·04709	9·83297	9·86483	9·96814	10·03186	6
55	9·82481	9·87164	9·95317	10·04683	9·83310	9·86472	9·96839	10·03161	5
56	9·82495	9·87153	9·95342	10·04658	9·83324	9·86460	9·96864	10·03136	4
57	9·82509	9·87141	9·95368	10·04632	9·83338	9·86448	9·96890	10·03110	3
58	9·82523	9·87130	9·95393	10·04607	9·83351	9·86436	9·96915	10·03085	2
59	9·82537	9·87119	9·95418	10·04582	9·83365	9·86425	9·96940	10·03060	1
60	9·82551	9·87107	9·95444	10·04556	9·83378	9·86413	9·96966	10·03034	0
′	Cosine	Sine	Cotang.	Tang.	Cosine	Sine	Cotang.	Tang.	′
	48°				47°				

	43°				44°				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	′
0	9·83378	9·86413	9·96966	10·03034	9·84177	9·85693	9·98484	10·01516	60
1	9·83392	9·86401	9·96991	10·03009	9·84190	9·85681	9·98509	10·01491	59
2	9·83405	9·86389	9·97016	10·02984	9·84203	9·85669	9·98534	10·01466	58
3	9·83419	9·86377	9·97042	10·02958	9·84216	9·85657	9·98560	10·01440	57
4	9·83432	9·86366	9·97067	10·02933	9·84229	9·85645	9·98585	10·01415	56
5	9·83446	9·86354	9·97092	10·02908	9·84242	9·85632	9·98610	10·01390	55
6	9·83459	9·86342	9·97118	10·02882	9·84255	9·85620	9·98635	10·01365	54
7	9·83473	9·86330	9·97143	10·02857	9·84269	9·85608	9·98661	10·01339	53
8	9·83486	9·86318	9·97168	10·02832	9·84282	9·85596	9·98686	10·01314	52
9	9·83500	9·86306	9·97193	10·02807	9·84295	9·85583	9·98711	10·01289	51
10	9·83513	9·86295	9·97219	10·02781	9·84308	9·85571	9·98737	10·01263	50
11	9·83527	9·86283	9·97244	10·02756	9·84321	9·85559	9·98762	10·01238	49
12	9·83540	9·86271	9·97269	10·02731	9·84334	9·85547	9·98787	10·01213	48
13	9·83554	9·86259	9·97295	10·02705	9·84347	9·85534	9·98812	10·01188	47
14	9·83567	9·86247	9·97320	10·02680	9·84360	9·85522	9·98838	10·01162	46
15	9·83581	9·86235	9·97345	10·02655	9·84373	9·85510	9·98863	10·01137	45
16	9·83594	9·86223	9·97371	10·02629	9·84385	9·85497	9·98888	10·01112	44
17	9·83608	9·86211	9·97396	10·02604	9·84398	9·85485	9·98913	10·01087	43
18	9·83621	9·86200	9·97421	10·02579	9·84411	9·85473	9·98939	10·01061	42
19	9·83634	9·86188	9·97447	10·02553	9·84424	9·85460	9·98964	10·01036	41
20	9·83648	9·86176	9·97472	10·02528	9·84437	9·85448	9·98989	10·01011	40
21	9·83661	9·86164	9·97497	10·02503	9·84450	9·85436	9·99015	10·00985	39
22	9·83674	9·86152	9·97523	10·02477	9·84463	9·85423	9·99040	10·00960	38
23	9·83688	9·86140	9·97548	10·02452	9·84476	9·85411	9·99065	10·00935	37
24	9·83701	9·86128	9·97573	10·02427	9·84489	9·85399	9·99090	10·00910	36
25	9·83715	9·86116	9·97598	10·02402	9·84502	9·85386	9·99116	10·00884	35
26	9·83728	9·86104	9·97624	10·02376	9·84515	9·85374	9·99141	10·00859	34
27	9·83741	9·86092	9·97649	10·02351	9·84528	9·85361	9·99166	10·00834	33
28	9·83755	9·86080	9·97674	10·02326	9·84540	9·85349	9·99191	10·00809	32
29	9·83768	9·86068	9·97700	10·02300	9·84553	9·85337	9·99217	10·00783	31
30	9·83781	9·86056	9·97725	10·02275	9·84566	9·85324	9·99242	10·00758	30
31	9·83795	9·86044	9·97750	10·02250	9·84579	9·85312	9·99267	10·00733	29
32	9·83808	9·86032	9·97776	10·02224	9·84592	9·85299	9·99293	10·00707	28
33	9·83821	9·86020	9·97801	10·02199	9·84605	9·85287	9·99318	10·00682	27
34	9·83834	9·86008	9·97826	10·02174	9·84618	9·85274	9·99343	10·00657	26
35	9·83848	9·85996	9·97851	10·02149	9·84630	9·85262	9·99368	10·00632	25
36	9·83861	9·85984	9·97877	10·02123	9·84643	9·85250	9·99394	10·00606	24
37	9·83874	9·85972	9·97902	10·02098	9·84656	9·85237	9·99419	10·00581	23
38	9·83887	9·85960	9·97927	10·02073	9·84669	9·85225	9·99444	10·00556	22
39	9·83901	9·85948	9·97953	10·02047	9·84682	9·85212	9·99469	10·00531	21
40	9·83914	9·85936	9·97978	10·02022	9·84694	9·85200	9·99495	10·00505	20
41	9·83927	9·85924	9·98003	10·01997	9·84707	9·85187	9·99520	10·00480	19
42	9·83940	9·85912	9·98029	10·01971	9·84720	9·85175	9·99545	10·00455	18
43	9·83954	9·85900	9·98054	10·01946	9·84733	9·85162	9·99570	10·00430	17
44	9·83967	9·85888	9·98079	10·01921	9·84745	9·85150	9·99596	10·00404	16
45	9·83980	9·85876	9·98104	10·01896	9·84758	9·85137	9·99621	10·00379	15
46	9·83993	9·85864	9·98130	10·01870	9·84771	9·85125	9·99646	10·00354	14
47	9·84006	9·85851	9·98155	10·01845	9·84784	9·85112	9·99672	10·00328	13
48	9·84020	9·85839	9·98180	10·01820	9·84796	9·85100	9·99697	10·00303	12
49	9·84033	9·85827	9·98206	10·01794	9·84809	9·85087	9·99722	10·00278	11
50	9·84046	9·85815	9·98231	10·01769	9·84822	9·85074	9·99747	10·00253	10
51	9·84059	9·85803	9·98256	10·01744	9·84835	9·85062	9·99773	10·00227	9
52	9·84072	9·85791	9·98281	10·01719	9·84847	9·85049	9·99798	10·00202	8
53	9·84085	9·85779	9·98307	10·01693	9·84860	9·85037	9·99823	10·00177	7
54	9·84098	9·85766	9·98332	10·01668	9·84873	9·85024	9·99848	10·00152	6
55	9·84112	9·85754	9·98357	10·01643	9·84885	9·85012	9·99874	10·00126	5
56	9·84125	9·85742	9·98383	10·01617	9·84898	9·84999	9·99899	10·00101	4
57	9·84138	9·85730	9·98408	10·01592	9·84911	9·84986	9·99924	10·00076	3
58	9·84151	9·85718	9·98433	10·01567	9·84923	9·84974	9·99949	10·00051	2
59	9·84164	9·85706	9·98458	10·01542	9·84936	9·84961	9·99975	10·00025	1
60	9·84177	9·85693	9·98484	10·01516	9·84949	9·84949	10·00000	10·00000	0
′	Cosine	Sine	Cotang.	Tang.	Cosine	Sine	Cotang.	Tang.	′
	46°				45°				

TABLE IV.

Refraction in altitude, of the heavenly bodies.

Alt.	Refr.	Alt.	Refr.	Alt.	Refr.	Alt.	Refr.	Alt.	Refr.
° ′	′ ″	° ′	′ ″	° ′	′ ″	°	′ ″	°	′ ″
0. 0	33. 0	4. 0	11.51	12.20	4.16	30	1.38	60	0.33
0. 5	32.10	4.10	11.29	12.40	4. 9	31	1.35	61	0.32
0.10	31.22	4.20	11. 8	13. 0	4. 3	32	1.81	62	0.30
0.15	30.35	4.30	10.48	13.20	3.57	33	1.28	63	0.29
0.20	29.50	4.40	10.29	13.40	3.51	34	1.24	64	0.28
0.25	29. 6	4.50	10.11	14. 0	3.45	35	1.21	65	0.26
0.30	28.23	5. 0	9.54	14.20	3.40	36	1.18	66	0.25
0.35	27.41	5.10	9.38	14.40	3.35	37	1.16	67	0.24
0.40	27. 0	5.20	9.23	15. 0	3.30	38	1.13	68	0.23
0.45	26.20	5.30	9. 8	15.30	3.24	39	1.10	69	0.22
0.50	25.42	5.40	8.54	16. 0	3.17	40	1. 8	70	0.21
0.55	25. 5	5.50	8.41	16.30	3.10	41	1. 5	71	0.19
1. 0	24.29	6. 0	8.28	17. 0	3. 4	42	1. 3	72	0.18
1. 5	23.54	6.10	8.15	17.30	2.59	43	1. 1	73	0.17
1.10	23.20	6.20	8. 3	18. 0	2.54	44	0.59	74	0.16
1.15	22.47	6.30	7.51	18.30	2.49	45	0.57	75	0.15
1.20	22.15	6.40	7.40	19. 0	2.44	46	0.55	76	0.14
1.25	21.44	6.50	7.30	19.30	2.39	47	0.53	77	0.13
1.30	21.15	7. 0	7.20	20. 0	2.35	48	0.51	78	0.12
1.35	20.46	7.10	7.11	20.30	2.31	49	0.49	79	0.11
1.40	20.18	7.20	7. 2	21. 0	2.27	50	0.48	80	0.10
1.45	19.51	7.30	6.53	21.30	2.24	51	0.46	81	0. 9
1.50	19.25	7.40	6.45	22. 0	2.20	52	0.44	82	0. 8
1.55	19. 0	7.50	6.37	23. 0	2.14	53	0.43	83	0. 7
2. 0	18.35	8. 0	6.29	24. 0	2. 7	54	0.41	84	0. 6
2. 5	18.11	8.10	6.22	25. 0	2. 2	55	0.40	85	0. 5
2.10	17.48	8.20	6.15	26. 0	1.56	56	0.38	86	0. 4
2.15	17.26	8.30	6. 8	27. 0	1.51	57	0.37	87	0. 3
2.20	17. 4	8.40	6. 1	28. 0	1.47	58	0.35	88	0. 2
2.25	16.44	8.50	5.55	29. 0	1.42	59	0.34	89	0. 1
2.30	16.24	9. 0	5.48						
2.35	16. 4	9.10	5.42						
2.40	15.45	9.20	5.36						
2.45	15.27	9.30	5.31						
2.50	15. 9	9.40	5.25						
2.55	14.52	9.50	5.20						
3. 0	14.36	10. 0	5.15						
3. 5	14.20	10.15	5. 7						
3.10	14. 4	10.30	5. 0						
3.15	13.49	10.45	4.58						
3.20	13.34	11. 0	4.47						
3.25	13.20	11.15	4.40						
3.30	13. 6	11.30	4.34						
3.40	12.40	11.45	4.29						
3.50	12.15	12. 0	4.28						

TABLE V.

Depression of the horizon, at sea.

Height of the eye.	Dip.
Feet.	′ ″
1	0.57
2	1.21
3	1.39
4	1.55
5	2. 8
6	2.20
7	2.31
8	2.42
9	2.52
10	3. 1
11	3.10
12	3.18
13	3.26
14	3.34
15	3,42
16	3.49
17	3.56
18	4. 3
19	4.10
20	4.16
21	4.22
22	4.28
28	4.34
24	4.40
26	4,52
28	5. 3
30	5.14
35	5.39
40	6. 2

TABLE VI.

The Sun's parallax in altitude.

Alt.	Paral.	Alt.	Paral.
°	″	°	″
0	9	60	4
10	9	65	4
20	8	70	3
30	8	75	2
40	7	80	2
50	6	85	1
55	5	90	0

TABLE VII.

Augmentation of the moon's semidiameter.

Alt.	Aug.	Alt.	Aug.
°	″	°	″
0	0	40	10
5	1	45	11
10	3	50	12
15	4	55	13
20	6	60	14
25	7	70	15
30	8	80	16
35	9	—	—

TABLE VIII.

The mean right ascensions and declinations of 40 principal fixed Stars, from the *Nautical Almanac*; corrected to January 1. 1840. The Stars marked with an asterisk (*) are used to find the longitude at sea. (See page 339.)

Characters.	Constellations.	Proper Names.	Mag.	Right Ascension.			Annual Variation	Declinations.				Annual Variation
				h	m	s	s		°	′	″	″
α	Andromeda	Alpheratz	1	0	0	7·7	3·07	N.	28	12	27	+20·06
γ	Pegasi	Algenib	2.3	0	5	0·2	3·08		14	17	39	20·05
α	Arietis	Arietis	3	1	58	9·9	3·35		22	42	9	17·45
α	Ceti	Menkar	2.3	2	53	55·3	3·13		3	27	28	14·55
α	Tauri	Aldebaran	1	4	26	44·8	3·43		16	10	54	7·94
α	Aurigæ	Capella	1	5	4	52·8	4·41	N.	45	49	39	+ 4·77
β	Orionis	Rigel	1	5	6	51·1	2·88	S.	8	23	29	4·61
β	Tauri	- - -	2	5	16	11·0	3·78	N.	28	27	55	3·81
α	Orionis	Betilguese	1	5	46	30·7	3·24	N.	7	22	17	+ 1·18
α	Canis Majoris	Sirius	1	6	38	5·9	2·65	S.	16	30	7	— 4·46
α^2	Geminorum	Castor	3	7	24	23·0	3·86	N.	32	13	56	— 7·22
α	Canis Minoris	Procyon	1.2	7	30	55·3	3·14		5	37	47	8·73
β	Geminorum	Pollux	2	7	35	31·1	3·68	N.	28	24	23	8·12
α	Hydræ	Alphard	2	9	19	43·6	2·95	S.	7	58	5	15·35
α	Leonis	Regulus	1	9	59	50·8	3·22	N.	12	44	46	—17·36
α	Ursæ Majoris	Dubhe	1.2	10	53	47·8	3·81	N.	62	36	48	—19·23
β	Leonis	Deneb	2.3	11	40	53·7	3·07		15	27	58	19·99
γ	Ursæ Majoris	- - -	2	11	45	23·1	3·19	N.	54	35	3	20·02
α	Virginis	Spica	1	13	16	46·4	3·15	S.	10	19	26	18·94
η	Ursæ Majoris	Benetnasch	2.3	13	41	13·7	2·35	N.	50	6	50	18·13
α	Bootis	Arcturus	1	14	8	22·0	2·73	N.	20	1	5	18·95
α^2	Libræ	Zubenesch	3	14	42	2·3	3·31	S.	15	22	18	15·25
β	Libræ	Zubenelg	2.3	15	8	24·3	3·22	S.	8	47	15	13·65
α	Coronæ Borealis	Alphacca	2	15	27	54·9	2·53	N.	27	15	25	12·35
α	Serpentis	- - -	2.3	15	36	23·4	2·94	N.	6	56	3	11·76
α	Scorpii	Antares	1	16	19	36·5	3·66	S.	26	4	13	— 8·51
α	Herculis	Ras Algethi	3.4	17	7	21·3	2·73	N.	14	34	41	4·57
α	Ophiuchi	Ras Alhagus	2	17	27	30·6	2·77		12	40	59	2·83
γ	Draconis	Rastaban	2	17	52	53·6	1·39		51	30	37	0·62
α	Lyræ	Vega	1	18	31	31·3	2·01		38	38	17	2·75
γ	Aquilæ	- - -	3	19	38	39·1	2·85	N.	10	13	43	8·37
α	Aquilæ	Altair	1.2	19	42	58·6	2·93		8	27	0	8·71
β	Aquilæ	- - -	3.4	19	47	27·2	2·94	N.	6	0	44	8·52
α^2	Capricorni	- - -	3	20	9	10·3	3·33	S.	13	2	6	10·72
α	Cygni	Deneb	1	20	35	58·8	2·04	N.	44	42	40	12·62
α	Cephei	Alderamin	3	21	14	45·3	1·42	N.	61	54	33	15·07
β	Aquarii	- - -	3	21	23	7·9	3·16	S.	6	16	16	15·54
α	Aquarii	- - -	3	21	57	33·9	3·08	S.	1	5	38	17·26
α	Piscis. Aust.	Fomalhaut	1	22	48	47·6	3·31	S.	30	28	5	19·10
α	Pegasi	Marcab	2	22	56	47·8	2·98	N.	14	20	47	19·30

THE END.

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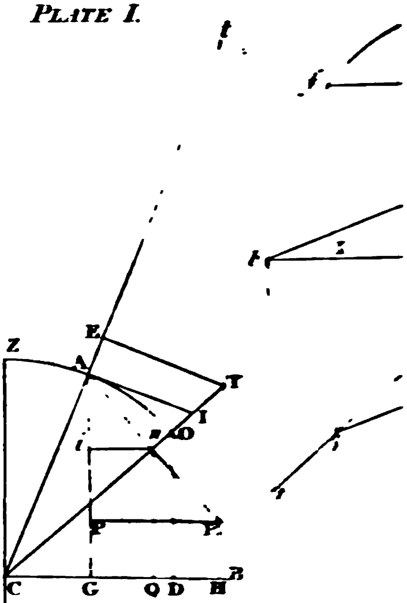
PLATE I.



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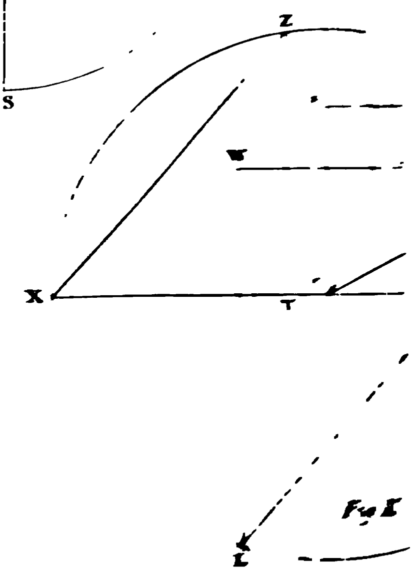

Fig I

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PLATE II.

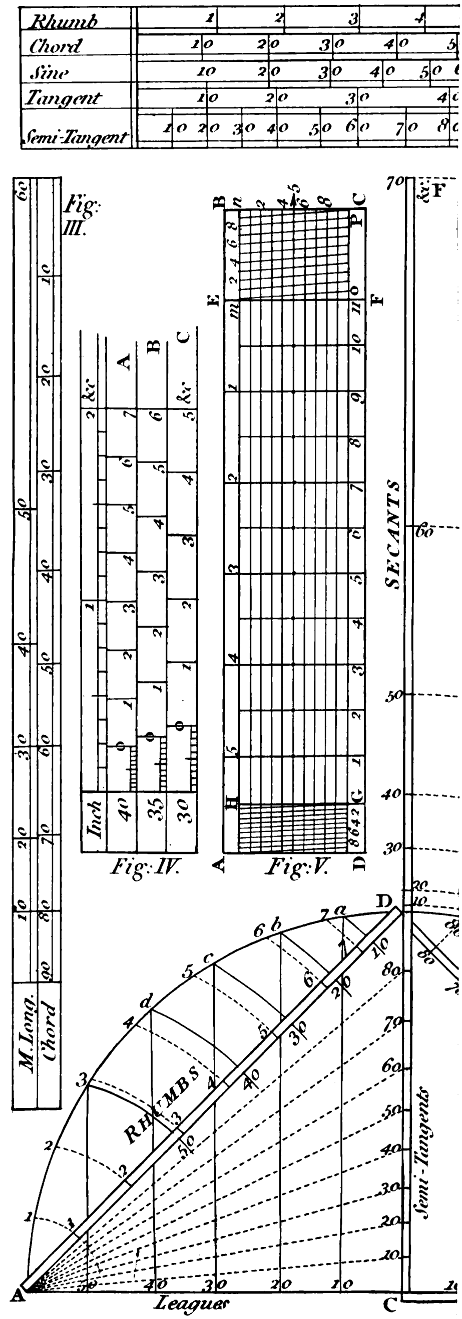

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PLATE III.

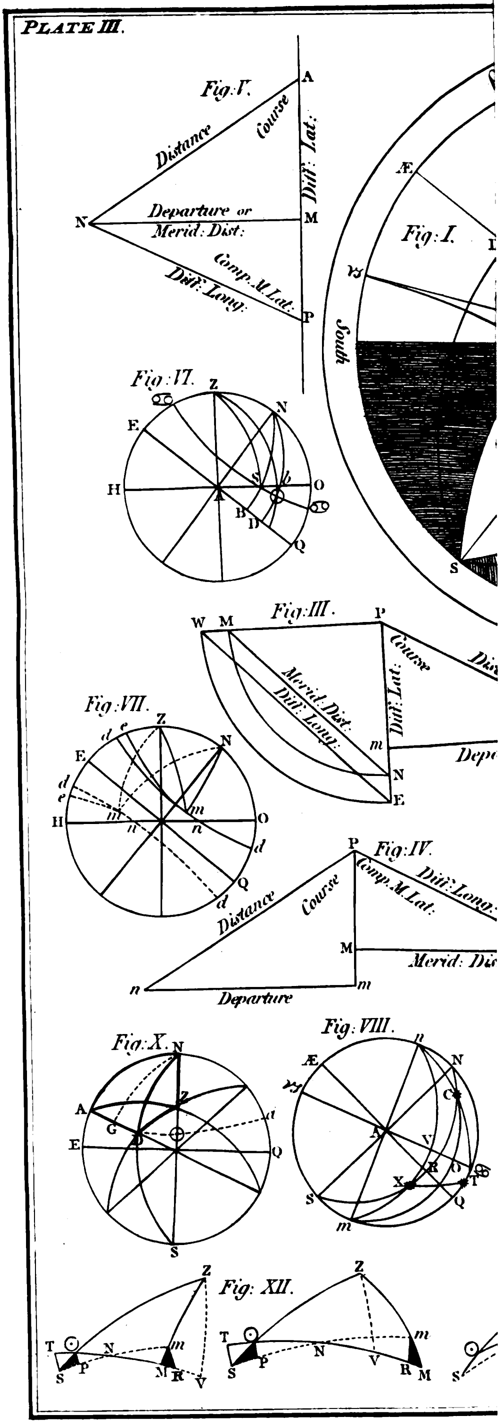
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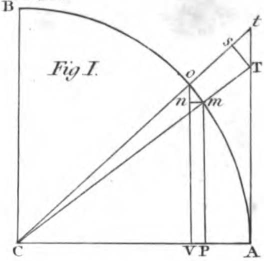
PLATE IV.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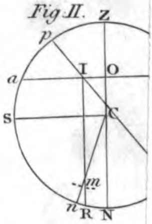
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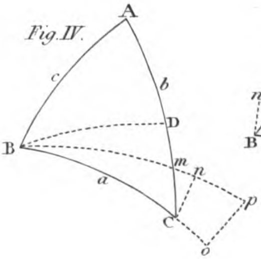
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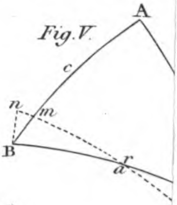
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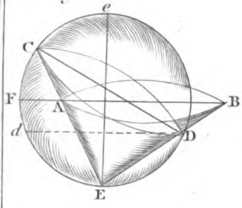
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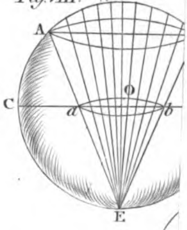
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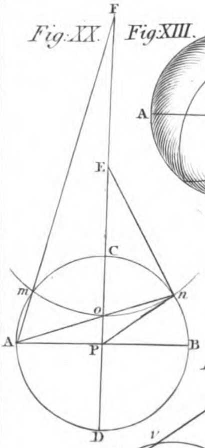
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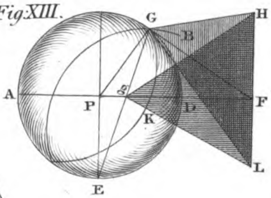
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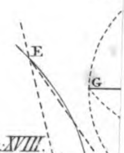
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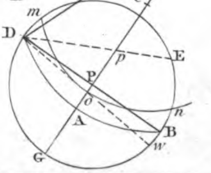
Fig: XXI.

Fig: XVIII.

Rusell sculp.

PLATE V.

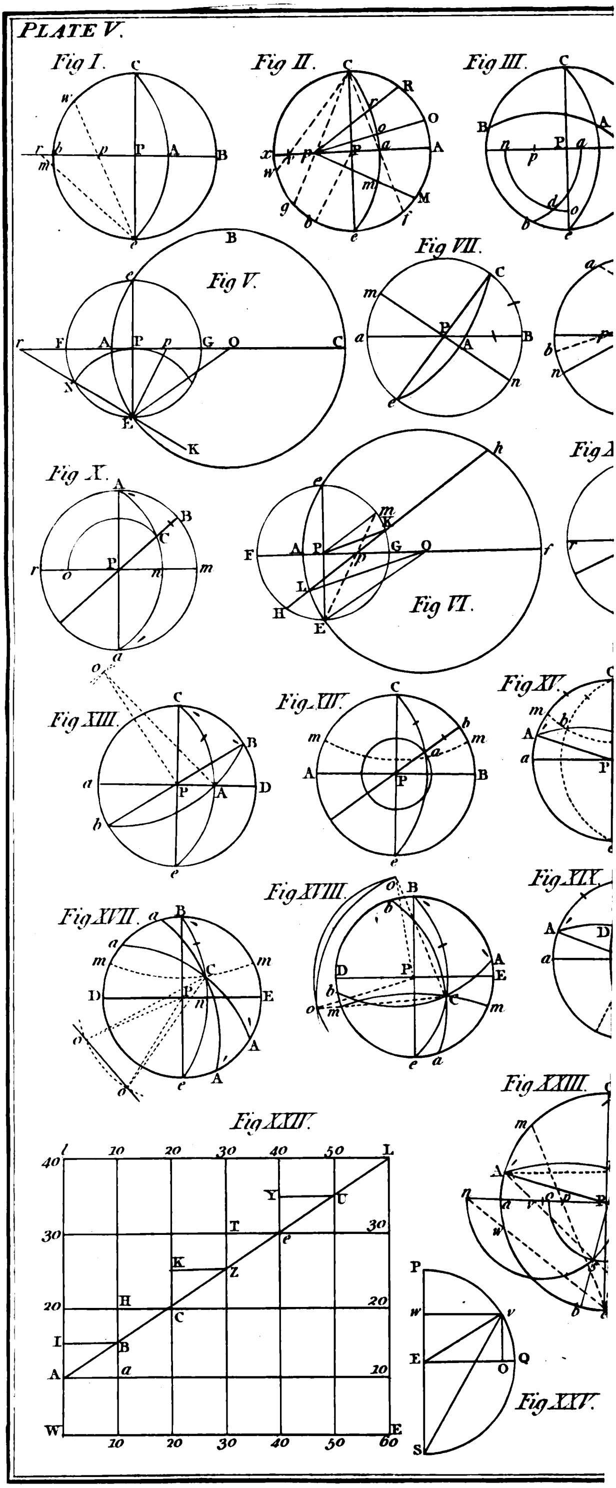
Fig I.
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